GEOMETRIC ANALYSIS ON MANIFOLDS

The lecture notes from my webpage and personal notes taken during the classes are allowed. Other material such as books or online material is prohibited. The exam lasts three hours and consists of three independent exercises and one problem.

Exercise 1

Let $n \geq 1$ be a positive integer. We denote by $(\mathbb{R}^n)^*$ the dual of $\mathbb{R}^n$, namely the set of continuous linear forms (covectors) $\xi : \mathbb{R}^n \to \mathbb{R}$, and we write (ξ, v) for the pairing of an element $\xi \in (\mathbb{R}^n)^*$ with an element $v \in \mathbb{R}^n$ (i.e. ξ applied to the vector v). We consider $A, B \in \mathrm{GL}_n(\mathbb{R})$ and a decomposition

$$\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}, \tag{0.1}$$

such that A and B act diagonally on (0.1), that is, we have $A(\mathbb{R}^k) = \mathbb{R}^k, A(\mathbb{R}^{n-k}) = \mathbb{R}^{n-k}$ and the same for B. Recall that, if $E \subset \mathbb{R}^n$ is a vector subspace, the *conormal* to E is defined as

$$N^*E := \{(x, \xi) \in T^*\mathbb{R}^n \mid x \in E, \forall v \in E, (\xi, v) = 0\}.$$

We set $N_0^*E := N^*E \setminus \{0\}$ (the conormal minus the zero section).

(1) Show that there exists a unique endomorphism $A^\top \in \mathrm{End}((\mathbb{R}^n)^*)$ such that for all $\xi \in (\mathbb{R}^n)^*$, $v \in \mathbb{R}^n$, the following equality holds: $(\xi, Av) = (A^\top \xi, v)$.

We define the following distributions: $\forall \varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$(\delta_A, \varphi) := \int_{\mathbb{R}^k} \varphi(A(x, 0))\mathrm{d}x, \quad (\delta_B, \varphi) := \int_{\mathbb{R}^{n-k}} \varphi(B(0, y))\mathrm{d}y.$$

(2) Show that $\mathrm{supp}(\delta_A) = \mathbb{R}^k$ and that δ_A is of order 0.
(3) Show that $\mathrm{WF}(\delta_A) = N_0^*\mathbb{R}^k$. *(A detailed answer is expected here.)*
(4) Show that the multiplication $u := \delta_A \times \delta_B$ is well-defined.
(5) Compute u.

Exercise 2

On $\mathbb{R}^n$, we define the Laplacian $\Delta := \sum_{i=1}^n \partial_{x_i}^2$. We consider the following Cauchy problem for the wave equation:

$$\begin{cases} \partial_t^2 f - \Delta f = 0 \\ f(t=0) = 0, \partial_t f(t=0) = u, \end{cases} \tag{0.2}$$

where $u \in C^\infty_{\text{comp}}(\mathbb{R}^n)$.

(1) Show existence and uniqueness of a solution $f \in C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^n))$ for (0.2) if $u \in C^\infty_{\text{comp}}(\mathbb{R}^n)$. *Hint: use the Fourier transform in the space variable to show that $\widehat{f}$ is the solution of a second-order differential equation and solve this equation.*

We let $\chi \in C^\infty(\mathbb{R}^n, [0,1])$ be a smooth cutoff function such that $\chi = 1$ for $|\xi| \leq 1$ and $\chi = 0$ for $|\xi| \geq 2$. For $\varepsilon > 0$, we define:

$$f_\varepsilon(t,x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|}) \chi(\varepsilon\xi) u(y) \mathrm{d}y \mathrm{d}\xi.$$

(2) Show that for all $t \geq 0, x \in \mathbb{R}^n$: $f_\varepsilon(t,x) \to_{\varepsilon \to 0} f(t,x)$.

In the following, we will thus write the solution as:

$$f(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|}) u(y) \mathrm{d}y \mathrm{d}\xi.$$

We decompose the solution as

$$f(t,x) = f_+(t,x) + f_-(t,x) + r(t,x),$$

where

$$f_+(t,x) = +\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} e^{+it|\xi|} (1-\chi(\xi)) u(y) \mathrm{d}y \mathrm{d}\xi,$$

$$f_-(t,x) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} e^{-it|\xi|} (1-\chi(\xi)) u(y) \mathrm{d}y \mathrm{d}\xi,$$

$$r(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|}) \chi(\xi) u(y) \mathrm{d}y \mathrm{d}\xi.$$

We introduce the operators $F_+(t), F_-(t)$ and $R(t)$ such that $f_\pm(t,\bullet) = F_\pm(t)u$ and $r(t,\bullet) = R(t)u$.

