EXERCISE CLASS: PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS

Pseudodifferential operators and principal symbols

Exercise 1. Let $\Delta := \partial_x^2 + \partial_y^2$ be the Laplacian in the standard coordinates (x, y) of $\mathbb{R}^2$. Let

$$[0, \infty) \times (\mathbb{R}/2\pi\mathbb{Z}) \ni (r, \theta) \mapsto re^{i\theta} \in \mathbb{R}^2,$$

be the polar coordinates.

(1) Show that, in polar coordinates, the expression of Δ is given by

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\,\partial_\theta^2.$$

(2) What is the full symbol of Δ in the standard coordinates?
(3) What is the full symbol of Δ in the polar coordinates?
(4) Is the full symbol invariant by this change of coordinates?
(5) Show that the principal symbol of Δ is invariant by this change of coordinates.

Exercise 2. Let $X \in C^\infty(M, TM)$ be a smooth vector field. It induces a first order differential operator by the formula $X\varphi(x) := d\varphi(X(x))$, for $\varphi \in C^\infty(M)$.

(1) Compute the principal symbol $\sigma_X \in S^1(T^*M)$.
(2) Compute $\mathrm{ell}(X) \subset T^*M \setminus \{0\}$, the set of ellipticity of the operator.
(3) Let $E \to M$ be a real vector bundle equipped with a connection ∇. Define the operator $\mathbf{X} := \nabla_X$ acting on $C^\infty(M, E)$ by $\mathbf{X}\varphi(x) := \nabla_X\varphi(x)$. Compute $\sigma_{\mathbf{X}} \in S^1(T^*M, \mathrm{End}(E))$. What is $\mathrm{ell}(X)$?

Exercise 3. Let M be a smooth closed manifold and μ be a smooth density on M. Let $P \in \Psi^m(X)$. Recall that the *formal adjoint* P^* of P is the (unique) operator satisfying the equality: for all $\varphi, \psi \in C^\infty(M)$:

$$\langle P\varphi, \psi\rangle_{L^2(M,\mathrm{d}\mu)} := \int_M (P\varphi)\psi \;\; \mathrm{d}\mu = \int_M \varphi(P^*\psi)\;\mathrm{d}\mu = \langle \varphi, P^*\psi\rangle_{L^2(M,\mathrm{d}\mu)}.$$

We showed in class that $P^* \in \Psi^m(X)$. Nevertheless, the formal adjoint P^* *depends* on a choice of density μ. As a consequence, we will rather write P^*_μ for the formal adjoint of P with respect to the density μ.

(1) Let $\mu' := a \cdot \mu$ be another smooth density on M, where $a \in C^\infty(M)$ is a positive function. Compute $P^*_{\mu'}$ in terms of a and P^*_μ.

(2) Deduce that the principal symbol σ_{P^*} is well-defined, independently of the choice of density μ.

Let $X \in C^\infty(M, TM)$ be a smooth vector field, seen as a differential operator of order 1 acting on $C^\infty(M)$. If μ is a smooth density on M, we define the *divergence* of X with respect to μ as the (unique) function satisfying the equality $\mathcal{L}_X \mu = \mathrm{div}_\mu(X)\mu$.

(3) Recall the value of $\sigma_X \in S^1(T^*M)$.
(4) Show that $\int_M \mathcal{L}_X \mu = 0$.
(5) Deduce the formal adjoint X^*_μ of X, computed with respect to μ.
(6) What is σ_{X^*}?

Exercise 4. Let g be a smooth metric on TM. Recall that the gradient $\nabla : C^\infty(M) \to C^\infty(M, T_{\mathbb{C}}M)$ (computed with respect to the metric g) is defined by $\nabla \varphi := (d\varphi)^\sharp$, where $\sharp : T^*M \to TM$ denotes the musical isomorphism[1]. It is a differential operator of order 1, that is $\nabla \in \Psi^1(M, \mathbb{C} \to T_{\mathbb{C}}M)$.

(1) Show that its principal symbol is given by $\sigma_\nabla(x, \xi) = i\xi^\sharp$.
(2) The metric g induces a metric on T^*M, denoted by g^{-1} and defined by $|\xi|^2_{g^{-1}} := |\xi^\sharp|^2_g$. Assume that g is given in a local patch of coordinates by the symmetric matrix $(g_{ij})_{1 \leq i,j \leq n}$. Compute the symmetric matrix defining the metric g^{-1} on T^*M in these coordinates.
(3) We define the (non-negative) Laplacian $\Delta := \nabla^* \nabla$. What is the order of this operator? Show that $\sigma_\Delta(x, \xi) = |\xi|^2_{g^{-1}}$.

