EXERCISE CLASS: WAVEFRONT SET OF DISTRIBUTIONS, SYMBOLS AND OSCILLATORY INTEGRALS

Distributions

Exercise 1. Let F_1, F_2 be two Fréchet spaces. What does it mean for a linear map $u : F_1 \to F_2$ to be continuous?

Exercise 2. Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Show that there exists $C, M > 0$ such that:

$$\forall \xi \in \mathbb{R}^n, \qquad |\widehat{u}(\xi)| \ \leq C\langle\xi\rangle^M.$$

Exercise 3. The principal value of $1/x$ is defined as the distribution $\mathrm{vp}(1/x) : C^\infty_{\mathrm{comp}}(\mathbb{R}) \to \mathbb{C}$ such that:

$$\mathrm{vp}(1/x) : \varphi \mapsto \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x} \mathrm{d}x.$$

(1) What is $\mathrm{supp}\,(\mathrm{vp}(1/x))$?
(2) What is the order of $\mathrm{vp}(1/x)$?
(3) Compute $\mathrm{WF}(\mathrm{vp}(1/x))$.

Exercise 4. Let $\delta_{\mathbb{R}^k}$ be the Dirac mass on the k-plane $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$, that is

$$(\delta_{\mathbb{R}^k}, \varphi) := \int_{\mathbb{R}^k} \varphi(x, 0)\mathrm{d}x.$$

(1) Show that $\mathrm{WF}(\delta_{\mathbb{R}^k}) = N^*\mathbb{R}^k \setminus \{0\}$, the *conormal to* $\mathbb{R}^k$, where

$$N^*\mathbb{R}^k := \Big\{((x,0);\xi) \ \mid \forall v \in \mathbb{R}^k \times \{0\}\,, (\xi, v) = 0\Big\}$$

More generally, given $E \subset \mathbb{R}^n$ a vector subspace of dimension k, we can define integration on E with respect to an arbitrary smooth measure as follows

$$(\delta_{E,a}, \varphi) := \int_{\mathbb{R}^k} \varphi(Ax)a(x)\mathrm{d}x,$$

where $A \in \mathrm{O}(\mathbb{R}^n)$ is some invertible matrix such that $A : \mathbb{R}^k \to E$ is an isometry and the function $a \in C^\infty(\mathbb{R}^k)$ defines the density.

(2) Show that $\delta_{E,a}$ has wavefront set contained in the conormal $N^*_0 E$ of E (minus the 0 section), where

$$N^*E := \{(x,\xi) \in T^*\mathbb{R}^n \mid x \in E, \forall v \in E, (\xi, v) = 0\}\,.$$

(3) Compute exactly $\mathrm{WF}(\delta_{E,a})$.

Exercise 5. Define for $z \in \mathbb{C}$ the function x_+^z by:

$$x_+^z = \begin{cases} 0 & \text{on } (-\infty, 0], \\ \exp(z \log(\bullet)) & \text{on } (0, \infty). \end{cases}$$

(1) Show that x_+^z defines a distribution on $\mathbb{R}$ for $\Re(z) > -1$. Compute its support and its wavefront set.
(2) Show that $\{\Re(z) > -1\} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ is holomorphic in the sense that for all $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R})$, the function $\{\Re(z) > -1\} \ni z \mapsto (x_+^z, \varphi) \in \mathbb{C}$ is holomorphic.

Our goal is to show that $\mathbb{C} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ extends to a meromorphic family of distributions. This means that there exists a maximal countable and isolated subset $\mathcal{P} \subset \mathbb{C}$ and a map $n : \mathcal{P} \to \mathbb{Z}_+^*$ such that for all $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R})$, the function $\mathbb{C} \ni z \mapsto (x_+^z, \varphi) \in \mathbb{C}$ is meromorphic, with poles contained in $\mathcal{P}$, and of order at most given by n.

For $z \in \mathbb{C} \setminus \mathbb{Z}_-^*$, define for $k > -\Re(z) - 1$:

$$(\mathrm{pf}(x_+^z), \varphi) := (-1)^k \int_0^{+\infty} \frac{x^{z+k}}{(z+1)...(z+k)} \partial_x^k \varphi(x) \mathrm{d}x.$$

(3) Show that the definition of $\mathrm{pf}(x_+^z)$ is independent of k as long as $k > -\Re(z) - 1$. Show that it coincides with x_+^z when $\Re(z) > -1$.