(3) Show that the operator $R(t)$ is smoothing, that is, it is bounded as a map $\mathcal{E}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

(4) Write the Schwartz kernel $K_\pm(t)$ of the operators $F_\pm(t)$ as an oscillatory integral. What is the phase function and the amplitude?

(5) Show that

$$\mathrm{WF}(K_+(t)) \subset \left\{(x, x + t\tfrac{\xi^\sharp}{|\xi|}; \xi, -\xi) \mid (x,\xi) \in T_0^*\mathbb{R}^n\right\},$$
$$\mathrm{WF}(K_-(t)) \subset \left\{(x, x - t\tfrac{\xi^\sharp}{|\xi|}; \xi, -\xi) \mid (x,\xi) \in T_0^*\mathbb{R}^n\right\},$$

where $\xi^\sharp \in \mathbb{R}^n$ is the vector such that for all $v \in \mathbb{R}^n$ $(\xi, v) = g_{\mathrm{euc}}(\xi^\sharp, v)$ (here g_{euc} is the Euclidean metric).

(6) Deduce from the previous question that $F_\pm(t) : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is continuous. Given $u \in \mathcal{E}'(\mathbb{R}^n)$, bound $\mathrm{WF}(F_\pm(t)u)$ in terms of $\mathrm{WF}(u)$.

(7) Show existence and uniqueness of a solution $f \in C^\infty([0,\infty), \mathcal{D}'(\mathbb{R}^n))$ if $u \in \mathcal{E}'(\mathbb{R}^n)$.

(8) Take $u := \delta_0$. Bound $\mathrm{WF}(f(t, \bullet))$.

(9) *(Cultural question)* Can you explain the result of question (8) from a physical perspective? *Hint: think of a stone thrown in water or a firework exploding.*

For **Exercise 3** and the **Problem**, we recall the following point. If M is a closed manifold, $E \to M$ is a vector bundle of finite rank and $A \in \mathrm{Diff}^m(M, E \to E)$ is a **differential** operator of order $m \geq 0$, the following holds: given $S \in C^\infty(M)$, $f \in C^\infty(M, E)$ such that $dS \neq 0$ on $\mathrm{supp}(f)$, and $x \in M$,

$$\left(h^m e^{-\frac{i}{h}S} A(e^{\frac{i}{h}S} f)\right)(x) = \sigma_A(x, dS(x)) f(x) + \mathcal{O}(h), \tag{0.3}$$

where $\sigma_A \in S^m(T^*M, \mathrm{End}(E))$ denotes a certain representative for the principal symbol of A.

Exercise 3

Let M be a smooth orientable closed (i.e. compact, without boundary) manifold. Let μ be a smooth **non-vanishing** volume form on M such that $\int_M \mu > 0$. Let $P \in \Psi^m(M)$. Recall that the *formal adjoint* P^* of P is the unique operator in $\Psi^m(M)$ satisfying the equality: for all $\varphi, \psi \in C^\infty(M)$,

$$\int_M (P\varphi)(x)\overline{\psi}(x)\ \mu = \int_M \varphi(x)\overline{(P^*\psi)}(x)\ \mu,$$

that is $\langle P\varphi, \psi\rangle_{L^2(M,\mu)} = \langle \varphi, P^*\psi\rangle_{L^2(M,\mu)}$. The formal adjoint P^* *depends* on the choice of volume form μ. Hence, throughout this exercise, we will rather write P^*_μ for the formal adjoint of P with respect to the volume form μ in order to insist on that dependence.

(1) Let $\mu' := a\mu$ be another smooth positive volume form on M, where $a \in C^\infty(M)$ is a positive function (beware of that: *positive* in English means *strictly positive* for the French). Compute $P^*_{\mu'}$ in terms of a and P^*_μ.
(2) Deduce that the principal symbol σ_{P^*} is well-defined, independently of the choice of density μ.

Recall that if $\alpha \in C^\infty(M, \Lambda^k T^*M)$ is a smooth k-form on M, the Lie derivative $\mathcal{L}_X\alpha \in C^\infty(M, \Lambda^k T^*M)$ is defined by:

$$\mathcal{L}_X\alpha := d\iota_X\alpha + \iota_X d\alpha,$$

where d is the exterior derivative and $\iota_X : \Lambda^\bullet T^*M \to \Lambda^{\bullet-1}T^*M$ is the interior product defined such that $(\iota_X\alpha)(v_1, ..., v_k) = \alpha(X, v_1, ..., v_k)$. We will **admit** that it satisfies the following relation:

$$\forall \eta \in T^*M, \alpha \in \Lambda^k T^*M, \quad \iota_X(\eta \wedge \alpha) = \eta(X)\alpha - \eta \wedge \iota_X\alpha.$$

Let $X \in C^\infty(M, TM)$ be a smooth vector field, seen as a differential operator of order 1 acting on functions by

$$X : C^\infty(M) \to C^\infty(M), \quad X\varphi(x) := d\varphi(X(x))$$

If μ is a smooth volume-form on M, we define the *divergence* $\mathrm{div}_\mu(X) \in C^\infty(M)$ of X with respect to μ as the unique smooth function satisfying the equality $\mathcal{L}_X\mu = \mathrm{div}_\mu(X)\mu$.

