Solution of Exercise 2

Jiasheng Lin

(1) Show that the number of fixed points is finite.

Proof. We first show that the fixed points are all isolated, that is, if $\Psi(x_0) = x_0$ then there is a neighborhood $U \ni x_0$ such that x_0 is the only fixed point in U. First we show this in the Euclidean space, namely if X, $Y \subset \mathbb{R}^n$ are open sets containing 0 and $\psi : X \to Y$ a diffeomorphism with $\psi(0) = 0$ and $\mathrm{d}\psi_0 - \mathbb{1}$ invertible, then there is some (say, ball) neighborhood $B_\delta(0) \ni 0$ such that 0 is the only fixed point in $B_\delta(0)$.

Indeed, since $\mathrm{d}\psi_0 - \mathbb{1}$ is invertible, there is some $c > 0$ such that

$$\|\mathrm{d}\psi_0(v) - v\| \geqslant c\,\|v\| \tag{0.1}$$

for all $v \in \mathbb{R}^n$. By definition of the total differential,

$$\begin{aligned}\frac{\|\psi(v) - v\|}{\|v\|} &\geqslant \left|\frac{\|\psi(v) - v + v - \mathrm{d}\psi_0(v)\|}{\|v\|} - \frac{\|v - \mathrm{d}\psi_0(v)\|}{\|v\|}\right| \\ &= \left|\frac{\|\psi(v) - \mathrm{d}\psi_0(v)\|}{\|v\|} - \frac{\|v - \mathrm{d}\psi_0(v)\|}{\|v\|}\right| \\ &\geqslant c - \varepsilon > 0,\end{aligned}$$

when $\|v\| < \delta$ for some $\delta > 0$, in which case $\|\psi(v) - \mathrm{d}\psi_0(v)\| / \|v\| < \varepsilon$. This means

$$\|\psi(v) - v\| \geqslant (c - \varepsilon)\,\|v\| \tag{0.2}$$

for $v \in B_\delta(0)$, which shows that there cannot be any other fixed point except 0 in $B_\delta(0)$. So we proved the fixed point 0 is isolated.

Now suppose $x_0 \in M$ is a fixed point. Let $\kappa : V, V' \to \mathbb{R}^n$ be local coordinates containing x_0 such that $\Psi(V) \subset V'$ and $\kappa(x_0) = 0$. Then

$$\psi = \kappa \circ \Psi \circ \kappa^{-1} : \kappa(V) \to \kappa(V') \tag{0.3}$$

is a diffeomorphism of Euclidean spaces with 0 being a fixed point. Now

$$\mathrm{d}\psi_0 - \mathbb{1} = \mathrm{d}\kappa_{x_0} \circ (\mathrm{d}\Psi_{x_0} - \mathbb{1}) \circ \mathrm{d}\kappa_0^{-1} \tag{0.4}$$

is also invertible. Thus by the previous discussion there is $B_\delta(0) \subset \mathbb{R}^n$ on which ψ has no fixed point other than 0, which means that $U := \kappa^{-1}(B_\delta(0))$ is a neighborhood of x_0 where Ψ has no fixed point other than x_0. So fixed points of Ψ are isolated.

Next we claim the set of fixed points of Ψ cannot have limit points. Indeed, if $x_i \to y$ with x_i being fixed points, then by continuity of Ψ, $\Psi(y) = \Psi(\lim x_i) = \lim \Psi(x_i) = y$, namely y must also be a fixed point. This contradicts that fixed points are all isolated. Consequently, the set of fixed points of Ψ is closed.

Finally, for each fixed point $x_i \in M$ we pick neighborhood $U_i \ni x_i$ with no fixed point other than x_i. Then

$$M \subset (M \setminus \{x_i\}) \cup \bigcup_i U_i \tag{0.5}$$

forms an open covering of M. This will contain a finite subcovering only if $\{x_i\}$ is finite. Thus we proved the result. □

(2) Explain roughly why $\Psi^* : C^\infty(M) \to C^\infty(M)$ admits a continuous extension to $\mathcal{D}'(M) \to \mathcal{D}'(M)$.