Exercise 5. The exterior derivative acts as $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ by the usual formula in coordinates:

$$d(f_I dx_I) = \sum_{i=1}^n \partial_{x_i} f_I dx_i \wedge dx_I,$$

where $dx_I = dx_{i_1} \wedge ... \wedge dx_{i_k}$ for some indices $i_1, ..., i_k \in \{1, ..., n\}$.

(1) Show that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, if α is a p-form.
(2) Compute $\sigma_d \in S^1(T^*M, \mathrm{Hom}(\Lambda^k T^*M, \Lambda^{k+1} T^*M))$.
(3) Is it injective/surjective? Describe the kernel and the range.

Let g be a smooth metric on TM. For all $k \in \{0, ..., n\}$, it induces a metric on $\Lambda^k T^*M$ by declaring $\mathbf{e}^*_{i_1} \wedge ... \wedge \mathbf{e}^*_{i_k}$ to be an orthonormal basis for all $i_1 < ... < i_k$, if $(\mathbf{e}_1, ..., \mathbf{e}_n)$ is a local orthonormal basis of TM.

(4) We let $d^* : C^\infty(M, \Lambda^{k+1} T^*M) \to C^\infty(M, \Lambda^k T^*M)$ be the adjoint (divergence operator). Compute σ_{d^*}.
(5) We define $\Delta := (d + d^*)^2$, the Hodge Laplacian acting on k-forms. Compute σ_Δ.

[1] $\xi^\sharp = Z$ is defined such that $(\xi, \bullet) = g(Z, \bullet)$.

Exercise 6. Let $X \subset \mathbb{R}^n$ and $A \in \Psi^m(X)$ be a properly supported pseudodifferential operator. We say that A is *classical* if its full symbol $\sigma_A^{\mathrm{full}}(x,\xi) = e^{-i\xi\cdot x}A(e^{i\xi\bullet})$ admits a *polyhomogeneous expansion*, namely, there exists a sequence of symbols $(a_{m-j})_{j\geq 0}$ such that $a_{m-j} \in S^{m-j}(T^*X)$ such that a_{m-j} is $(m-j)$-homogeneous in the ξ-variable for $|\xi| \geq 1$ and

$$\sigma_A^{\mathrm{full}} \sim \sum_{j\geq 0} a_{m-j}.$$

(1) Show that differential operators are classical.
(2) Show that the pullback (by a diffeomorphism) of a classical pseudodifferential operator is still classical.
(3) Define classical pseudodifferential operators on manifolds.
(4) Show that, given $A \in \Psi^m(M, E \to F)$, a classical pseudodifferential operator acting on sections of the vector bundle $E \to M$, the following holds: given $f \in C^\infty(M,E)$, $S \in C^\infty(M)$ such that $dS \neq 0$ on $\mathrm{supp}(f)$, show that for all $x \in \mathrm{supp}(f)$:

$$\lim_{h\to 0} e^{-\frac{i}{h}S(x)} h^m A(e^{\frac{i}{h}S} f)(x) = \sigma_A(x, dS(x)) f(x).$$

Exercise 7. Let M be a closed oriented manifold and let $X \in C^\infty(M, TM)$ be a smooth vector field. Define the first order differential operator $P := -i\mathcal{L}_X \in \Psi^1(M, \Lambda^k T^*M)$ on k-forms, where $0 \leq k \leq n := \dim M$. We let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow on M generated by X and define its *symplectic lift* as the flow $(\Phi_t)_{t\in\mathbb{R}}$ on T^*M given by:

$$\Phi_t(x,\xi) := (\varphi_t(x), (d\varphi_t)_x^{-\top}(\xi)).$$

Define $\Sigma := \{(x,\xi) \in T^*X \mid (\xi, X(x)) = 0\}$. The vector field X acts naturally on distributions, seen as "generalized" volume forms, by setting for $u \in \mathcal{D}'(M)$, $\psi \in C^\infty(M)$,

$$(\mathcal{L}_X u, \psi) := -(u, X\psi). \tag{0.1}$$

The action by pullback by the flow extends to distributions by setting for $u \in \mathcal{D}'(M), \psi \in C^\infty(M)$,

$$(\varphi_t^* u, \psi) := (u, (\varphi_t^{-1})^*\psi).$$

Note that $\varphi_t^{-1} = \varphi_{-t}$.