Let Γ be the Euler function. Recall that $\Gamma(n+1) = n!$ and that Γ admits a meromorphic extension to $\mathbb{C}$. We define for $z \in \mathbb{C} \setminus \mathbb{Z}_-^*$:

$$\chi_+^z := \frac{\mathrm{pf}(x_+^z)}{\Gamma(z+1)}.$$

(4) Show that $\partial_x \chi_+^z = \chi_+^{z-1}$ in $\mathcal{D}'(\mathbb{R})$ for all $z \in \mathbb{C}$ such that $\{\Re(z) > 0\}$.
(5) Deduce that $\mathbb{C} \ni z \mapsto \chi_+^z \in \mathcal{D}'(\mathbb{R})$ is holomorphic.
(6) Conclude that $\mathbb{C} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ admits a meromorphic extension from $\{\Re(z) > -1\}$ to $\mathbb{C}$.

Exercise 6. Let $f \in C^\infty(X), \Im(f) \geq 0$, where $X \subset \mathbb{R}^n$ an open subset. Fix $\varepsilon > 0$.

(1) Show that

$$\frac{1}{f(x) + i\varepsilon} = \frac{1}{i} \int_0^{+\infty} e^{i(f(x)+i\varepsilon)\tau} \mathrm{d}\tau.$$

(2) We assume that $df(x) \neq 0$ when $f(x) = 0$. Show that the limit

$$\frac{1}{f(x) + i0} := \lim_{\varepsilon \to 0} \frac{1}{f(x) + i\varepsilon}$$

exists in $\mathcal{D}'(X)$.
(3) What is $\mathrm{singsupp}\left((f(x) + i0)^{-1}\right)$?
(4) Compute $\mathrm{WF}\left((f(x) + i0)^{-1}\right)$.

(5) For $n = 1$, show that
$$\frac{1}{x \pm i0} = \mp i\pi\delta_0 + \mathrm{vp}(1/x),\ \delta_0 = \frac{1}{2i\pi}\left(\frac{1}{x - i0} - \frac{1}{x + i0}\right).$$

Wavefront set

Exercise 7. Let X be an open subset of $\mathbb{R}^n$ and let $u \in \mathcal{D}'(X)$.

(1) What does it mean for a distribution to be real?
(2) Show that if u is real, then $\mathrm{WF}(u)$ is invariant by the action of the fiberwise antipodal map $(x, \xi) \mapsto (x, -\xi)$.
(3) Assume X is such that $R(X) = X$, where $R(x) = -x$. What does it mean for u to be even or odd?
(4) Show that if u is even or odd, then: $(x, \xi) \in \mathrm{WF}(u)$ iff $(-x, -\xi) \in \mathrm{WF}(u)$.

Exercise 8: tensor product and wavefront set. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be open subsets and $\Gamma_1 \subset T_0^*X, \Gamma_2 \subset T_0^*Y$ be two closed cones. Show that the map
$$C^\infty(X) \otimes C^\infty(Y) \ni (u, v) \mapsto u \otimes v \in C^\infty(X \times Y),$$
extends uniquely to a continuous distribution
$$\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(Y) \ni (u, v) \mapsto u \otimes v \in \mathcal{D}'_{\Gamma_3}(X \times Y)$$
where:
$$\Gamma_3 = (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times O_Y) \cup (O_X \times \Gamma_2).$$

Exercise 9: multiplication of distributions. Let $X \subset \mathbb{R}^n$ be an open subset and let $u_1, u_2 \in \mathcal{D}'(X)$.

(1) Define, when possible, the product
$$\mathcal{D}'(X) \times \mathcal{D}'(X) \ni (u_1, u_2) \mapsto u_1 \times u_2 \in \mathcal{D}'(X),$$
as the unique continuous extension of the product $C^\infty(X) \times C^\infty(X) \to C^\infty(X)$. *Hint: You can either mimic the proof of Lemma 1.1.21 in the lecture notes (integration of a product of distributions), or use the previous exercise by considering the embedding map of the diagonal $\iota : X \to X \times X, x \mapsto (x, x)$ and $\iota^*(u_1 \otimes u_2)$.*
(2) Show that
$$\mathrm{WF}(u_1 \times u_2) \subset \mathrm{WF}(u_1) \cup \mathrm{WF}(u_2) \cup (\mathrm{WF}(u_1) \oplus \mathrm{WF}(u_2)). \tag{0.1}$$
Hint: You can either adapt the proof of Lemma 1.1.21 or use Theorem 1.1.23 (continuous extension of linear operators to distributions).
(3) For $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, compute $\delta_{x_1} \times \delta_{x_2}$. Explain heuristically why $\delta_{x_1}^2$ cannot be well-defined.

