EXERCISE CLASS: PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS

Pseudodifferential operators and principal symbols

Exercise 1. Let $\Delta := \partial_x^2 + \partial_y^2$ be the Laplacian in the standard coordinates (x, y) of $\mathbb{R}^2$. Let

$$[0, \infty) \times (\mathbb{R}/2\pi\mathbb{Z}) \ni (r, \theta) \mapsto re^{i\theta} \in \mathbb{R}^2,$$

be the polar coordinates.

(1) Show that, in polar coordinates, the expression of Δ is given by

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\ \partial_\theta^2.$$

(2) What is the full symbol of Δ in the standard coordinates?
(3) What is the full symbol of Δ in the polar coordinates?
(4) Is the full symbol invariant by this change of coordinates?
(5) Show that the principal symbol of Δ is invariant by this change of coordinates.

Exercise 2. Let $X \in C^\infty(M, TM)$ be a smooth vector field. It induces a first order differential operator by the formula $X\varphi(x) := d\varphi(X(x))$, for $\varphi \in C^\infty(M)$.

(1) Compute the principal symbol $\sigma_X \in S^1(T^*M)$.
(2) Compute $\mathrm{ell}(X) \subset T^*M \setminus \{0\}$, the set of ellipticity of the operator.
(3) Let $E \to M$ be a real vector bundle equipped with a connection ∇. Define the operator $\mathbf{X} := \nabla_X$ acting on $C^\infty(M, E)$ by $\mathbf{X}\varphi(x) := \nabla_X\varphi(x)$. Compute $\sigma_{\mathbf{X}} \in S^1(T^*M, \mathrm{End}(E))$. What is $\mathrm{ell}(X)$?

Exercise 3. Let M be a smooth closed manifold and μ be a smooth density on M. Let $P \in \Psi^m(X)$. Recall that the *formal adjoint* P^* of P is the (unique) operator satisfying the equality: for all $\varphi, \psi \in C^\infty(M)$:

$$\langle P\varphi, \psi\rangle_{L^2(M,\mathrm{d}\mu)} := \int_M (P\varphi)\psi\ \ \mathrm{d}\mu = \int_M \varphi(P^*\psi)\ \mathrm{d}\mu = \langle \varphi, P^*\psi\rangle_{L^2(M,\mathrm{d}\mu)}.$$

We showed in class that $P^* \in \Psi^m(X)$. Nevertheless, the formal adjoint P^* *depends* on a choice of density μ. As a consequence, we will rather write P^*_μ for the formal adjoint of P with respect to the density μ.

(1) Let $\mu' := a \cdot \mu$ be another smooth density on M, where $a \in C^\infty(M)$ is a positive function. Compute $P^*_{\mu'}$ in terms of a and P^*_μ.
(2) Deduce that the principal symbol σ_{P^*} is well-defined, independently of the choice of density μ.

Let $X \in C^\infty(M, TM)$ be a smooth vector field, seen as a differential operator of order 1 acting on $C^\infty(M)$. If μ is a smooth density on M, we define the *divergence* of X with respect to μ as the (unique) function satisfying the equality $\mathcal{L}_X \mu = \mathrm{div}_\mu(X)\mu$.

(3) Recall the value of $\sigma_X \in S^1(T^*M)$.
(4) Show that $\int_M \mathcal{L}_X \mu = 0$.
(5) Deduce the formal adjoint X^*_μ of X, computed with respect to μ.
(6) What is σ_{X^*}?

Exercise 3. Let g be a smooth metric on TM. Recall that the gradient $\nabla : C^\infty(M) \to C^\infty(M, T_{\mathbb{C}}M)$ (computed with respect to the metric g) is defined by $\nabla\varphi := (d\varphi)^\sharp$, where $\sharp : T^*M \to TM$ denotes the musical isomorphism[1]. It is a differential operator of order 1, that is $\nabla \in \Psi^1(M, \mathbb{C} \to T_{\mathbb{C}}M)$.