For $u \in \mathcal{D}'(M)$, define

$$(\Psi^* u, \varphi) := (u, \Psi_* \varphi), \tag{0.6}$$

for $\varphi \in C^\infty(M)$, where $(\Psi_*\varphi)(y) := |\det \mathrm{d}\Psi_{\Psi^{-1}(y)}|^{-1} \varphi(\Psi^{-1}(y))$, where $|\det \mathrm{d}\Psi|$ is the smooth determinant function such that $\Psi^*\mathrm{d}\mu = |\det \mathrm{d}\Psi| \mathrm{d}\mu$. Now clearly Ψ^* is continuous under the weak* topology of $\mathcal{D}'(M)$ (follows from definition and the fact that Ψ, Ψ^{-1} are smooth).

We need to verify that $\Psi^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$ restricts to the usual pullback on $C^\infty(M)$. This is to say

$$(\psi \mathrm{d}\mu, \Psi_*\varphi) = (\Psi^*(\psi)\mathrm{d}\mu, \varphi), \tag{0.7}$$

for all ψ, $\varphi \in C^\infty(M)$. But we check, by the definition of $\det \mathrm{d}\Psi$, that

$$\begin{aligned}
(\Psi^*(\psi)\mathrm{d}\mu, \varphi) &= \int_M \psi(\Psi(x))\varphi(x)\mathrm{d}\mu(x) \\
&= (\mathrm{d}\mu, \psi(\Psi(\bullet))\varphi(\bullet)) \\
&= \Big(\mathrm{d}\mu, \Psi_*\big(\psi(\Psi(\bullet))\varphi(\bullet)\big)\Big) \\
&= \int_M |\det \mathrm{d}\Psi_{\Psi^{-1}(x)}|^{-1} \psi(\Psi(\Psi^{-1}(x)))\varphi(\Psi^{-1}(x))\mathrm{d}\mu(x) \\
&= \int_M \psi(x)(\Psi_*\varphi)(x)\mathrm{d}\mu(x) \\
&= (\psi\mathrm{d}\mu, \Psi_*\varphi).
\end{aligned}$$

Thus the map $\Psi^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$ given by (0.6) is what we want.

(3) Find $\operatorname{supp} K$ and $\mathrm{WF}(K)$.

Claim 1. $\operatorname{supp} K = G(\Psi) = \{(x, \Psi(x)) \in M \times M \mid x \in M\}$.

Proof. By definition, we have

$$(\Psi^*(\psi)\mathrm{d}\mu, \varphi) =: (K, \varphi \otimes \psi), \tag{0.8}$$

for all ψ, $\varphi \in C^\infty(M)$. Define $\widetilde{K} \in \mathcal{D}'(M \times M)$ such that

$$(\widetilde{K}, \theta) := \int_M \theta(x, \Psi(x))\mathrm{d}\mu(x), \tag{0.9}$$

for all $\theta(x,y) \in C^\infty(M \times M)$. We check that $\widetilde{K} \equiv K$. Indeed,

$$\begin{aligned}
(\widetilde{K}, \varphi \otimes \psi) &= \int_M \varphi(x)\psi(\Psi(x))\mathrm{d}\mu(x) \\
&= (\Psi^*(\psi)\mathrm{d}\mu, \varphi) =: (K, \varphi \otimes \psi).
\end{aligned}$$

So by uniqueness of the Schwartz kernel we have $\widetilde{K} \equiv K$. From the expression (0.9) it is then clear that $\operatorname{supp} K = G(\Psi)$. □

Claim 2. $\mathrm{WF}(K) = N_0^* G(\Psi) = \{(x, \Psi(x), J) \in T_0^*(M \times M) \mid J(v, \mathrm{d}\Psi(v)) = 0 \text{ for all } v \in TM\}$.

Proof. From standard manifold theory we know that $G(\Psi)$ is an embedded submanifold of $M \times M$, and for each $(x_0, \Psi(x_0)) \in G(\Psi)$, there exist a coordinate neighborhood $U \ni (x_0, \Psi(x_0))$ of $M \times M$ and $\kappa : U \to \mathbb{R}^{2n}$ such that $\kappa(U \cap G(\Psi)) \subset \{(x, 0) \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}$. This induces a diffeomorphism $\mathrm{d}\kappa^{-\top} : T_0^* U \to T_0^*(\kappa(U)) \subset T_0^*(\mathbb{R}^{2n})$ such that

$$\mathrm{d}\kappa^{-\top}(z, \xi) = (\kappa(z), (\kappa^{-1})^*\xi). \tag{0.10}$$

We would like to know which points of T_0^*U belong to $\mathrm{WF}(K)$.

