GEOMETRIC ANALYSIS ON MANIFOLDS

Exercise 1

(1) We define $A^\top \in \mathrm{End}((\mathbb{R}^n)^*)$ such that $(A^\top \xi, v) := (\xi, Av)$, for $\xi \in (\mathbb{R}^n)^*, v \in \mathbb{R}^n$. This is a well-defined endomorphism of $(\mathbb{R}^n)^*$. Uniqueness follows from the fact that $(B\xi, v) = 0$ for all $\xi \in (\mathbb{R}^n)^*, v \in \mathbb{R}^n$ implies $B = 0$.

(2) If $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n \setminus \mathbb{R}^k)$, then $(\delta_A, \varphi) = 0$ so $\mathrm{supp}(\delta_A) \subset \mathbb{R}^k$. Conversely, given $x_0 \in \mathbb{R}^k$, we take $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that $\varphi \geq 0, \varphi(x_0) = 1$, then $(\delta_A, \varphi) > 0$ so $\mathrm{supp}(\delta_A) = \mathbb{R}^k$. Moreover, if $K \subset \mathbb{R}^n$ is compact and φ has support in K, then

$$|(\delta_A, \varphi)| \ \leq \mathrm{vol}_{\mathbb{R}^k}(K \cap \mathbb{R}^k) \|\varphi\|_{C^0},$$

so δ_A is of order 0.

(3) Clearly, $\mathrm{WF}(\delta_A) \subset T^*_{\mathbb{R}^k}\mathbb{R}^n$. We use the coordinates $z = (x, y), x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$ and $\xi = (\xi_x, \xi_y)$. For $(x_0, 0) \in \mathbb{R}^k$, pick $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that $\varphi(x_0, 0) = 1$. Then:

$$\widehat{\varphi\delta_A}(\xi) = (\delta_A, \varphi e^{-i\xi \ \bullet}) = \int_{\mathbb{R}^k} e^{-i\xi_x \cdot Ax} \varphi(A(x,0)) \mathrm{d}x = \widehat{\chi}(A^\top \xi_x),$$

where $\chi := \varphi(A(\bullet, 0))$. If $\xi \in N^*_{(x_0,0)}\mathbb{R}^k$, that is $\xi = (0, \xi_y)$, then $\widehat{\varphi\delta_A}(\xi) = \widehat{\chi}(0)$ does not decrease as $|\xi| \to 0$ so $\mathrm{WF}(\delta_A) \supset N^*_0\mathbb{R}^k$. Moreover, if $\xi_0 \notin N^*_{x_0,0}\mathbb{R}^k$, we can find an open conic neighborhood $V \subset \mathbb{R}^n$ of ξ_0 and $C > 0$ such that for all $\xi \in V, |\xi_y| \leq C|\xi_x|$. Hence, $\langle \xi \rangle \lesssim \langle \xi_x \rangle$ on V. Using that $\chi \in \mathcal{S}(\mathbb{R}^k)$, we obtain for all $N > 0$:

$$|\widehat{\varphi\delta_A}(\xi)| \leq C_N \langle \xi_x \rangle^{-N} \leq C'_N \langle \xi \rangle^{-N},$$

so $\mathrm{WF}(\delta_A) = N^*_0\mathbb{R}^k$.

(4) Similarly, $\mathrm{WF}(\delta_B) = N^*_0\mathbb{R}^{n-k}$ and thus $\mathrm{WF}(\delta_A) \cap (-\mathrm{WF}(\delta_B)) = \emptyset$ so the multiplication is well-defined.

(5) We write in the decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

Let $\chi_\varepsilon := \varepsilon^{-n}\chi(\bullet/\varepsilon)$, where $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n), \chi \geq 0, \int \chi = 1$. By the previous question, we know that $\delta_A \times (\delta_B \star \chi_\varepsilon) \to \delta_A \times \delta_B$ in $\mathcal{D}'(\mathbb{R}^n)$. But:

$$(\delta_B \ \star \chi_\varepsilon)(z) = (\delta_B, \chi_\varepsilon(z - \bullet)) = \int_{\mathbb{R}^{n-k}} \chi_\varepsilon(z - (0, B_2 y)) \mathrm{d}y.$$