(1) Show that its principal symbol is given by $\sigma_\nabla(x, \xi) = i\xi^\sharp$.
(2) The metric g induces a metric on T^*M, denoted by g^{-1} and defined by $|\xi|^2_{g^{-1}} := |\xi^\sharp|^2_g$. Assume that g is given in a local patch of coordinates by the symmetric matrix $(g_{ij})_{1\leq i,j\leq n}$. Compute the symmetric matrix defining the metric g^{-1} on T^*M in these coordinates.
(3) We define the (non-negative) Laplacian $\Delta := \nabla^*\nabla$. What is the order of this operator? Show that $\sigma_\Delta(x, \xi) = |\xi|^2_{g^{-1}}$.

Exercise 4. The exterior derivative acts as $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1}T^*M)$ by the usual formula in coordinates:

$$d(f_I dx_I) = \sum_{i=1}^n \partial_{x_i} f_I dx_i \wedge dx_I,$$

[1] $\xi^\sharp = Z$ is defined such that $(\xi, \bullet) = g(Z, \bullet)$.

where $dx_I = dx_{i_1} \wedge ... \wedge dx_{i_k}$ for some indices $i_1, ..., i_k \in \{1, ..., n\}$.

(1) Show that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, if α is a p-form.
(2) Compute $\sigma_d \in S^1(T^*M, \mathrm{Hom}(\Lambda^k T^*M, \Lambda^{k+1}T^*M))$.
(3) Is it injective/surjective? Describe the kernel and the range.

Let g be a smooth metric on TM. For all $k \in \{0, ..., n\}$, it induces a metric on $\Lambda^k T^*M$ by declaring $\mathbf{e}_{i_1}^* \wedge ... \wedge \mathbf{e}_{i_k}^*$ to be an orthonormal basis for all $i_1 < ... < i_k$, if $(\mathbf{e}_1, ..., \mathbf{e}_n)$ is a local orthonormal basis of TM.

(4) We let $d^* : C^\infty(M, \Lambda^{k+1}T^*M) \to C^\infty(M, \Lambda^k T^*M)$ be the adjoint (divergence operator). Compute σ_{d^*}.
(5) We define $\Delta := (d + d^*)^2$, the Hodge Laplacian acting on k-forms. Compute σ_Δ.

Exercise 5. Let $X \subset \mathbb{R}^n$ and $A \in \Psi^m(X)$ be a properly supported pseudodifferential operator. We say that A is *classical* if its full symbol $\sigma_A^{\mathrm{full}}(x, \xi) = e^{-i\xi \cdot x} A(e^{i\xi \bullet})$ admits a *polyhomogeneous expansion*, namely, there exists a sequence of symbols $(a_{m-j})_{j \geq 0}$ such that $a_{m-j} \in S^{m-j}(T^*X)$ such that a_{m-j} is $(m-j)$-homogeneous in the ξ-variable for $|\xi| \geq 1$ and

$$\sigma_A^{\mathrm{full}} \sim \sum_{j \geq 0} a_{m-j}.$$

(1) Show that differential operators are classical.
(2) Show that the pullback (by a diffeomorphism) of a classical pseudodifferential operator is still classical.
(3) Define classical pseudodifferential operators on manifolds.
(4) Show that, given $A \in \Psi^m(M, E \to F)$, a classical pseudodifferential operator acting on sections of the vector bundle $E \to M$, the following holds: given $f \in C^\infty(M, E)$, $S \in C^\infty(M)$ such that $dS \neq 0$ on $\mathrm{supp}(f)$, show that for all $x \in \mathrm{supp}(f)$:

$$\lim_{h \to 0} e^{-\frac{i}{h}S(x)} h^m A(e^{\frac{i}{h}S} f)(x) = \sigma_A(x, dS(x)) f(x).$$

Fredholm operators

Exercise 7. Let $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and define $P_s := D_x - s$ on $\mathbb{S}^1$, where $D_x := i^{-1}\partial_x$.

(1) Show that $\mathbb{R} \ni s \mapsto P_s \in \mathcal{L}(H^1(\mathbb{S}^1), L^2(\mathbb{S}^1))$ is an analytic family of Fredholm operators of index 0.
(2) Compute $\ker P_s$ and $\mathrm{ran}\, P_s$ for $s \in \mathbb{R}$. Are there special values?

Exercise 8. Let H be a (separable) Hilbert space. Let $(\pi_0(\mathcal{F}(H)), \circ)$ be the monoid[2] of connected components of $\mathcal{F}(H)$ endowed with the composition law $\circ$.