According to definition 1.2.26 in the poly, $(z,\xi) \in T_0^*U$ belongs to $\mathrm{WF}(K)$ if and only if $\mathrm{d}\kappa^{-\top}(z,\xi)$ belongs to $\mathrm{WF}((\kappa^{-1})^*(\chi K))$ for some $\chi \in C_c^\infty(U)$ with $\chi(z) > 0$. Thus we must now compute $\mathrm{WF}((\kappa^{-1})^*(\chi K))$. By definition, for $\varphi \in C_c^\infty(\kappa(U))$, we have

$$\begin{aligned}
((\kappa^{-1})^*(\chi K), \varphi) &= (\chi K, (\kappa^{-1})_*\varphi) \\
&= (K, \chi \cdot |\det \mathrm{d}\kappa_{(\bullet)}^{-1}|^{-1}\varphi(\kappa(\bullet))) \\
&= \int_M \chi(x,\Psi(x))\varphi(\kappa(x,\Psi(x))) \cdot |\det \mathrm{d}\kappa_{(x,\Psi(x))}| \mathrm{d}\mu(x) \\
&= \int_{G(\Psi)} \chi(z)\varphi(\kappa(z))|\det \mathrm{d}\kappa_z|\pi^*(\mathrm{d}\mu) \\
&= \int_{\mathbb{R}^n} \chi(\kappa^{-1}(y))\varphi(y,0)|\det \mathrm{d}\kappa_{\kappa^{-1}(y)}|(\kappa^{-1})^*\pi^*(\mathrm{d}\mu) \\
&= \int_{\mathbb{R}^n} \chi(\kappa^{-1}(y))\varphi(y,0)\mathrm{d}\widetilde{\mu},
\end{aligned}$$

where $\pi : G(\Psi) \to M$ is the first factor projection (which is a diffeomorphism), $z = \pi^{-1}(x) = (x, \Psi(x))$, and $\mathrm{d}\widetilde{\mu} = |\det \mathrm{d}\kappa_{\kappa^{-1}(\bullet)}|(\kappa^{-1})^*\pi^*(\mathrm{d}\mu)$ is a measure (density) on $\kappa(U \cap G(\Psi)) \subset \mathbb{R}^{2n}$. Since $\mathrm{d}\mu$ is nowhere zero, $\mathrm{d}\widetilde{\mu}$ is nonzero where it is defined. Thus we see that $(\kappa^{-1})^*(\chi K)$ is a Dirac mass on $\mathbb{R}^n \subset \mathbb{R}^{2n}$ multiplied by a smooth cutoff. By the result of exercise 1.1.5, we have

$$\begin{aligned}
&\mathrm{WF}((\kappa^{-1})^*(\chi K)) \cap T^*_{\kappa(z)}(\kappa(U)) \\
= &\begin{cases} \{(\kappa(z),\eta) \mid \eta \neq 0,\ \eta(v) = 0 \text{ for all } v \in T_{\kappa(z)}\mathbb{R}^n\} & \text{if } z \in G(\Psi), \\ \varnothing & \text{if } z \notin G(\Psi). \end{cases}
\end{aligned}$$

Since $z \in U$ and $\chi \in C_c^\infty(U)$, $\chi(z) > 0$, are arbitrary, we conclude $\mathrm{WF}(K) = N^*G(\Psi)$. □

(4A) Show that $\mathrm{WF}(K) \cap N^*\Delta = \varnothing$, where

$$N^*\Delta := \{(x,\xi,x,-\xi) \mid (x,\xi) \in T_0^*M\}. \tag{0.11}$$

Proof. By the previous result, we need to show $N_0^*G(\Psi) \cap N^*\Delta = \varnothing$. Clearly $G(\Psi)$ and Δ intersects only at fixed points of Ψ. Let $(x_0,x_0) \in G(\Psi) \cap \Delta$ be a fixed point, we ask which points $(x_0,x_0,\xi,\eta) \in T^*_{(x_0,x_0)}(M \times M)$ is in $N^*G(\Psi)$ and which is in $N^*\Delta$.

