HOMEWORK: ON THE SCHWARTZ KERNEL THEOREM

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Let $X, Y \subset \mathbb{R}^n$ be open subsets. Recall that, given $\varphi \in C^\infty_{\text{comp}}(X)$ and $(\varphi_n)_{n\geq 0}$, a sequence of functions in $C^\infty_{\text{comp}}(X)$, $\varphi_n \to \varphi$ if for all $n \geq 0$ large enough, all functions φ_n have support in a fixed compact subset $K \subset X$ and $\|\varphi_n - \varphi\|_{C^m(K)} \to 0$ for all $m \geq 0$. Denote by $\mathcal{L}(C^\infty_{\text{comp}}(Y), \mathcal{D}'(X))$ the space of continuous linear operators acting on compactly supported functions on Y, that is, $A \in \mathcal{L}(C^\infty_{\text{comp}}(Y), \mathcal{D}'(X))$ if and only if $A\varphi_n \to A\varphi$ in $\mathcal{D}'(X)$ for all $\varphi_n \to \varphi$ in $C^\infty_{\text{comp}}(Y)$. The aim of this homework is to prove the Schwartz kernel Theorem:

Theorem (Schwartz, 1952)**.** *The map*

$$\Phi : \mathcal{D}'(X \times Y) \ni K \mapsto A_K \in \mathcal{L}(C^\infty_{\text{comp}}(Y), \mathcal{D}'(X)),$$

defined by $(A_K(\varphi), \psi) := (K, \psi \otimes \varphi)$ *for* $\psi \in C^\infty_{\text{comp}}(X)$ *and* $\varphi \in C^\infty_{\text{comp}}(Y)$*, is an isomorphism.*

We will not care much about the continuity of Φ, so by isomorphism, we simply mean a bijective linear map.

We recall that Sobolev spaces $H^s(\mathbb{R}^n)$ are defined as the completion of $C^\infty_{\text{comp}}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|^2_{H^s} := \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

For $\ell \in \mathbb{R}$, the space $\langle x \rangle^\ell H^s(\mathbb{R}^n)$ consists of all functions $f = \langle x \rangle^\ell \widetilde{f}$, for some $\widetilde{f} \in H^s(\mathbb{R}^n)$. The natural norm on this space is

$$\|f\|_{\langle x \rangle^\ell H^s(\mathbb{R}^n)} := \|\langle x \rangle^{-\ell} f\|_{H^s(\mathbb{R}^n)}.$$

(1) **Definition of Φ.** Show that Φ is well-defined.
(2) **Weighted Sobolev spaces.**
  (a) Show that the pairing given by

$$C^\infty_{\text{comp}}(\mathbb{R}^n) \times C^\infty_{\text{comp}}(\mathbb{R}^n) \ni \varphi, \psi \mapsto (\varphi, \psi) := \int_{\mathbb{R}^n} \varphi(x)\psi(x)\mathrm{d}x \in \mathbb{C} \tag{0.1}$$

  extends continuously to a pairing $\langle x \rangle^\ell H^s(\mathbb{R}^n) \times \langle x \rangle^{-\ell} H^{-s}(\mathbb{R}^n) \to \mathbb{C}$, for all $s, \ell \in \mathbb{R}$.
  (b) Show that for all $s,\ \ell \in \mathbb{R}$, there exists a natural identification of $(\langle x \rangle^\ell H^s(\mathbb{R}^n))'$ with $\langle x \rangle^{-\ell} H^{-s}(\mathbb{R}^n)$ using the extension of the pairing (0.1).
  In other words, show that there is a natural isometry $\Psi : \big(\langle x \rangle^\ell H^s(\mathbb{R}^n)\big)' \to \langle x \rangle^{-\ell} H^{-s}(\mathbb{R}^n)$ *such that for all* $T \in \big(\langle x \rangle^\ell H^s(\mathbb{R}^n)\big)'$, $T(\varphi) = (\Psi(T), \varphi)$*, where the last pairing is understood as the continuous extension obtained in* (a).

As a first step towards proving the general Schwartz kernel Theorem, we want to prove it on Schwartz spaces. Recall that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$ if all Schwartz semi-norms are convergent. We say that $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is continuous if $A\varphi_n \to A\varphi$ for all $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$, that is, for all $\psi \in \mathscr{S}(\mathbb{R}^n)$,

$$(A\varphi_n, \psi) \to (A\varphi, \psi).$$

For $M \geq 0$, we denote by $\|\cdot\|_{(M)}$ the norm $\|\cdot\|_{\langle x\rangle^{-M} H^{+M}(\mathbb{R}^n)}$.

(2) **Schwartz kernel Theorem on Schwartz spaces.**
   (a) Show that $A \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n))$ if and only if there exists $M \geq 0$ large enough and $C > 0$ such that for all $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$:
   $$|(A\varphi, \psi)| \leq C\|\varphi\|_{(M)}\|\psi\|_{(M)}.$$
   (b) Deduce that $A : \langle x\rangle^{-M} H^M(\mathbb{R}^n) \to \langle x\rangle^{+M} H^{-M}(\mathbb{R}^n)$ is bounded.
   (c) We now consider a continuous map $A : L^2(\mathbb{R}^n) \to H^m(\mathbb{R}^n)$. Show that for $m > 0$ large enough, and $x \in \mathbb{R}^n$, the map $L^2(\mathbb{R}^n) \ni \varphi \mapsto (A\varphi)(x) \in \mathbb{C}$ can be well-defined and is continuous.
   (d) Deduce that that $A\varphi(x) = (T_x, \varphi) = \int_{\mathbb{R}^n} T_x(y)\varphi(y)\, \mathrm{d}y$ for a certain map $T : \mathbb{R}^n \ni x \mapsto T_x \in L^2(\mathbb{R}^n_y)$ such that $T \in C^0_{\mathrm{bded}}(\mathbb{R}^n_x, L^2(\mathbb{R}^n_y))$. (The last space denotes continuous functions with values in L^2 such that the C^0 norm is uniformly bounded on $\mathbb{R}^n$.)
   (e) Prove that $\mathscr{S}(\mathbb{R}^n_x) \otimes \mathscr{S}(\mathbb{R}^n_y)$ is dense in $\mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_y)$.
   (f) Eventually, deduce a version of the Schwartz kernel Theorem for Schwartz spaces: the map $\mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n) \ni K \mapsto A_K \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n))$ is an isomorphism.

(3) **Schwartz kernel Theorem.**
   (a) Show that $C^\infty_{\mathrm{comp}}(X) \otimes C^\infty_{\mathrm{comp}}(Y)$ is dense in $C^\infty_{\mathrm{comp}}(X \times Y)$.
   (b) Prove the Schwartz kernel Theorem.

(4) **Wavefront set computations.**
   (a) What is the Schwartz kernel $K_{\mathrm{id}} \in \mathcal{D}'(X \times X)$ of the identity map $\mathrm{id} : C^\infty(X) \to C^\infty(X)$?
   (b) Compute $\mathrm{WF}(K_{\mathrm{id}})$.
   (c) Let $P := \sum_{|\alpha| \leq M} a_\alpha(x) D^\alpha_x$. Express K_P in terms of K_{id}. Deduce $\mathrm{WF}(K_P)$.

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