(1) Show that $(\pi_0(\mathcal{F}(H)), \circ)$ is a well-defined monoid. What is the identity element?
(2) Prove that $(\pi_0(\mathcal{F}(H)), \circ)$ can actually be turned into a group. What is the inverse of an element?
(3) Deduce that $\mathrm{ind} : \pi_0(\mathcal{F}(H)) \to \mathbb{Z}$ is a well-defined group homomorphism.
(4) Show that $0 \to \pi_0(\mathcal{F}(H)) \to \mathbb{Z} \to 0$ is exact.

Elliptic operators

Exercise 6. [Sobolev spaces] Let M be a smooth closed manifold. We fix an arbitrary smooth measure $\mathrm{d}\mu$ on M and denote by $L^2(M)$ the L^2-space induced by this measure. Let $\mathrm{Op} : S^\bullet(T^*M) \to \Psi^\bullet(M)$ be a quantization on M. For $s > 0$, we introduce the following operator:

$$\Lambda_s := \mathbb{1} + \mathrm{Op}(\langle \xi \rangle^{s/2})^* \, \mathrm{Op}(\langle \xi \rangle^{s/2}) \in \Psi^s(M),$$

(1) Compute the principal symbol σ_{Λ_s} of Λ_s.

For $s > 0$ and $\varphi, \psi \in C^\infty(M)$, we set:

$$\|\varphi\|_{H^s(M)} := \|\Lambda_s \varphi\|_{L^2(M)}, \ \langle \varphi, \psi \rangle_{H^s(M)} := \langle \Lambda_s \varphi, \Lambda_s \psi \rangle_{L^2(M)}$$

and define $H^s(M) := \overline{C^\infty(M)}^{\|\cdot\|_{H^s(M)}}$, the completion of $C^\infty(M)$ with respect to the norm $H^s(M)$.

(2) Show that $H^s(M)$ is a Hilbert space such that $C^\infty(M) \hookrightarrow H^s(M)$ embeds densely and $H^s(M) \hookrightarrow L^2(M)$ densely and continuously.
(3) Show that $\Lambda_s : H^s(M) \to L^2(M)$ is an isometry and an isomorphism. *Hint: In order to prove surjectivity, use a parametrix for Λ_s.*
(4) Show that the inverse operator $Q : L^2(M) \to H^s(M)$ (i.e. the operator Q such that $Q \circ \Lambda_s = \mathbb{1}_{H^s(M)}$ and $\Lambda_s \circ Q = \mathbb{1}_{L^2(M)}$) is the restriction of an operator $\Lambda_s^{-1} \in \Psi^{-s}(M)$ to $L^2(M)$.
(5) Show that the embedding $H^s(M) \hookrightarrow L^2(M)$ is compact.

[2] A monoid is a set endowed with an associative binary operation together with an identity element. As opposed to groups, there is *a priori* no inverses in monoids.

(6) Show that the space $H^s(M)$ is independent of the choice of quantization Op and that another choice of quantization produce the same Hilbert space with equivalent norm.
(7) More generally, show that for all $0 \leq s < t$, one has the compact embedding $H^t(M) \hookrightarrow H^s(M)$ and that $\Lambda_s^{-1}\Lambda_t : H^t(M) \to H^s(M)$ is an isometry.

For $s > 0$, we define $\Lambda_{-s} := \Lambda_s^{-1} \in \Psi^{-s}(M)$ and we let $H^{-s}(M)$ be the completion of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{H^{-s}} := \|\Lambda_{-s}\varphi\|_{L^2(M)}.$$

We now show that $H^{-s}(M)$ can be identified with the dual $(H^s(M))'$.

(8) Show that $\Lambda_s : L^2(M) \to H^{-s}(M)$ is an isometric isomorphism.
(9) Show that the L^2-pairing

$$L^2(M) \times L^2(M) \ni (\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_{L^2(M)} \in \mathbb{C}$$

extends continuously to a pairing $H^{-s}(M) \times H^s(M) \to \mathbb{C}$, for $s > 0$. Deduce that there exists a natural continuous embedding $\iota : H^{-s}(M) \to (H^s(M))'$.
(10) Using the Riesz representation theorem, construct a natural map $j : (H^s(M))' \to H^{-s}(M)$. Show that j is an isometric isomorphism and that $j^{-1} = \iota$.
(11) Show the compact embedding $L^2(M) \hookrightarrow H^{-s}(M)$.

