HOMEWORK: FLAT TRACE OF DIFFEOMORPHISMS WITH NON-DEGENERATE FIXED POINTS

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Exercise 1: Preliminaries of linear algebra. Let $A \in \mathcal{M}_n(\mathbb{C})$. The action of A on $\mathbb{C}^n$ extends naturally to $\Lambda^k \mathbb{C}^n$ (for $k = 0, ..., n$) by setting on pure elements:

$$A(\eta_1 \wedge ... \wedge \eta_k) := (A\eta_1) \wedge ... \wedge (A\eta_k),$$

where $\eta_1, ..., \eta_k \in \mathbb{C}^n$. Show that:

$$\det(\mathbb{1} - A) = \sum_{k=1}^{n} (-1)^k \operatorname{Tr}(\Lambda^k A).$$

Exercise 2: Flat trace of Morse diffeomorphisms. Let M be a smooth closed manifold and $\Psi : M \to M$ be a smooth diffeomorphism with non-degenerate fixed points. By this, we mean that for every fixed point $x_\star$ of Ψ (i.e. such that $\Psi(x_\star) = x_\star$), $d\Psi(x_\star) - \mathbb{1}$ is invertible. The operator Ψ induces a map $\Psi^* : C^\infty(M) \to C^\infty(M)$ by pullback, $\Psi^* f := f(\Psi(\bullet))$, whose Schwartz kernel is denoted by K.

(1) Show that, under these assumptions, the number of fixed points is finite.
(2) Explain quickly why $\Psi^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$ is bounded.
(3) What is $\operatorname{supp}(K)$? Compute $\operatorname{WF}(K)$?
(4) Show that $\operatorname{WF}(K) \cap N^*\Delta = \emptyset$. Deduce that $\operatorname{Tr}^\flat(\Psi^*)$ is well-defined.

We now want to compute $\operatorname{Tr}^\flat(\Psi^*)$. For that, let $X, Y \subset \mathbb{R}^n$ be open sets containing 0 and let $\psi : X \to Y$ be a smooth diffeomorphism such that $\psi(0) = 0, d\psi(0) - \mathbb{1}$ is invertible and 0 is the only fixed point in X. Let $\chi : \mathbb{R}^n \to [0, 1]$ be a smooth nonnegative cutoff function with support near 0, such that

$$\int_{\mathbb{R}^n} \chi = 1.$$

We define $\widetilde{\chi}_\varepsilon := \varepsilon^{-n} \chi(\bullet/\varepsilon)$ and $\chi_\varepsilon \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\chi_\varepsilon(x, y) := \widetilde{\chi}_\varepsilon(x)\widetilde{\chi}_\varepsilon(y).$$

We let $K_\psi \in \mathcal{D}'(Y \times X)$ be the Schwartz kernel of ψ^* and let $\imath : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal embedding $x \mapsto (x, x)$. We define

$$K_\varepsilon := \chi_\varepsilon \star K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n),$$

where $\star$ is the usual convolution product. Eventually, we consider $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n, [0,1])$, a smooth cutoff function with support near 0 such that $\varphi(0) = 1$.

(5) Why is $\mathrm{Tr}^\flat(\varphi\psi^*\varphi)$ well-defined if the support of φ is close enough to 0?
(6) Show that $(\varphi \otimes \varphi)K_\varepsilon \to (\varphi \otimes \varphi)K_\psi$ in $\mathcal{D}'_\Gamma(X \times X)$ as $\varepsilon \to 0$, where Γ is some well-chosen closed conic subset of $T^*(X \times X)$ that you will introduce.
(7) Show that $\iota^* : C^\infty(X \times X) \to C^\infty(X)$ extends uniquely to a continuous map $\iota^* : \mathcal{D}'_\Gamma(X \times X) \to \mathcal{D}'(X)$.
(8) Compute $\imath^*((\varphi \otimes \varphi)K_\varepsilon)$.
(9) Show that
$$\mathrm{Tr}^\flat(\varphi\psi^*\varphi) = |\det(d\psi(0) - \mathbb{1})|^{-1}.$$
(10) Deduce the value of $\mathrm{Tr}^\flat(\Psi^*)$.