(3) Show that $\mathcal{L}_X$ satisfies the usual Leibniz rule: for $\varphi \in C^\infty(M), \alpha \in C^\infty(M, \Lambda^k T^*M)$, $\mathcal{L}_X(\varphi\alpha) = (X\varphi)\alpha + \varphi\mathcal{L}_X\alpha$.
(4) Compute the principal symbols: a) σ_X of X acting on functions and b) $\sigma_{\mathcal{L}_X}$ of $\mathcal{L}_X$ acting on k-forms. *Hint: use* (0.3).
(5) Compute the formal adjoint X^*_μ of X with respect to μ. *Hint: consider the quantity* $\int_M \mathcal{L}_X(\varphi\overline{\psi}\mu)$.
(6) When is $i^{-1}X$ formally self-adjoint with respect to μ, namely, $(i^{-1}X)^*_\mu = i^{-1}X$? *(Cultural question)* What does it imply for the vector field X?
(7) Compute σ_{X^*}.

Problem: operators of gradient/divergence type

Notation. Let (M,g) be a smooth closed Riemannian manifold. The metric g is a metric on the tangent bundle TM. It induces a smooth metric on T^*M, denoted by g^{-1}, such that $|\xi|^2_{g^{-1}} := |\xi^\sharp|^2_g$, where $\sharp : T^*M \to TM$ is the musical isomorphism defined such that $(\xi, v) = g_x(\xi^\sharp, v)$ for all $x \in M, v \in T_xM$. We denote by dvol_g the smooth Riemannian measure induced by g.

Let $E, F \to M$ be two smooth real vector bundles over M of finite rank equipped with (fiberwise) metrics g_E, g_F. Without further notice, we will also denote by E and F their complexification $E \otimes_{\mathbb{R}} \mathbb{C}$ and $F \otimes_{\mathbb{R}} \mathbb{C}$. The L^2-scalar product for sections of (the complexification of) E is then defined as:

$$\forall \varphi, \psi \in C^\infty(M,E), \qquad \langle \varphi, \psi \rangle_{L^2(M,E)} := \int_M g_E(\varphi(x), \overline{\psi}(x)) \, \mathrm{dvol}_g(x),$$

and similarly for sections of F. We denote by $\Psi^m(M, E \to F)$, the space of pseudodifferential operators of order $m \in \mathbb{R}$ mapping smooth sections $C^\infty(M,E)$ to $C^\infty(M,F)$. Given $P \in \Psi^m(M, E \to F)$, we will denote by $P^* \in \Psi^m(M, F \to E)$ its formal adjoint, defined as the unique operator such that: $\forall \varphi \in C^\infty(M,E), \psi \in C^\infty(M,F)$,

$$\int_M g_F(P\varphi(x), \overline{\psi}(x)) \, \mathrm{dvol}_g(x) = \int_M g_E(\varphi(x), \overline{P^*\psi}(x)) \, \mathrm{dvol}_g(x),$$

that is, $\langle P\varphi, \psi \rangle_{L^2(M,F)} = \langle \varphi, P^*\psi \rangle_{L^2(M,E)}$.

Recall that the principal symbol σ_P of P is defined as the equivalence class in $S^m(T^*M, \mathrm{Hom}(E,F))/S^{m-1}(T^*M, \mathrm{Hom}(E,F))$ of its full symbol (defined in coordinates). When talking about *the* principal symbol, we will therefore assume that a representative $p \in S^m(T^*M, \mathrm{Hom}(E,F))$ was chosen. Eventually, recall that the principal symbol σ_{P^*} of the adjoint P^* is given by the adjoint of the principal symbol of P, namely, $p^*(x,\xi) \in \mathrm{Hom}(F_x, E_x)$ is a representative for the principal symbol of P^*. Here $p^*(x,\xi)$ denotes the algebraic adjoint of $p(x,\xi)$, i.e. the homomorphism satisfying the relation

$$\forall f \in F_x, e \in E_x, \quad g_F(p(x,\xi)e, f) = g_E(e, p^*(x,\xi)f).$$

Preliminaries. Let $P \in \mathrm{Diff}^m(M, E \to F)$ be a **differential** operator of order $m \geq 0$.