For $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$:

$$\begin{aligned}(\delta_A(\delta_B \star \chi_\varepsilon), \varphi) &= (\delta_A, (\delta_B \star \chi_\varepsilon)\varphi) = \int_{\mathbb{R}^k} (\delta_B \star \chi_\varepsilon)\varphi(A_1 x, 0)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \varphi(A_1 x, 0)\chi_\varepsilon(A_1 x, -B_2 y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n} \varphi(x')\chi_\varepsilon(z')\mathrm{d}z' |\det F|^{-1},\end{aligned}$$

where $F = \begin{pmatrix} A_1 & 0 \\ 0 & -B_2 \end{pmatrix}$, $z' = (x', y') = F(x, y)$. Using that $\chi_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$, we then obtain that

$$\delta_A \times \delta_B = |\det A_1|^{-1} |\det B_2|^{-1} \delta_0.$$

Exercise 2

(1) If a solution f exists, then $\partial_t^2 \widehat{f} + |\xi|^2 \widehat{f} = 0$. Using the initial conditions, we obtain $\widehat{f}(\xi, t) = \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}(\xi)$. This proves uniqueness. Conversely, it is clear that

$$f(t, \bullet) := \mathcal{F}^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}u\right)$$

is a solution to the Cauchy problem and that $f \in C^\infty([0, \infty), \mathcal{S}(\mathbb{R}^n))$.

(2) $f_\varepsilon(t, x) = \mathcal{F}^{-1}\left(\chi(\varepsilon\xi)\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}u\right) \to f(t, x)$ by dominated convergence.

(3) The Schwartz kernel of $R(t)$ is given by

$$K_{R(t)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}\frac{e^{it|\xi|} - e^{-it|\xi|}}{2i|\xi|}(1 - \chi(\xi))\mathrm{d}\xi.$$

Since the integrand has compact support, this is a smooth function of (x, y), that is, $K_{R(t)} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ so $R(t)$ is bounded $\mathcal{E}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

(4) $K_\pm(t) = \pm\dfrac{1}{(2\pi)^n}\int e^{i((x-y)\xi \pm t|\xi|)}\frac{\chi(\xi)}{2i|\xi|}\mathrm{d}\xi$ with phase $\phi_\pm(x, y, \xi) = (x - y)\xi \pm t|\xi|$, amplitude $a(\xi) := \frac{\chi(\xi)}{2i|\xi|}$.

(5) We deduce

$$\begin{aligned}\mathrm{WF}(K_\pm(t)) \subset \Lambda_{\phi_\pm} &= \{(x, y, d_x\phi_\pm, d_y\phi_\pm) \mid d_\xi\phi_\pm = 0\} \\ &= \left\{(x, y, \xi, -\xi) \mid x - y \pm t\xi^\sharp/|\xi| = 0\right\} \\ &= \left\{(x, x \pm t\xi^\sharp/|\xi|, \xi, -\xi) \mid (x, \xi) \in T_0^*\mathbb{R}^n\right\}.\end{aligned}$$

(6) Write $X = \mathbb{R}^n, Y = \mathbb{R}^n$ so that $K_\pm(t) \in \mathcal{D}'(X \times Y)$. It is immediate to check that $\mathrm{WF}'_Y(K_\pm(t)) = \emptyset = \mathrm{WF}'_X(K_\pm(t))$. Hence $F_\pm(t) : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is bounded by the

extension theorem. Moreover, if $u \in \mathcal{E}'(\mathbb{R}^n)$, then:

$$\begin{aligned}\mathrm{WF}(F_\pm(t)u) &\subset \mathrm{WF}(K'_\pm(t)) \circ \mathrm{WF}(u)\\ &= \{(x,\xi) \in T_0^*\mathbb{R}^n \mid \exists (y,\eta) \in \mathrm{WF}(u), (x,y,\xi,-\eta) \in \mathrm{WF}(K_\pm(t))\}\\ &= \left\{(y \mp t\tfrac{\xi^\sharp}{|\xi|}, \xi) \mid (y,\xi) \in \mathrm{WF}(u)\right\}\end{aligned}$$

(7) If $u \in \mathcal{E}'(\mathbb{R}^n)$, then by construction $f(t) := (R(t) + F_+(t) + F_-(t))u$ solves the Cauchy problem. Uniqueness follows from (1).

(8) $\mathrm{WF}(\delta_0) = \{0\} \times (\mathbb{R}^n \setminus \{0\})$ so by (6), we obtain that

$$\mathrm{WF}(f(t,\bullet)) \subset \left\{(\pm t\xi^\sharp/|\xi|, \xi) \mid \ \xi \in \mathbb{R}^n \setminus \{0\} \ \right\}.$$

(9) The wavefront set of the previous question describes the shock wave generated by a punctual explosion at $x = 0$, time $t = 0$ and propagating at constant speed 1.