(1) Let $p \in M$ and define the Dirac mass at p by $(\delta_p, \varphi) := \varphi(p)$. Show that $\varphi_t^*\delta_p = \delta_{\varphi_{-t}p}$.
(2) Show that (0.1) holds if u is a smooth volume form, that is,

$$(\mathcal{L}_X u, \psi) = -(u, X\psi),$$

for all $u \in C^\infty(M, \Lambda^n T^*M), \psi \in C^\infty(M)$.
(3) Show that the principal symbol $\sigma_P \in S^1(M, \mathrm{End}(\Lambda^k T^*M))$ of P is given by

$$\sigma_P : (x,\xi) \mapsto (\xi, X(x))\mathbf{1}_{\Lambda^k T_x^* M}.$$

(4) Let $u \in \mathcal{D}'(M)$. Show that $\mathcal{L}_X u = 0$ if and only if $\varphi_t^* u = u$, for all $t \in \mathbb{R}$.

From now on, we will say that $u \in \mathcal{D}'(M)$ is **flow-invariant** if it satisfies $\varphi_t^* u = u$ for all $t \in \mathbb{R}$.

(5) Show that a flow-invariant distribution $u \in \mathcal{D}'(M)$ satisfies $\mathrm{WF}(u) \subset \Sigma \setminus \{0\}$.
(6) Given $u \in \mathcal{D}'(M)$, compute $\mathrm{WF}(\varphi_{-t}^* u)$ in terms of $\mathrm{WF}(u)$. Deduce that for a flow-invariant $u \in \mathcal{D}'(M)$, $\mathrm{WF}(u)$ is invariant by the flow $(\Phi_t)_{t\in\mathbb{R}}$.
(7) Let $x_0 \in M$ be a periodic point for the flow, that is, such that there exists $T > 0$ such that $\varphi_T(x_0) = x_0$. Define the distribution $\delta_\gamma \in \mathcal{D}'(M)$ by

$$\forall \psi \in C^\infty(M), \qquad (\delta_\gamma, \psi) := \int_0^T \psi(\varphi_t x_0) \mathrm{d}t.$$

Show that δ_γ is flow-invariant. Compute $\mathrm{WF}(\delta_\gamma)$.

Fredholm operators

Exercise 8. Let $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and define $P_s := D_x - s$ on $\mathbb{S}^1$, where $D_x := i^{-1}\partial_x$.

(1) Show that $\mathbb{R} \ni s \mapsto P_s \in \mathcal{L}(H^1(\mathbb{S}^1), L^2(\mathbb{S}^1))$ is an analytic family of Fredholm operators of index 0.
(2) Compute $\ker P_s$ and $\operatorname{ran} P_s$ for $s \in \mathbb{R}$. Are there special values?

Exercise 9. Let H be a (separable) Hilbert space. Let $(\pi_0(\mathcal{F}(H)), \circ)$ be the monoid[2] of connected components of $\mathcal{F}(H)$ endowed with the composition law $\circ$.

(1) Show that $(\pi_0(\mathcal{F}(H)), \circ)$ is a well-defined monoid. What is the identity element?
(2) Prove that $(\pi_0(\mathcal{F}(H)), \circ)$ can actually be turned into a group. What is the inverse of an element?
(3) Deduce that $\mathrm{ind} : \pi_0(\mathcal{F}(H)) \to \mathbb{Z}$ is a well-defined group homomorphism.
(4) Show that $0 \to \pi_0(\mathcal{F}(H)) \to \mathbb{Z} \to 0$ is exact.

Elliptic operators

Exercise 10. Let M be a smooth closed manifold, $E \to M$ a Hermitian vector bundle and $P \in \Psi^m(M, E)$ a pseudodifferential operator. Let $u \in \mathcal{D}'(M, E)$ such that $Pu = 0$. Show that

$$\mathrm{WF}(u) \subset \mathrm{Char}(P) := T_0^* M \setminus \mathrm{ell}(P).$$

[2] A monoid is a set endowed with an associative binary operation together with an identity element. As opposed to groups, there is *a priori* no inverses in monoids.