We see that $(x_0,x_0,\xi,\eta) \in N^*G(\Psi)$ iff $\xi(v)+\eta(\mathrm{d}\Psi(v)) = 0$ for all $v \in T_{x_0}M$; and $(x_0,x_0,\xi,\eta) \in N^*\Delta$ iff $\eta = -\xi$. Thus for (x_0,x_0,ξ,η) in the intersection, one has

$$\begin{cases} \xi(v) + \eta(\mathrm{d}\Psi(v)) = 0, \\ \xi(v) + \eta(v) = 0, \end{cases} \tag{0.12}$$

for all $v \in T_{x_0}M$, meaning $\eta(\mathrm{d}\Psi(v) - v) = 0$. But $\mathrm{d}\Psi - \mathbb{1}$ is invertible at x_0; this implies $\eta \equiv 0$. But $(x_0,x_0,0,0)$ is not in $N_0^*G(\Psi)$. This proves our result. □

(4B) Deduce that

$$\mathrm{tr}^\flat(\Psi^*) := (\iota^*K, \mathbf{1}), \tag{0.13}$$

where $\iota : M \to M \times M$ is the diagonal embedding and $\mathbf{1} : M \to \mathbb{R}$ the constant function with value 1 on M, is well-defined.

Proof. This is stated in lemma 1.2.30 of the polycopy. Basically, it amounts to observing that the Schwartz kernel K_{ι^*} of the diagonal restriction $\iota^* : C^\infty(M \times M) \to C^\infty(M)$ has

$$\mathrm{WF}(K_{\iota^*}) = \{(x,x,x,-(\xi+\eta),\xi,\eta) \mid x \in M, \xi,\eta \in T_0^*M\}, \tag{0.14}$$

and thus

$$\begin{aligned}\mathrm{WF}'_{M\times M}(K_{\iota^*}) &= \left\{(y,z,\xi_1,\xi_2)\in T_0^*(M\times M) \;\middle|\; \begin{array}{l}\exists\, x\in M \text{ such that}\\ (x,y,z,0,-\xi_1,-\xi_2)\in \mathrm{WF}(K_{\iota^*})\end{array}\right\}\\ &= \left\{(y,y,\xi,-\xi)\in T_0^*(M\times M)\mid y\in M, \xi\in T_y^*M\right\}\\ &= N^*\Delta.\end{aligned}$$

The result then follows from the extension theorem 1.1.25. □

Now we localize near one fixed point and move to the Euclidean space.

- X, $Y\subset\mathbb{R}^n$ two Euclidean open sets, containing 0;
- $\psi: X\to Y$ a smooth diffeomorphism such that $\psi(0)=0$, $\mathrm{d}\psi_0-\mathbb{1}$ is invertible, and has no other fixed points except 0 in X;
- $\chi\in C_c^\infty(\mathbb{R}^n)$, $0\leqslant\chi\leqslant 1$, a smooth cutoff function supported near 0, with

$$\int_{\mathbb{R}^n}\chi = 1; \tag{0.15}$$

- Define an approximate unit $\{\widetilde{\chi}_\varepsilon\}$ by $\widetilde{\chi}_\varepsilon(x)=\varepsilon^{-n}\chi(x/\varepsilon)$, and put $\chi_\varepsilon\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ to be

$$\chi_\varepsilon(x,y) := \widetilde{\chi}_\varepsilon(x)\widetilde{\chi}_\varepsilon(y); \tag{0.16}$$

- Denote by $K\in\mathcal{D}'(Y\times X)$ the Schwartz kernel of ψ^* (abusing notation), and put

$$K_\varepsilon := \chi_\varepsilon * K \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n), \tag{0.17}$$

  the usual convolution product, namely

$$\begin{aligned}K_\varepsilon(x,y) &:= \iint \widetilde{\chi}_\varepsilon(x')\widetilde{\chi}_\varepsilon(y')K(x-x',y-y')\mathrm{d}x'\mathrm{d}y'\\ &= \big(K, \widetilde{\chi}_\varepsilon(x-\bullet)\otimes\widetilde{\chi}_\varepsilon(y-\bullet)\big)\\ &= \int_X \widetilde{\chi}_\varepsilon(x-w)\widetilde{\chi}_\varepsilon(y-\psi(w))\mathrm{d}w;\end{aligned}$$

- $\iota:\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n$ is the diagonal embedding;
- $\varphi\in C_c^\infty(\mathbb{R}^n)$, $0\leqslant\varphi\leqslant 1$, another smooth cutoff function supported near 0, with $\varphi(0)=1$.