Exercise 7. [The weak Gårding inequality] Let $A \in \Psi^m(M)$ with $m \geq 0$ and assume that there exists $C_0 > 0$ such that for all $|\xi| \geq C_0$, the following inequality holds:

$$\Re(\sigma_A(x,\xi)) \geq C_0\langle \xi \rangle^m.$$

(1) Show that there exists a constant $C > 0$ such that for all $\varphi \in C^\infty(M)$,

$$\Re\langle A\varphi, \varphi \rangle_{L^2} \geq 1/C \times \|\varphi\|^2_{H^{m/2}} - C\|\varphi\|^2_{L^2}.$$

(2) Show that the same inequality holds with $\varphi \in H^{m/2}(M)$.

Exercise 8. Let $E \to M$ be a vector bundle of finite rank over M. Let $P \in \Psi^m(M,E)$ be an elliptic pseudodifferential operator. Recall that this means that its principal symbol $\sigma_P \in S^m(T^*M, \mathrm{End}(E))$ satisfies: there exists $C_1, C_2 > 0$ such that for all $|\xi| \geq C_1$,

$$\|\sigma_P(x,\xi)f\|_{E_x} \geq C_2\langle \xi \rangle^m \|f\|_{E_x}.$$

In particular, $\sigma_P(x,\xi) : E_x \to E_x$ is an isomorphism for all $|\xi| \geq C_1$.

(1) Show that $P^* \in \Psi^m(M,E)$ is elliptic.
(2) Show that $PP^* \in \Psi^{2m}$ is elliptic.
(3) Show that $\ker P^* = \ker PP^*$.

The kernel of P^* is finite-dimensional and included in $C^\infty(M,E)$. Write

$$L^2(M,E) = \ker P^* \oplus^\perp G,$$

and denote by Π_0 the orthogonal projection onto $\ker P^*$.

(2) Show that $\Pi_0 \in \Psi^{-\infty}(M,E)$.
(3) Show that there exists $Q \in \Psi^{-m}(M,E)$ such that

$$QPP^* = \mathbb{1} - \Pi_0.$$

Hint: Consider the operator $PP^* : L^2(M,E) \to H^{-2m}(M,E)$ and construct Q abstractly. Then justify that $Q \in \Psi^{-2m}(M,E)$.

(4) Deduce from the previous question that for all $s \in \mathbb{R}$, for all $f \in H^s(M,E)$, there exists a unique pair $u \in H^s(M,E), v \in H^{s+m}(M,E)$ such that $\Pi_0 v = 0$ and

$$f = u + P^* v.$$

(5) Deduce that $H^s(M,E) = \ker P \oplus P^*(H^{s+m}(E,F))$.

Exercise 9. [Hodge's theorem] We define

$$E_+ := \oplus_{k\geq 0}\Lambda^{2k}T^*M, E_- := \oplus_{k\geq 0}\Lambda^{2k+1}T^*M,$$

and $E := E_+ \oplus E_- = \oplus_{k\geq 0}\Lambda^k T^*M$. We define

$$D_\pm : C^\infty(M,E_\pm) \to C^\infty(M,E_\mp),$$

by $D_\pm := d+d^*$. We define $D : C^\infty(M,E) \to C^\infty(M,E)$ by $D := D_+ \oplus D_-$. The operator D is called a *Dirac operator*. It can be represented in matrix form in the basis $E_+ \oplus E_-$ by:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

Eventually, we define $\Delta : C^\infty(M,E) \to C^\infty(M,E)$ as $\Delta := \oplus_{k\geq 0}\Delta^{(k)}$.

(1) Show that $D_+^* = D_-$, $D^* = D$ and $D^2 = \Delta$.
(2) Show that D is elliptic.
(3) Show that $\ker D = \oplus_{k\geq 0} \ker \Delta^{(k)}$.
(4) Using the previous exercise, prove Hodge's theorem.
(5) Show that $\operatorname{ind}(D_+) = \chi(M)$.