Exercise 11. [Sobolev spaces] Let M be a smooth closed manifold. We fix an arbitrary smooth measure $\mathrm{d}\mu$ on M and denote by $L^2(M)$ the L^2-space induced by this measure. Let $\mathrm{Op} : S^\bullet(T^*M) \to \Psi^\bullet(M)$ be a quantization on M. For $s > 0$, we introduce the following operator:

$$\Lambda_s := \mathbf{1} + \mathrm{Op}(\langle\xi\rangle^{s/2})^* \,\mathrm{Op}(\langle\xi\rangle^{s/2}) \in \Psi^s(M),$$

(1) Compute the principal symbol σ_{Λ_s} of Λ_s.

For $s > 0$ and $\varphi, \psi \in C^\infty(M)$, we set:

$$\|\varphi\|_{H^s(M)} := \|\Lambda_s\varphi\|_{L^2(M)}, \ \langle\varphi,\psi\rangle_{H^s(M)} := \langle\Lambda_s\varphi, \Lambda_s\psi\rangle_{L^2(M)}$$

and define $H^s(M) := \overline{C^\infty(M)}^{\|\cdot\|_{H^s(M)}}$, the completion of $C^\infty(M)$ with respect to the norm $H^s(M)$.

(2) Show that $H^s(M)$ is a Hilbert space such that $C^\infty(M) \hookrightarrow H^s(M)$ embeds densely and $H^s(M) \hookrightarrow L^2(M)$ densely and continuously.
(3) Show that $\Lambda_s : H^s(M) \to L^2(M)$ is an isometry and an isomorphism. *Hint: In order to prove surjectivity, use a parametrix for* Λ_s.
(4) Show that the inverse operator $Q : L^2(M) \to H^s(M)$ (i.e. the operator Q such that $Q\circ\Lambda_s = \mathbf{1}_{H^s(M)}$ and $\Lambda_s\circ Q = \mathbb{1}_{L^2(M)}$) is the restriction of an operator $\Lambda_s^{-1} \in \Psi^{-s}(M)$ to $L^2(M)$.
(5) Show that the embedding $H^s(M) \hookrightarrow L^2(M)$ is compact.
(6) Show that the space $H^s(M)$ is independent of the choice of quantization Op and that another choice of quantization produce the same Hilbert space with equivalent norm.
(7) More generally, show that for all $0 \le s < t$, one has the compact embedding $H^t(M) \hookrightarrow H^s(M)$ and that $\Lambda_s^{-1}\Lambda_t : H^t(M) \to H^s(M)$ is an isometry.

For $s > 0$, we define $\Lambda_{-s} := \Lambda_s^{-1} \in \Psi^{-s}(M)$ and we let $H^{-s}(M)$ be the completion of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{H^{-s}} := \|\Lambda_{-s}\varphi\|_{L^2(M)}.$$

We now show that $H^{-s}(M)$ can be identified with the dual $(H^s(M))'$.

(8) Show that $\Lambda_s : L^2(M) \to H^{-s}(M)$ is an isometric isomorphism.
(9) Show that the L^2-pairing

$$L^2(M) \times L^2(M) \ni (\varphi,\psi) \mapsto \langle\varphi,\psi\rangle_{L^2(M)} \in \mathbb{C}$$

extends continuously to a pairing $H^{-s}(M) \times H^s(M) \to \mathbb{C}$, for $s > 0$. Deduce that there exists a natural continuous embedding $\iota : H^{-s}(M) \to (H^s(M))'$.
(10) Using the Riesz representation theorem, construct a natural map $j : (H^s(M))' \to H^{-s}(M)$. Show that j is an isometric isomorphism and that $j^{-1} = \iota$.
(11) Show the compact embedding $L^2(M) \hookrightarrow H^{-s}(M)$.

Exercise 12. Let M be a smooth manifold and $p \in M$. Show that $\delta_p \in H^{-n/2-\varepsilon}(M)$ for every $\varepsilon > 0$. Is $\delta_p \in H^{-n/2}(M)$?

Exercise 13. Generalize the previous exercise to the following setting. Let M be a smooth manifold and $Y \subset M$ a smooth closed manifold equipped with a smooth nowhere vanishing measure $\mu \in C^\infty(Y, \Omega^1 Y)$. Define $u \in \mathcal{D}'(M)$ by

$$(u, \varphi) := \int_Y \varphi\ \mu.$$

Compute the Sobolev regularity of u.

Exercise 14. [The weak Gårding inequality] Let $A \in \Psi^m(M)$ with $m \geq 0$ and assume that there exists $C_0 > 0$ such that for all $|\xi| \geq C_0$, the following inequality holds: $\Re(\sigma_A(x,\xi)) \geq C_0 \langle \xi \rangle^m$.