(5) Denote by $\varphi\psi^*\varphi$ the map $C_c^\infty(Y)\to C_c^\infty(X)$ such that

$$f\longmapsto \varphi\cdot\psi^*(\varphi f)\in C_c^\infty(X)\subset\mathcal{D}'(X), \tag{0.18}$$

for all $f\in C_c^\infty(Y)$. Explain why $\mathrm{tr}^\flat(\varphi\psi^*\varphi)$ is well-defined when the support of φ is close enough to 0.

Proof. We show that the Schwartz kernel K_φ of $\varphi\psi^*\varphi$ is $(\varphi\otimes\varphi)K$. Indeed,

$$\begin{aligned}((\varphi\otimes\varphi)K,\theta\otimes\rho) &:= (K,(\varphi\otimes\varphi)(\theta\otimes\rho))\\ &= (K,(\varphi\theta)\otimes(\varphi\rho))\\ &= (\psi^*(\varphi\rho),\varphi\theta)\\ &= (\varphi\psi^*(\varphi\rho),\theta),\end{aligned}$$

for all $\rho\in C_c^\infty(Y)$ and $\theta\in C_c^\infty(X)$.

Moreover, $\mathrm{WF}(K_\varphi)\subset\mathrm{WF}(K)$ and similarly to the previous question, $\mathrm{WF}(K)\cap N^*\Delta=\emptyset$, thus ι^*K_φ is well-defined.

A priori, $K \in \mathcal{D}'(X \times Y)$ and we can only say that $\iota^* K \in \mathcal{D}'(X \cap Y)$; in this case

$$(\iota^* K, \mathbf{1}) \tag{0.19}$$

does not make sense because $\mathbf{1} \notin C_c^\infty(X \cap Y)$.

Therefore, as long as $\operatorname{supp}\varphi \subset (X \cap Y)$, one then has $\iota^*(\varphi \otimes \varphi) = \varphi^2 \in C_c^\infty(X \cap Y)$ and consequently

$$\begin{aligned}(\iota^*((\varphi \otimes \varphi)K), \mathbf{1}) &= (\iota^*(\varphi \otimes \varphi)\iota^* K, \mathbf{1}) \\ &= (\varphi^2(\iota^* K), \mathbf{1}) \\ &= (\iota^* K, \varphi^2)\end{aligned}$$

is well-defined. □

(6) Show that as $\varepsilon \to 0$, $(\varphi \otimes \varphi)K_\varepsilon \to (\varphi \otimes \varphi)K$ in $\mathcal{D}'_\Gamma(X \times X)$ for some closed conic subset Γ of $T^*(X \times X)$.

Proof. Here we take $\Gamma = \mathrm{WF}(K)$. This is really the same proof as that of lemma 1.1.13 in the lecture notes. □

(7) Show that $\iota^* : C^\infty(X \times X) \to C^\infty(X)$ has a unique continuous extension $\iota^* : \mathcal{D}'_\Gamma(X \times X) \to \mathcal{D}'(X)$.

Proof. This is basically the same as part (4). One has $\Gamma = \mathrm{WF}(K) = N_0^* G(\psi)$, the conormal to the graph, while $\mathrm{WF}'_{X \times X}(K_{\iota^*}) = N^*\Delta$. The two have no intersection because $G(\Psi) \cap \Delta = \{0\}$ and $\mathrm{d}\psi(0) - \mathbb{1}$ is invertible. □

(8) Find $\iota^*\left((\varphi \otimes \varphi)K_\varepsilon\right)$.

Proof. By the previous formula for K_ε, one has

$$(\varphi \otimes \varphi)K_\varepsilon(x, y) = \varphi(x)\varphi(y) \int_X \widetilde{\chi}_\varepsilon(x - w)\widetilde{\chi}_\varepsilon(y - \psi(w))\mathrm{d}w, \tag{0.20}$$

and thus

$$\iota^*\left((\varphi \otimes \varphi)K_\varepsilon\right)(x) = \varphi(x)^2 \int_X \widetilde{\chi}_\varepsilon(x - w)\widetilde{\chi}_\varepsilon(x - \psi(w))\mathrm{d}w. \quad \square \tag{0.21}$$

(9) Show that

$$\mathrm{tr}^\flat(\varphi\psi^*\varphi) = |\det(\mathrm{d}\psi(0) - \mathbb{1})|^{-1}. \tag{0.22}$$