(1) Show that there exists a unique representative of the principal symbol σ_P, denoted by $p \in S^m(T^*M, \mathrm{Hom}(E, F))$, which is m-homogeneous in the ξ-variable, namely, such that $p(x, \lambda\xi) = \lambda^m p(x, \xi)$ for all $(x, \xi) \in T^*M \setminus \{0\}\,, \lambda > 0$. *Hint: Use* (0.3).

Definition and examples. In the following, $P \in \mathrm{Diff}^m(M, E \to F)$ denotes a **differential** operator of order $m \geq 0$. With some slight abuse of notation, we will denote **indistinctly** by the same letter σ_P the principal symbol of P and its unique m-homogeneous representative obtained from question (1).

We will say that P is of **gradient type** if for all $(x, \xi) \in T^*M \setminus \{0\}$, its principal symbol

$$\sigma_P(x, \xi) \in \mathrm{Hom}(E_x, F_x)$$

is injective. We will say that it is of **divergence type** if for all $(x, \xi) \in T^*M \setminus \{0\}$, the principal symbol $\sigma_P(x, \xi)$ is surjective.

(2) Let $\nabla_g : C^\infty(M) \to C^\infty(M, T^*_{\mathbb{C}}M)$ be the *gradient* of the metric g defined as $\nabla_g \varphi := (d\varphi)^\sharp$. Compute the principal symbols of ∇_g and of its adjoint (the *divergence*) ∇_g^*.
(3) Show that the gradient ∇_g is of gradient type while the divergence ∇_g^* is of divergence type.
(4) *(Cultural question)* Can you give another example of a differential operator of gradient type?

First properties. Let $P \in \Psi^m(M, E \to F)$ be a differential operator.

(5) Show that P is of gradient type if and only if its formal adjoint $P^* \in \Psi^m(M, F \to E)$ is of divergence type.
(6) Show that P is of gradient type if and only if there exists a constant $C > 0$ such that for all $x \in M, \xi \in T^*_x M$ such that $|\xi|_{g^{-1}} \geq 1$ and $f \in E_x$, the following inequality holds:

$$\|\sigma_P(x, \xi) f\|_{F_x} \geq C |\xi|_{g^{-1}}^m \|f\|_{E_x},$$

where $\| \bullet \|_{E_x} := |g_E(\bullet, \bullet)|^{1/2}, \| \bullet \|_{F_x} := |g_F(\bullet, \bullet)|^{1/2}$.

From now on, we assume that $P \in \Psi^m(M, E \to F)$ is a given differential operator of gradient type.

(7) Show that the operator $P^*P \in \Psi^{2m}(M, E \to E)$ is elliptic.

(8) Deduce from the previous question that there exists a parametrix $Q \in \Psi^{-m}(M, F \to E)$ and $R \in \Psi^{-\infty}(M, E \to E)$ such that

$$QP = \mathbf{1}_E - R,$$

where $\mathbb{1}_E : C^\infty(M, E) \to C^\infty(M, E)$ is the identity operator. Show that the principal symbol σ_Q of Q is given by

$$\sigma_Q(x, \xi) = [\sigma_P^*(x, \xi)\sigma_P(x, \xi)]^{-1}\sigma_P^*(x, \xi). \tag{0.4}$$

(9) Using the parametrix, prove that for all $s \in \mathbb{R}$:
  (a) $\ker P|_{H^s(M,E)}$ is finite-dimensional, independent of s and contained in $C^\infty(M, E)$,
  (b) $P(H^{s+m}(M, E))$ is a closed subspace in $H^s(M, F)$.

(10) Consider the case $P = \nabla_g$ of question (2). Compute $\ker \nabla_g|_{H^s(M)}$ for all $s \in \mathbb{R}$.

Decomposition of the domain and target space of P. We decompose the Hilbert space $L^2(M, E)$ as:

$$L^2(M, E) = \ker P \oplus^\perp G,$$

where G is the orthogonal of $\ker P$ with respect to the scalar product $\langle \bullet, \bullet \rangle_{L^2(M,E)}$. We let Π_0 be the orthogonal projection onto $\ker P$ (parallel to G). We **admit** that, up to a modification by a smoothing operator in $\Psi^{-\infty}(M, F \to E)$, the operator Q in question (8) can be constructed so that

$$QP = \mathbb{1}_E - \Pi_0, \quad \Pi_0 Q = 0, \quad \forall s \in \mathbb{R},\ Q(\ker P^*|_{H^s(M,F)}) = 0.$$

(11) Show that $\Pi_0^2 = \Pi_0$, $\Pi_0^* = \Pi_0$ and that $\Pi_0 \in \Psi^{-\infty}(M, E \to E)$.