(1) Show that there exists a constant $C > 0$ such that for all $\varphi \in C^\infty(M)$,

$$\Re\langle A\varphi, \varphi\rangle_{L^2} \geq 1/C \times \|\varphi\|^2_{H^{m/2}} - C\|\varphi\|^2_{L^2}.$$

(2) Show that the same inequality holds with $\varphi \in H^{m/2}(M)$.

Exercise 15. Let $E \to M$ be a real vector bundle of finite rank over M equipped with a metric g_E. Let $P \in \Psi^m(M, E)$ be an elliptic pseudodifferential operator. Recall that this means that its principal symbol $\sigma_P \in S^m(T^*M, \mathrm{End}(E))$ satisfies: there exists $C_1, C_2 > 0$ such that for all $|\xi| \geq C_1$,

$$\|\sigma_P(x,\xi) f\|_{E_x} \geq C_2 \langle \xi \rangle^m \|f\|_{E_x}.$$

In particular, $\sigma_P(x,\xi) : E_x \to E_x$ is an isomorphism for all $|\xi| \geq C_1$.

(1) Show that $P^* \in \Psi^m(M, E)$ is elliptic.
(2) Show that $PP^* \in \Psi^{2m}$ is elliptic.
(3) Show that $\ker P^* = \ker PP^*$.

The kernel of P^* is finite-dimensional and included in $C^\infty(M, E)$. Write

$$L^2(M, E) = \ker P^* \oplus^\perp G,$$

and denote by Π_0 the orthogonal projection onto $\ker P^*$.

(4) Show that $\Pi_0 \in \Psi^{-\infty}(M, E)$.
(5) Show that $PP^*(L^2(M, E)) = P(H^{-m}(M, E))$.
(6) Show that there exists $Q \in \Psi^{-m}(M, E)$ such that

$$QPP^* = \mathbb{1} - \Pi_0, \quad PP^*Q = \Pi_{\mathrm{ran}\, P}, \quad \Pi_0 Q = 0.$$

Hint: Consider the operator $PP^* : L^2(M, E) \to H^{-2m}(M, E)$ and construct Q abstractly. Then justify that $Q \in \Psi^{-2m}(M, E)$.

(7) Deduce from the previous question that for all $s \in \mathbb{R}$, for all $f \in H^s(M, E)$, there exists a unique pair $u \in H^s(M, E), v \in H^{s+m}(M, E)$ such that $Pu = 0, \Pi_0 v = 0$ and

$$f = u + P^* v.$$

(8) Deduce that $H^s(M, E) = \ker P \oplus P^*(H^{s+m}(E, F))$.

Exercise 16. [Hodge's theorem] We define

$$E_+ := \oplus_{k\geq 0} \Lambda^{2k} T^*M, E_- := \oplus_{k\geq 0} \Lambda^{2k+1} T^*M,$$

and $E := E_+ \oplus E_- = \oplus_{k\geq 0} \Lambda^k T^*M$. We define

$$D_\pm : C^\infty(M, E_\pm) \to C^\infty(M, E_\mp),$$

by $D_\pm := d + d^*$. We define $D : C^\infty(M, E) \to C^\infty(M, E)$ by $D := D_+ \oplus D_-$. The operator D is called a *Dirac operator*. It can be represented in matrix form in the basis $E_+ \oplus E_-$ by:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

Eventually, we define $\Delta : C^\infty(M, E) \to C^\infty(M, E)$ as $\Delta := \oplus_{k\geq 0} \Delta^{(k)}$.

(1) Show that $D_+^* = D_-$, $D^* = D$ and $D^2 = \Delta$.
(2) Show that D is elliptic.
(3) Show that $\ker D = \oplus_{k\geq 0} \ker \Delta^{(k)}$.
(4) Using the previous exercise, prove Hodge's theorem.
(5) Show that $\mathrm{ind}(D_+) = \chi(M)$.

Exercise 17. Let M be a closed manifold. The wavefront set $\mathrm{WF}(A) \subset T^*M \setminus \{0\}$ of a pseudodifferential operator $A \in \Psi^m(M)$ is defined as the closed conic subset such that the following holds: $(x_0, \xi_0) \notin \mathrm{WF}(A)$ if and only if, there exists a small conic neighborhood V of (x_0, ξ_0) such that for every $k \in \mathbb{R}$, for every symbol $b \in S^k(T^*M)$ with b supported in V, $\mathrm{Op}(b)A \in \Psi^{-\infty}(M)$. In other words, the complement of the wavefront set of A is the set of (co-)directions where A behaves like an operator in $\Psi^{-\infty}(M)$.