Proof. Without loss of generality, suppose $\operatorname{supp}\chi \subset B_1(0)$, the (open) unit ball; consequently $\operatorname{supp}\widetilde{\chi}_\varepsilon \subset B_\varepsilon(0)$, the ball of radius ε centered at 0. Define

$$\begin{aligned}\Phi : X &\longrightarrow \mathbb{R}^n, \\ x &\longmapsto \psi(x) - x.\end{aligned}$$

Then one has $\mathrm{d}\Phi(0) = \mathrm{d}\psi(0) - \mathbb{1}$. Therefore our condition says that $\mathrm{d}\Phi(0)$ is invertible. By the inverse function theorem, there exist neighborhoods $0 \in U \subset X$ and $V \ni 0$ such that $\Phi|_U$ is a diffeomorphism of U onto V. Now pick $\delta > 0$ such that $B_\delta(0) \subset V$, and put $U_1 = \Phi^{-1}(B_\delta(0)) \subset U$. Consequently, when $\varepsilon < \delta/2$, one has

$$\widetilde{\chi}_\varepsilon(x - w)\widetilde{\chi}_\varepsilon(x - \psi(w)) \neq 0 \quad \Longrightarrow \quad |x - w| < \varepsilon,\ |x - \psi(w)| < \varepsilon, \tag{0.23}$$

which implies $|\psi(w) - w| < 2\varepsilon < \delta$. This in turn implies $w \in U_1$. Thus under the condition $\varepsilon < \delta/2$ one has

$$(\iota^*((\varphi \otimes \varphi)K_\varepsilon), \mathbf{1}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x)^2 \widetilde{\chi}_\varepsilon(x - w)\widetilde{\chi}_\varepsilon(x - \psi(w))\mathrm{d}w\mathrm{d}x$$

$$= \int_{U_1 \times U_2} \varphi(x)^2 \widetilde{\chi}_\varepsilon(x-w)\widetilde{\chi}_\varepsilon(x-\psi(w))\mathrm{d}w\mathrm{d}x,$$

where U_2, the domain of x, is a slightly larger neighborhood of U_1.

Now consider the change of variables

$$\begin{cases} r = \Phi(w) = \psi(w) - w, \\ s = x - w, \end{cases} \quad \text{that is,} \quad \begin{cases} w = \Phi^{-1}(r), \\ x = s + \Phi^{-1}(r). \end{cases} \tag{0.24}$$

This has Jacobian matrix

$$\begin{pmatrix} \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \end{pmatrix} = \begin{pmatrix} \mathrm{d}\Phi^{-1} & 0 \\ * & \mathbb{1} \end{pmatrix}, \tag{0.25}$$

which has determinant $\det \mathrm{d}\Phi_r^{-1}$. Hence:

$$\begin{aligned} \text{the integral} &= \int_{\mathbb{R}^{2n}} \varphi^2(s+\Phi^{-1}(r))\widetilde{\chi}_\varepsilon(s)\widetilde{\chi}_\varepsilon(s-r)\cdot|\det \mathrm{d}\Phi_r^{-1}|\mathrm{d}r\mathrm{d}s \\ &= \int_{\mathbb{R}^{2n}} \varphi^2(s+\Phi^{-1}(s-s'))\underbrace{\widetilde{\chi}_\varepsilon(s)\widetilde{\chi}_\varepsilon(s')}_{=\chi_\varepsilon(s,s')}\cdot|\det \mathrm{d}\Phi_{s-s'}^{-1}|\mathrm{d}s\mathrm{d}s', \end{aligned} \tag{0.26}$$

where the integrand is supported for $|s| < \varepsilon, |s'| < \varepsilon$. Then, using that $\chi_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^{2n})$, we get that the integral converges to the value of the integrand at $s = s' = 0$, that is

$$(\iota^*((\varphi\otimes\varphi)K_\varepsilon), \mathbf{1}) \to |\det(\mathrm{d}\psi(0) - \mathbb{1})|^{-1}.$$

□

(10) Compute $\mathrm{tr}^\flat(\Psi^*)$.

First we deal with two technicalities of coordinate charts (please skip them).