Exercise 3: Flat trace and fixed points. The action of Ψ on functions/distributions by pullback can be naturally extended to k-forms. More precisely, if $f \in C^\infty(M, \Lambda^k T^*M)$, $x \in M, v_1, ..., v_k \in T_xM$, then we can define:
$$\left[\Psi^*_{(k)} f\right]_x (v_1, ..., v_k) := f_{\Psi(x)} \left(d\Psi(x)(v_1), ..., d\Psi(x)(v_k)\right).$$
Similarly to the pullback of functions, this action naturally extends by continuity as a map
$$\Psi^*_{(k)} : \mathcal{D}'(M, \Lambda^k T^*M) \to \mathcal{D}'(M, \Lambda^k T^*M).$$
We let $K_{(k)}$ be the Schwartz kernel of the operator acting on these bundles.

(1) Given $k = 0, ..., n$, what is $\mathrm{supp}(K_{(k)})$? $\mathrm{WF}(K_{(k)})$?
(2) Show that the flat trace $\mathrm{Tr}^\flat(\Psi^*_{(k)})$ is well-defined and that
$$\mathrm{Tr}^\flat(\Psi^*_{(k)}) = \sum_{j=1}^N \frac{\mathrm{Tr}(\Lambda^k d\Psi(x_j)^\top)}{|\det(\mathbb{1} - d\Psi(x_j))|},$$
where $x_1, ..., x_N$ are the fixed points and $\Lambda^k d\Psi(x_j)^\top$ denotes the linear operator induced by $d\Psi(x_j) : T_{x_j}M \to T_{x_j}M$ on $\Lambda^k T^*_{x_j}M$.
(3) Let
$$\Lambda^\star T^*M := \oplus_{k=0}^n \Lambda^k T^*M.$$
Let $L_\Psi : C^\infty(M, \Lambda^\star TM) \to C^\infty(M, \Lambda^\star TM)$ be the operator defined as
$$L_\Psi \left(\sum_{k=0}^n f_k\right) = \sum_{k=0}^n (-1)^k \Psi^*_{(k)} f_k,$$
where $f_k \in C^\infty(M, \Lambda^k T^*M)$. We denote by the same letter L its continuous extension as a map
$$L_\Psi : \mathcal{D}'(M, \Lambda^\star T^*M) \to \mathcal{D}'(M, \Lambda^\star T^*M),$$

Show that $\mathrm{Tr}^\flat(L_\Psi)$ is well-defined and that

$$\mathrm{Tr}^\flat(L_\Psi) = \sum_{j=1}^{N} \mathrm{sgn}\det(\mathbb{1} - d\Psi(x_j)),$$

that is the number of fixed points of Ψ counted with signs.

Exercise 4: Examples. Consider the sphere

$$\mathbb{S}^2 := \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\right\} \subset \mathbb{R}^3.$$

(1) Let $R : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map, given by $R(v) := -v$. Show that $\mathrm{Tr}^\flat(L_R)$ is well-defined and compute it.

(2) Let $N := (0,0,1)$ be the North pole. Consider the exponential map

$$\mathbb{S}^1 \times [0,\pi] \in (u,r) \mapsto \exp_N(ru) \in \mathbb{S}^2,$$

where $\mathbb{S}^1$ is identified here with $\{v \in T_N\mathbb{S}^2 \mid |v| = 1\}$. Show that the vector field defined in coordinates by

$$X(u,r) := -r(\pi - r)\partial_r$$

is well-defined on $\mathbb{S}^2$ and smooth.

(3) Let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow generated by X. Describe (and draw on a picture) its flowlines.

(4) Define $\Psi := \varphi_1$. Show that $\mathrm{Tr}^\flat(L_\Psi) = 2$. What does 2 represent for the sphere?

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