(1) *Egorov's Theorem for flows:* Let $A \in \Psi^m(M)$ for some $m \in \mathbb{R}$. Given a diffeomorphism $\kappa : M \to M$, explain quickly why $\kappa^* A(\kappa^{-1})^* \in \Psi^m(M)$ is still a pseudodifferential operator of same order and compute its principal symbol. Deduce that $\varphi_t^* A \varphi_{-t}^* \in \Psi^m(M)$ for all $t \in \mathbb{R}$ and compute its principal symbol.

We **admit** that $\mathbb{R} \ni t \mapsto \varphi_t^* A \varphi_{-t}^* \in \Psi^m(M)$ is smooth with respect to t.

(2) Show that there exists a constant $C > 0$ such that:

$$\forall \psi \in L^2(M), \qquad \|\varphi_t^* \psi\|_{L^2(M)} \leq C e^{Ct} \|\psi\|_{L^2(M)}.$$

(3) Using Egorov's Theorem and the previous question, show that there exists a constant $C > 0$ such that

$$\forall t \geq 0, \forall \psi \in H^s(M), \qquad \|\varphi_t^* \psi\|_{H^s(M)} \leq Ce^{Ct} \|\psi\|_{H^s(M)}.$$

(4) Show that for all $s \in \mathbb{R}$, there exists a constant $C > 0$ such that for all $t \in [0, 1]$, the following holds:

$$\forall \psi \in H^{s+m}(M), \qquad \|\varphi_t^* A \varphi_{-t}^* \psi\|_{H^s(M)} \leq C\|\psi\|_{H^{s+m}(M)}.$$

(5) Show that for all $\psi \in C^\infty(M)$, for all $T \geq 0$:

$$\varphi_{-T}^* \psi = -\int_0^T \varphi_{-t}^* X\psi \, \mathrm{d}t + \psi. \tag{0.2}$$

Let $(x_0, \xi_0) \in T^*M \setminus \{0\}$ and $A \in \Psi^0(M)$ such that $(x_0, \xi_0) \in \mathrm{ell}(A)$.

(6) Show that there exists $B \in \Psi^0(M)$ such that $\Phi_{-T}(x_0, \xi_0) \in \mathrm{ell}(B)$, and $C \in \Psi^0(M)$ such that $\mathrm{WF}(C)$ is contained in a small conic neighborhood of the trajectory $(\Phi_{-t}(x_0, \xi_0))_{t\in[0,T]}$ such that the following holds: for all $s \in \mathbb{R}, N > 0$, there exists a constant $C > 0$ such that for all $\psi \in C^\infty(M)$,

$$\|A\psi\|_{H^s} \leq C\left(\|B\psi\|_{H^s} + \|CX\psi\|_{H^s}\right) \tag{0.3}$$

Hint: you may start by multiplying (0.2) by the operator A and then use Egorov's theorem.

(7) Show that (0.3) generalizes in the following way: given $(x_0, \xi_0) \in T^*M \setminus \{0\}$ and $A \in \Psi^0(M)$ with $\mathrm{WF}(A)$ contained in a small conic neighborhood of (x_0, ξ_0) (but A is not assumed to be elliptic at (x_0, ξ_0)), there exists $B \in \Psi^0(M)$ such that $\mathrm{WF}(B)$ is contained in a small conic neighborhood of the point $\Phi_{-T}(x_0, \xi_0)$ and $\Phi_{-T}(x_0, \xi_0) \in \mathrm{ell}(B)$, and $C \in \Psi^0(M)$ such that $\mathrm{WF}(C)$ is contained in a small conic neighborhood of $(\Phi_{-t}(x_0, \xi_0))_{t\in[0,T]}$ such that the following holds: for all $s \in \mathbb{R}, N > 0$, there exists a constant $C > 0$ such that for all $\psi \in C^\infty(M)$,

$$\|A\psi\|_{H^s} \leq C\left(\|B\psi\|_{H^s} + \|CX\psi\|_{H^s} + \|\psi\|_{H^{-N}}\right) \tag{0.4}$$

(8) Show that (0.4) extends to all $\psi \in H^{s+1}(M)$.