Exercise 3

(1) We have:

$$\int (P\varphi)\psi\mu' = \int (P\varphi)a\psi\mu = \int \varphi P_\mu^*(a\psi)\mu = \int \varphi a^{-1} P_\mu^*(a\psi)a\mu = \int \varphi(P_{\mu'}^*\psi)\mu',$$

that is, $P_{\mu'}^* = a^{-1}P_\mu^* a$.

(2) $\sigma_{P_{\mu'}^*} = \sigma_{a^{-1}}\sigma_{P_\mu^*}\sigma_a = \sigma_{P_\mu^*}$.

(3) We have:

$$\begin{aligned}\mathcal{L}_X(\varphi\alpha) &= (d\iota_X + \iota_X d)(\varphi\alpha) = d(\varphi\iota_X\alpha) + \iota_X(d\varphi \wedge \alpha + \varphi d\alpha)\\ &= d\varphi \wedge \iota_X\alpha + \varphi d\iota_X\alpha + d\varphi(X)\alpha - d\varphi \wedge \iota_X\alpha + \varphi\iota_X d\alpha = (X\varphi)\ \alpha + \varphi\mathcal{L}_X\alpha.\end{aligned}$$

(4) Take $(x_0, \xi_0) \in T^*M \ \setminus \{0\}$, $S \in C^\infty(M)$ such that $dS(x_0) = \xi_0$, $\alpha \in C^\infty(M, \Lambda^k T^*M)$ with support near x_0 such that $dS \neq 0$ on the support of α.

a) We have:

$$\sigma_X(x_0,\xi_0) = \lim_{h\to 0}\left(he^{-\frac{i}{h}S}X(e^{+\frac{i}{h}S})\right)(x_0) = idS(X(x_0)) = i(\xi_0, X(x_0))$$

b) By Leibniz's rule: $he^{-\frac{i}{h}S}\mathcal{L}_X(e^{+\frac{i}{h}S}\alpha) = i(XS)\alpha + h\mathcal{L}_X\alpha$ so taking the limit $h \to 0$ yields $\sigma_{\mathcal{L}_X}(x_0,\xi_0) = i(\xi_0, X(x_0))\mathbb{1}_{\Lambda^k T_{x_0}^* M}$.

(5) First of all, observe that by Stoke's theorem:

$$\int \mathcal{L}_X\mu = \int d\iota_X\mu = 0.$$

Hence:

$$\int \mathcal{L}_X(\varphi\overline{\psi}\mu) = \int (X\varphi)\overline{\psi}\mu + \varphi\overline{X\psi}\mu + \varphi\overline{\psi}\mathrm{div}_\mu(X)\mu = 0.$$

This yields: $X_\mu^* = -X - \mathrm{div}_\mu(X)$.

(6) This is formally self-adjoint if and only if the divergence vanishses, $\mathrm{div}_\mu(X) = 0$. This means that X preserves the volume μ, namely $\mu(\varphi_t(U)) = \mu(U)$, if $(\varphi_t)_{t\in\mathbb{R}}$ is the flow generated by X and $U \subset M$ is an open subset.

(7) $\sigma_{X^*} = \overline{\sigma_X} = -i(\xi, X(x))$.

Problem: operators of gradient/divergence type

(1) We check that the representative given by (0.3) is m-homogeneous:

$$\begin{aligned}\sigma_P(x_0,\lambda\xi_0)f_{x_0} &= \lim_{h\to 0}\left(h^m e^{-\frac{i}{h}\lambda S}P(e^{+\frac{i}{h}\lambda S}f)\right)(x_0)\\ &= \lambda^m \lim_{h\to 0}\left((h/\lambda)^m e^{-\frac{i}{h/\lambda}S}P(e^{+\frac{i}{h/\lambda}S}f)\right)(x_0) = \lambda^m\sigma_P(x_0,\xi_0)f_x.\end{aligned}$$

If there are two representatives p and p' that are m-homogeneous, then so is the difference $p-p'$. But $p - p' \in S^{m-1}(T^*M, \mathrm{Hom}(E,F))$ and thus

$$\frac{(p-p')(x,\xi)}{|\xi|^m} = (p-p')(x,\xi/|\xi|) \to_{|\xi|\ \to\infty} 0,$$

that is $p = p'$.

(2) We saw in class that $\sigma_{\nabla_g}(x,\xi) = i\xi^\sharp$ (this can be obtained directly using (0.3)) and $\sigma_{\nabla_g^*}(x,\xi) = \sigma_{\nabla_g}(x,\xi)^* = -ig(\xi^\sharp, \bullet)$.