(4) In $\mathbb{R}^2$, define

$$(\delta_x, \varphi) := \int_{\mathbb{R}^2} \varphi(x,0)\mathrm{d}x, \quad (\delta_y, \varphi) := \int_{\mathbb{R}^2} \varphi(0,y)\mathrm{d}y.$$

Show that $\delta_x \times \delta_y$ is well-defined and compute it. Compare $\mathrm{WF}(\delta_x), \mathrm{WF}(\delta_y)$ and $\mathrm{WF}(\delta_x \times \delta_y)$.

(5) Find an example where the inclusion (0.1) is an equality and an example where it is not.

Exercise 10: pullback and pushforward of distributions. Let $X \subset \mathbb{R}^n, F \subset \mathbb{R}^m$ be two open subsets. Let $\pi : X \times F \to X$ be given by $\pi(x,y) = x$. For $u \in C^\infty_{\mathrm{comp}}(X \times F), f \in C^\infty_{\mathrm{comp}}(X)$, define:

$$\pi^* f(x,y) := f(x),\ \pi_* u(x) := \int_F u(x,y)\mathrm{d}y.$$

(1) Show that $\pi^* : \mathcal{E}'(X) \to \mathcal{D}'(X \times F)$ extends continuously and bound $\mathrm{WF}(\pi^* f)$ in terms of $\mathrm{WF}(f)$.
(2) Deduce that $\pi^* : \mathcal{E}'(X) \to \mathcal{D}'_\Gamma(X \times F)$ extends continuously for some well-chosen conic subset $\Gamma \subset T^*_0(X \times F)$.
(3) Show that $\pi_* : \mathcal{E}'(X \times F) \to \mathcal{E}'(X)$ extends continuously and bound $\mathrm{WF}(\pi_* u)$ in terms of $\mathrm{WF}(u)$.

Exercise 11. Let M^n be a smooth closed oriented n-dimensional manifold and $\pi : E \to M$ be an oriented fiber bundle, with fiber diffeomorphic to F^k, a closed oriented k-dimensional manifold. Let ω_E be a smooth volume form on E and ω_M be a smooth volume form on M.

(1) Recall the definition of a fiber bundle.
(2) Show the existence of $\nu \in C^\infty(E, \Lambda^k T^* E)$ such that $\omega_E = \nu \wedge \pi^* \omega_M$. Show that the restriction of ν to each fiber $E_x \hookrightarrow E$ is a (positive) volume form.
(3) Consider the pullback operator $\pi^* : C^\infty(M) \to C^\infty(E)$, defined by $\pi^* f(x,v) := f(x)$. Show that it extends uniquely to a continuous map $\pi^* : L^2(M, \omega_M) \to L^2(E, \omega_E)$.
(4) We let $\pi_* : L^2(E, \omega_E) \to L^2(M, \omega_M)$ be the adjoint of π^*. Compute π_*.
(5) Show that $\pi^* : \mathcal{D}'(M) \to \mathcal{D}'(E)$ extends continuously. Bound $\mathrm{WF}(\pi^* f)$ in terms of $\mathrm{WF}(f)$.
(6) Show that $\pi_* : \mathcal{D}'(E) \to \mathcal{D}'(M)$ extends continuously. Bound $\mathrm{WF}(\pi_* u)$ in terms of $\mathrm{WF}(u)$.

Exercise 12. Let M^n be a smooth closed manifold and let $X \in C^\infty(M, TM)$ be a smooth vector field. Let $(\varphi_t)_{t \in \mathbb{R}}$ be the flow generated by X. It acts by pullback on smooth functions as $\varphi_t^* : C^\infty(M) \to C^\infty(M)$, $\varphi_t^* f(x) := f(\varphi_t(x))$.

(1) Show that $\varphi_t^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$ extends continuously.
(2) Compute $\mathrm{WF}(\varphi_t^* f)$ in terms of $\mathrm{WF}(f)$. Explain this heuristically.

Let $\chi \in C^\infty(\mathbb{R})_{\mathrm{comp}}$ be a smooth cutoff function. Define the operator

$$E := \int_{-\infty}^{+\infty} \chi(t)\varphi_t^* \mathrm{d}t.$$

(3) Compute $\mathrm{WF}(Eu)$ in terms of $\mathrm{WF}(u)$. *Hint: Consider the projection* $\pi : M \times \mathbb{R} \to M, \pi(x,t) = x$.