(12) Show that for all $(x, \xi) \in T^*M$ such that $|\xi| \geq 1$, for all $f \in F_x$, there exists a unique pair $(u, v) \in E_x \times F_x$ such that

$$f = \sigma_P(x, \xi)u + v,$$

and $v \in \ker \sigma_P^*(x, \xi)$, $g_F(\sigma_P(x, \xi)u, v) = 0$. Show that $u = \sigma_Q(x, \xi)f$, where σ_Q is given in (0.4).

Let $\pi_{\operatorname{ran} \sigma_P}(x, \xi)$ be the orthogonal projection onto $\operatorname{ran} \sigma_P(x, \xi)$ (with respect to the metric g_F) and $\pi_{\ker \sigma_P^*}(x, \xi)$ be the orthogonal projection onto $\ker \sigma_P^*(x, \xi)$.

(13) Give an expression of $\pi_{\operatorname{ran} \sigma_P}, \pi_{\ker \sigma_P^*}$ in terms of σ_P, σ_P^*. Show that $\pi_{\operatorname{ran} \sigma_P}, \pi_{\ker \sigma_P^*} \in S^0(T^*M, \operatorname{End}(F))$.

We **admit** that any section $f \in H^s(M, F)$ can be decomposed as

$$f = Ph + k, \tag{0.5}$$

where $k \in H^s(M, F) \cap \ker P^*, h \in H^{s+m}(M, E)$ and $\Pi_0 h = 0$.

(14) Show that the decomposition (0.5) is unique. Compute h and k in terms of f.

(15) Show that, in the decomposition (0.5), we can write $Ph = \Pi_{\mathrm{ran}\, P} f$ and $k = \Pi_{\ker P^*} f$, where $\Pi_{\mathrm{ran}\, P}, \Pi_{\ker P^*} \in \Psi^0(M, F \to F)$ are pseudodifferential operators of order 0 such that $\Pi^2_{\mathrm{ran}\, P} = \Pi_{\mathrm{ran}\, P}, \Pi^2_{\ker P^*} = \Pi_{\ker P^*}$ and $\Pi_{\mathrm{ran}\, P} + \Pi_{\ker P^*} = \mathbb{1}$. Compute the principal symbols $\sigma_{\Pi_{\mathrm{ran}\, P}}, \sigma_{\Pi_{\ker P^*}}$ of these operators.

Operators of uniform divergence type. Let $x_\star \in M$ be an arbitrary point. The goal of this section is to study the *evaluation map*:

$$\mathrm{ev}_{x_\star} : C^\infty(M, F) \cap \ker P^* \to F_{x_\star}, \quad \mathrm{ev}_{x_\star}(f) := f(x_\star).$$

We will **admit** that all pseudodifferential operators involved in the problem are **classical** (the technical definition of classical is not needed) and that classical operators still satisfy the expansion (0.3).

We will say that the operator P^* is of **uniform divergence type** if for all $x \in M$,

$$F_x = \sum_{|\xi|_{g^{-1}}=1} \ker \sigma_P^*(x, \xi), \tag{0.6}$$

where the sum in (0.6) is understood in the following sense:

$$\sum_{|\xi|_{g^{-1}}=1} \ker \sigma_P^*(x, \xi) = \mathrm{Span}\left\{ v \mid \exists \xi \in T_x^* M, |\xi|_{g^{-1}} = 1, v \in \ker \sigma_P^*(x, \xi) \right\}.$$

(16) Show that the divergence ∇_g^* is of uniform divergence type.

(17) Show that, under the assumption that P^* is of uniform divergence type, the evaluation map $\mathrm{ev}_{x_\star}$ is surjective. *Hint: Apply (0.3) with the operator* $\Pi_{\ker P^*}$.

(18) Show that if $\pi_{\ker \sigma_P^*}(x, \xi) \neq 0$ for all $(x, \xi) \in T^*M \setminus \{0\}$, then $C^\infty(M, F) \cap \ker P^*$ is infinite-dimensional (without assuming that P^* is of uniform divergence type). *Hint: You may consider N distinct points $x_1, ..., x_N$ on the manifold M and the evaluation map*

$$\mathrm{ev} : C^\infty(M, F) \cap \ker P^* \to \bigoplus_{i=1}^N F_{x_i},$$

given by $\mathrm{ev}(f) := (f(x_1), ..., f(x_N))$.