(3) We assume that $\xi \neq 0$. Given $z \in \mathbb{C}$, we have $\sigma_{\nabla_g}(x,\xi)z = iz\xi^\sharp = 0$ if and only if $z = 0$ so ∇_g is of gradient type. Conversely, given $z \in \mathbb{C}$, we have $\sigma_{\nabla_g^*}(x,\xi)(iz\xi^\sharp/|\xi|^2) = z|\xi|^2/|\xi|^2 = z$ so ∇_g^* is of divergence type.

(4) Another example: if $E \to M$ is a vector bundle equipped with a connection $\nabla^E : C^\infty(M,E) \to C^\infty(M, T^*M\otimes E)$, then ∇^E is of gradient type since $\sigma_{\nabla^E}(x,\xi) = i\xi\otimes \mathbb{1}_E$.

(5) The question boils down to showing that $\sigma_{P^*} = \sigma_P^*$ is surjective if and only if σ_P is injective. But this is immediate since $\dim\ker\sigma_P = \dim(\mathrm{ran}\,\sigma_P^*)^\perp$.

(6) By m-homogeneity, it suffices to show that gradient type is equivalent to

$$\|\sigma_P(x,\xi)f\|_{F_x} \geq C\|f\|_{E_x}, \tag{0.1}$$

for all $|\xi| = 1$. Now, if (0.1) is satisfied, σ_P is injective and P is of gradient type. Conversely, if $f \in E_x, \|f\|_{E_x} = 1$, then injectivity and compactness of the unit sphere of E_x ensures the existence of $C > 0$ such that $\|\sigma_P(x,\xi)f\|_{F_x} \geq C$. Hence, by linearity, for a general f, we get (0.1). This is pointwise in $(x,\xi) \in T^*M$ such that $|\xi| = 1$ but by compactness of the unit sphere of T^*M, such a constant $C > 0$ can be taken uniform.

(7) We can use the previous question: P^*P is of elliptic if and only if $\|\sigma_{P^*P}(x,\xi)f\|_{E_x} \geq C|\xi|^m\|f\|_{E_x}$ if and only if $\sigma_{P^*P}(x,\xi)$ is injective (which is equivalent to invertibility). But $\sigma_P(x,\xi)$ is injective by assumption and if $f \in \ker \sigma_{P^*P}(x,\xi)$, then

$$g_E(\sigma_{P^*P}(x,\xi)f, f) = 0 = \|\sigma_P(x,\xi)f\|^2_{F_x}$$

that is $\sigma_P(x,\xi)f = 0$, thus $f = 0$.

(8) We can find a parametrix $L \in \Psi^{-2m}$ of P^*P such that $LP^*P = 1 - R$, where $R \in \Psi^{-\infty}$. Take $Q = LP^*$. Then it is clear that $\sigma_Q = \sigma_{P^*P}^{-1}\sigma_{P^*} = (\sigma_P^*\sigma_P)^{-1}\sigma_P^*$.

(9) a) Take $u \in H^s(M,E)$ such that $Pu = 0$. Then by the parametrix, $u = Ru \in C^\infty(M,E)$. Moreover, R is compact (on L^2 for instance) and $\ker(R - \mathbb{1})$ is finite-dimensional so this proves the claim.

b) If $u_n \in H^{s+m}$ and $Pu_n \to v$ in H^s, then by the parametrix and the continuity of $Q : H^s \to H^{s+m}$, we get $QPu_n = u_n - Ru_n \to v$ in H^{s+m}. By compactness of R on H^{s+m}, we have that, up to extraction $Ru_n \to w$. Hence $u_n \to w + v \in H^{s+m}$ in H^{s+m} and $P(w+v) = v$, that is $v \in P(H^{s+m})$ so the range is closed.

(10) If $u \in H^s$, $\nabla_g u = 0$, then u is smooth by the previous question. Moreover, u is locally constant, thus constant on each connected component of M. Hence $\ker \nabla_g = H^0_{\mathrm{dR}}(M,\mathbb{C})$.

(11) $\Pi_0^2 = \Pi_0$ since it is a projection and $\Pi_0^* = \Pi_0$ since it is orthogonal (with respect to the L^2 inner product). Moreover, it is clear that the range of Π_0 is contained in C^∞ and it thus suffices to check that it is bounded as a map $\mathcal{D}' \to C^\infty$. If we denote by Q_0 the operator of question (8) such that $Q_0 P = \mathbb{1} - R$, then Q_0 differs from Q by an operator K in $\Psi^{-\infty}$. Hence, $\Pi_0 QP = 0 = \Pi_0(Q_0 + K)P = \Pi_0(1-R) + \Pi_0 KP$, that is $\Pi_0 = \Pi_0 R - \Pi_0 KP \in \Psi^{-\infty}$ since $R, K \in \Psi^{-\infty}$.