Exercise 13. Let $X \subset \mathbb{R}^n$ be an open subset and $S \subset T_0^*X$ be a closed conic subset. Show that there exists $u \in \mathcal{D}'(X)$ such that $\mathrm{WF}(u) = S$.

Symbols

Exercise 14.

(1) Show that $a(x,\theta) := \langle\theta\rangle^m \in S^m(X \times \mathbb{R}^N)$.
(2) Let $a \in C^\infty(X \times \mathbb{R}^N)$ be positively homogeneous of order m for $|\theta| \geq 1$, namely $a(x, \lambda\theta) = \lambda^m a(x,\theta)$ for all $\lambda \geq 1, |\theta| \geq 1$. Show that $a \in S^m_{1,0}(X \times \mathbb{R}^N)$.
(3) Let $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all $x \in X, a(x, \bullet)$ has compact support in $\mathbb{R}^N$. Show that $a \in S^{-\infty}(X \times \mathbb{R}^N)$.

Exercise 15. Define $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, $\xi'^2 = \sum_{j=1}^{n-1} \xi_j^2, \xi^2 = \xi'^2 + \xi_n^2$. To which symbol space do the following belong?

(1) $(\xi'^2 + i\xi_n)^{-1}$,
(2) $\chi(|\xi|)(\xi'^2 + i\xi_n)^{-1}$, where $\chi \in C^\infty(\mathbb{R})$ is such that $\chi = 1$ for $|x| \geq 1$ and $\chi = 0$ for $|x| \leq 1/2$,
(3) $(\xi^2 + 1)^{-1}$,
(4) $(\xi'^2 + 1)^{-1}$,
(5) $e^{i\xi^2}$,
(6) $e^{ix\cdot\xi}$.

Exercise 16. Assume that $a \in S^m_{\rho,0}(X \times \mathbb{R}^N)$ with $m < 0$ and $\rho > 1$. Show that $a \in S^{-\infty}(X \times \mathbb{R}^N)$.

Hint: Apply $|\theta|\partial_\theta$ *many times and then integrate by parts to recover the expression of* a.

Exercise 17: Borel's theorem. The goal of this exercise is to show the following:

Theorem 0.1 (Borel)**.** *For every sequence $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ of complex numbers $a_\alpha \in \mathbb{C}$, there exists $u \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha u(0) = a_\alpha$.*

Let $\chi \in C^\infty(\mathbb{R}^n, [0,1])$ be such that $\chi = 0$ for $|x| \geq 1$ and $\chi = 1$ for $|x| \leq 1$. Define for $\lambda > 0$:

$$u_N(x,\lambda) := \chi(\lambda x) \sum_{|\alpha|=N} \frac{a_\alpha}{\alpha!} x^\alpha.$$

(1) Compute $\partial_x^\beta u_N(0)$.
(2) Show that $\|u_N(x,\lambda)\|_{C^{N-1}} \leq 2^{-N}$ if $\lambda \geq \lambda_N$, where λ_N is chosen large enough.
(3) Show that $u(x) := \sum_{N\geq 0} u_N(x,\lambda_N)$ solves the problem.

Oscillatory integrals

Exercise 18: The Cauchy problem for the wave equation. We consider the following Cauchy problem for the wave equation:

$$\begin{cases} \partial_t^2 f - \Delta f = 0 \\ f(t=0) = 0, \partial_t f(t=0) = u, \end{cases} \tag{0.2}$$

where $u \in C^\infty_{\text{comp}}(\mathbb{R}^n)$.

(1) Show existence and uniqueness of a solution $f \in C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^n))$ for (0.2) if $u \in C^\infty_{\text{comp}}(\mathbb{R}^n)$.
(2) Show that for all $t \geq 0, x \in \mathbb{R}^n$:

$$f(t,x) = \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|}) u(y) \mathrm{d}y \mathrm{d}\xi$$

We let $\chi \in C^\infty(\mathbb{R}^n)$ be a smooth cutoff function such that $\chi = 0$ near $\xi = 0$ and $\chi = 1$ for $|\xi| \geq 1$. We decompose the solution as

$$f(t,x) = f_+(t,x) + f_-(t,x) + k(t,x),$$

where

$$f_\pm(t,x) = \pm \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} e^{\pm it|\xi|} \chi(\xi) u(y) \mathrm{d}y \mathrm{d}\xi =: F_\pm(t) u$$

$$k(t,x) = \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|})(1-\chi(\xi)) u(y) \mathrm{d}y \mathrm{d}\xi =: K(t) u$$

(3) Show that the operator $K(t)$ is smoothing.
(4) Show that $F_\pm(t)$ is an operator whose Schwartz kernel $K_\pm(t)$ is given by an oscillatory integral. Compute $\mathrm{WF}(K_\pm(t))$.