(a) The embedding $C^\infty(M) \hookrightarrow \mathcal{D}'(M)$ on the manifold M depends on the choice of a smooth nonvanishing density $\mathrm{d}\mu \in |\Omega^1|(M)$. We remark that on any coordinate neighborhood $U \subset M$ it is always possible to choose the coordinate map $\kappa : U \to V \subset \mathbb{R}^n$ so that $\mathrm{d}\mu = \kappa^*(\mathrm{d}L)$, pullback of the Lebesgue measure. This boils down to a question about Euclidean spaces. If $F : X \to Y$ is a diffeomorphism of Euclidean open sets, and given an arbitrary smooth measure $a(x)\mathrm{d}L_X$ on X, where $a : X \to \mathbb{R}$ is a nonvanishing positive smooth function, we want to determine F so that $F^*(\mathrm{d}L_Y) = a(x)\mathrm{d}L_X$. But we know that $F^*(\mathrm{d}L_Y) = |\det \mathrm{d}F_x|\mathrm{d}L_X$. Thus we just need to make $|\det \mathrm{d}F_x| \equiv a(x)$. One way is to put

$$F(x_1,\ldots,x_n) = \left(\int_{t_0}^{x_1} a(t,x_2,\ldots,x_n)\mathrm{d}t, x_2,\ldots,x_n\right), \tag{0.27}$$

provided that X is convex in the x_1 direction (t_0 is some fixed point so that the formula makes sense).

(b) For a fixed point x_0 of Ψ, $\det(\mathrm{d}\Psi_{x_0} - \mathbb{1})$ is always well-defined and independent of coordinates. This implies that we would get the same determinant if we view Ψ under a coordinate chart around x_0.

Now let $\{x_1,\ldots,x_N\}$ be fixed points of Ψ. Consider the open covering

$$M \subset (M \setminus \{x_1,\ldots,x_N\}) \cup \bigcup_{i=1}^N U_i \tag{0.28}$$

mentioned in part (1), here we will suppose each $U_i \subset \widetilde{U}_i$, where $\widetilde{U}_i$ is a larger coordinate neighborhood, with coordinates sending $\mathrm{d}\mu$ to $\mathrm{d}L$, and such that $\Psi(U_i) = V_i \subset \widetilde{U}_i$.

Let $\{\rho_j^2\}_{j=0}^N$ be a smooth partition of unity subordinate to the above covering, where ρ_0^2 corresponds to $U_0 = M \setminus \{x_1, \dots, x_N\}$. One then has

$$(\iota^* K, \mathbf{1}) = \sum_j (\rho_j^2 \iota^* K, \mathbf{1}) = \sum_j (\iota^*((\rho_j \otimes \rho_j)K), \mathbf{1}). \tag{0.29}$$

First we deal with $(\rho_0 \otimes \rho_0)K \in \mathcal{E}'(U_0 \times U_0)$. Observe that $\Psi|_{U_0} : U_0 \to U_0$ has no fixed points. This means that $\operatorname{supp}((\rho_0 \otimes \rho_0)K)$, which is the part of $G(\Psi)$ above U_0, does not intersect the diagonal Δ. This would imply that we will be able to approximate $(\rho_0 \otimes \rho_0)K$ with smooth $(\rho_0 \otimes \rho_0)R_\varepsilon$ so that the support of $(\rho_0 \otimes \rho_0)R_\varepsilon$ does not intersect the diagonal when ε is small enough. This shows (heuristically) that $\iota^*((\rho_0 \otimes \rho_0)K)$ would be zero.

Next we look at $(\rho_j \otimes \rho_j)K =: K_j \in \mathcal{E}'(\widetilde{U}_j \times \widetilde{U}_j)$, $j > 0$. By our previous remarks on compatibility, $(\iota^* K_j, \mathbf{1})$ would be the same as $(\iota^*(\widetilde{\kappa}_j^{-1})^* K_j, \mathbf{1})$, where $\widetilde{\kappa}_j = \kappa_j \times \kappa_j$ is the induced coordinate map on $\widetilde{U}_j \times \widetilde{U}_j$; and $(\widetilde{\kappa}_j^{-1})^* K_j$ would exactly be (a cutoff times) the Schwartz kernel of the Euclidean diffeomorphism $\kappa_j \circ \Psi \circ \kappa_j^{-1} : U_j \to V_j$. By the Euclidean result, we have

$$(\iota^*(\rho_j \otimes \rho_j)K, \mathbf{1}) = |\det(\mathrm{d}\Psi_{x_j} - \mathbb{1})|^{-1}. \tag{0.30}$$

Consequently,

$$\boxed{\operatorname{tr}^\flat(\Psi^*) = \sum_{j=1}^N |\det(\mathrm{d}\Psi_{x_j} - \mathbb{1})|^{-1}.} \tag{0.31}$$