(12) If such a decomposition exists, then $\sigma_P^* f = \sigma_P^*\sigma_P u + 0$. By ellipticity of P^*P, this is invertible and $u = (\sigma_P^*\sigma_P)^{-1}\sigma_P^* f = \sigma_Q f$, $v = f - \sigma_P\sigma_Q f$. This shows uniqueness. Conversely, we set $u := \sigma_Q f, v := v = f - \sigma_P u$ and it is straighforward to check that these satisfy the required conditions.

(13) By the previous question, $\pi_{\operatorname{ran}\sigma_P} = \sigma_P(\sigma_P^*\sigma_P)^{-1}\sigma_P^*, \pi_{\ker\sigma_P^*} = \mathbb{1} - \pi_{\operatorname{ran}\sigma_P}$. By the symbolic calculus, it is immediate that $\pi_{\operatorname{ran}\sigma_P} \in S^0(T^*M, \operatorname{End}(F))$ and the same holds for $\pi_{\ker\sigma_P^*}$.

(14) If such a decomposition exists, applying Q, we get $Qf = QPh + 0 = h - \Pi_0 h = h$ and $k = f - PQf$ which proves uniqueness.

(15) $Ph = PQf = \Pi_{\operatorname{ran}P} f$ with $\Pi_{\operatorname{ran}P} = PQ \in \Psi^0(M, F \to F)$ and with principal symbol $\sigma_{\Pi_{\operatorname{ran}P}} = \sigma_P\sigma_Q = \pi_{\operatorname{ran}\sigma_P}$. Similarly, $\Pi_{\ker P^*} = \mathbb{1} - \Pi_{\operatorname{ran}P} \in \Psi^0$ with principal symbol $\pi_{\ker\sigma_P^*}$.

Since $\Pi_{\mathrm{ran}\, P}, \Pi_{\ker P^*}$ are two complementary projections, the identities are immediate.

(16) By question (2), $\ker \sigma_{\nabla_g^*}(x,\xi) = \ker \iota_{\xi^\sharp} = \ker \xi$. Hence, taking ξ_1 and ξ_2 such that $\ker \xi_1 + \ker \xi_2 = T_x M$ solves the question.

(17) By assumption, we can find $v_i \in F_{x_\star}, \xi_i \in T^*_{x_\star}M, |\xi_i| = 1$ such that $v_i \in \ker \sigma^*_P(x_\star, v_i)$ and $F_{x_\star} = \mathrm{Span}(v_1, ..., v_n)$. Let $S \in C^\infty(M)$ such that $dS(x_\star) = \xi_i$ and $f \in C^\infty(M, F)$ with support near $x_\star$ (and $dS \neq 0$ on $\mathrm{supp}(f)$, $S(x_\star) = 0$) such that $f(x_0) = v_i$. By (0.3), we get:

$$\mathrm{ev}_{x_\star}\left(\Pi_{\ker \sigma^*_P}(e^{\frac{i}{h}S}f)\right) = \left(\Pi_{\ker \sigma^*_P}(e^{\frac{i}{h}S}f)\right)(x_\star) = \sigma_{\ker \sigma^*_P}(x_\star, \xi_i) f(x_\star) + \mathcal{O}(h) = v_i + \mathcal{O}(h)$$

Since the $\{v_i\}$ span $F_{x_\star}$, then so do $\{v_i + \mathcal{O}(h)\}$ for h small enough. This proves the claim.

(18) Take $x_1, ..., x_N \in M$, N distinct points and $\xi_i \in T^*_{x_i}M$ with $|\xi_i| = 1$. Let $v_i \in \ker \sigma^*_P(x_i, \xi_i)$ with $v_i \neq 0$ (this is possible by assumption). We can find $f_i \in C^\infty(M, F)$ with support localized near x_i (such that f_i vanishes near x_j for $j \neq i$), $S_i \in C^\infty(M)$ such that $dS_i \neq 0$ on $\mathrm{supp}(f_i)$ but the support of S_i is only slightly larger than that of f_i, $dS_i(x_i) = \xi_i$, $S_i(x_i) = 0$. We then take $f = \sum f_i, S = \sum S_i$. By (0.3):

$$\left(\Pi_{\ker \sigma^*_P}(e^{\frac{i}{h}S}f)\right)(x_i) = v_i + \mathcal{O}(h),$$

so the range of the evaluation map on the N points is larger than N. Since N is arbitrary, this proves that $C^\infty(M, F) \cap \ker P^*$ is infinite-dimensional.