(5) Show that $F_\pm(t) : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is continuous. Given $u \in \mathcal{E}'(\mathbb{R}^n)$, compute $\mathrm{WF}(F_\pm(t)u)$ in terms of $\mathrm{WF}(u)$.
(6) Show existence and uniqueness of a solution $f \in C^\infty([0,\infty), \mathcal{S}'(\mathbb{R}^n))$ if $u \in \mathcal{E}'(\mathbb{R}^n)$.
(7) Take $u := \delta_0$. Compute $\mathrm{WF}(f(t,\bullet))$. Can you explain this from a physical perspective?

Stationary phase

Exercise 19. Show that there exists $C > 0$ such that for all $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$\|\widehat{\varphi}\|_{L^1(\mathbb{R}^n)} \ \leq C\|\varphi\|_{C^{n+1}(\mathbb{R}^n)}.$$

Exercise 20: The Morse lemma. Let $X \subset \mathbb{R}^n$ and $\phi \in C^\infty(X)$. Assume that $\nabla\phi(x_0) = 0$ and that the Hessian $\nabla^2\phi(x_0)$ is non-degenerate (i.e. invertible). Show that there exists a diffeomorphism $\kappa : U \to V$, where U is a small neighborhood of x_0 and V is a small neighborhood of $0 \in \mathbb{R}^n$ such that for all $y \in V$:

$$\kappa^*\phi(y) = \phi(x_0) + \frac{1}{2}\left(y_1^2 + \dots + y_r^2 - (y_{r+1}^2 + \dots + y_n^2)\right).$$

The quantity $\mathrm{sgn}(\nabla^\phi(x_0)) := r-(n-r)$ is called the signature of the Hessian.

Exercise 21. Let $Q \in \mathcal{M}_n(\mathbb{R})$ be non-degenerate symmetric. Show that the following identities holds for all $\xi \in \mathbb{R}^n$:

(1) Further assuming that Q is definite positive:

$$\mathcal{F}\left(e^{-\frac{1}{2}\langle Q\bullet,\bullet\rangle}\right)(\xi) = (2\pi)^{n/2}|\det Q|^{-1/2}e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle}.$$

(Fourier transform of a Gaussian.)

(2)

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Q\bullet,\bullet\rangle}\right)(\xi) = (2\pi)^{n/2}e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}|\det Q|^{-1/2}e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}.$$

(Fourier transform of an imaginary quadratic phase function.)

Exercise 22. Given $\phi \in C^\infty(\mathbb{R})$ such that $\phi' \neq 0$ except at 0 where $\phi(0) = \phi'(0) = 0$, $\phi''(0) > 0$, and $a \in C^\infty_{\mathrm{comp}}(\mathbb{R})$, compute the Taylor expansion up to $\mathcal{O}(h^{3/2})$ of

$$\int_{\mathbb{R}} e^{\frac{i}{h}\phi(x)}a(x)\mathrm{d}x.$$

Exercise 23. Study the convergence in $\mathcal{D}'(\mathbb{R}^n)$ as $h \to 0$ of:

(1) $u_h(x) := h^{-N}e^{-\frac{i}{h}x}, v_h(x) := h^{-1/2}e^{-\frac{i}{h}x^2/2}, w_h(x) := h^{-1/2}e^{+\frac{i}{h}x^2/2}$,
(2) $u_h(x) := h^{-N}e^{-\frac{i}{h}f(x)}, v_h(x) := h^{-1/2}e^{-\frac{i}{h}(f(x))^2/2}$, where $f \in C^\infty(\mathbb{R})$ and $f' \neq 0$.

Exercise 22: Stirling's formula. Define

$$F(\lambda) := \Gamma(\lambda + 1) = \int_0^{+\infty} e^{-t} t^{\lambda} \mathrm{d}t,$$

for $\lambda \geq 0$. Recall that $F(n) = \Gamma(n+1) = n!$ for all $n \geq 0$. We want to find an asymptotic of F as $\lambda \to \infty$.

(1) Rewrite this integral by means of the change of variable $t = \lambda(1+s)$.

(2) Show that

$$F(\lambda) = (\lambda e^{-1})^{\lambda} \sqrt{2\pi\lambda}(1 + a_1\lambda^{-1} + a_2\lambda^{-2} + ...),$$

and give a precise meaning to "...".

(3) Deduce Stirling's formula.

(4) Compute a_1, a_2.