Exercice : Changement de coordonnées pour le laplacien.

$$\kappa : (0,\infty)\times\left(\mathbb{R}/2\pi\mathbb{Z}\right) \ni (r,\theta) \longmapsto (r\cos\theta, r\sin\theta).$$

1) On calcule pour $f \in C^\infty_{comp}(\mathbb{R}^2)$:

$$\partial_r f(r\cos\theta, r\sin\theta) = \cos\theta\, \partial_{x_1} f(r\cos\theta, r\sin\theta) + \sin\theta\, \partial_{x_2} f.$$

$$\begin{aligned}\partial_r^2 f(r\cos\theta, r\sin\theta) &= \cos\theta\left(\cos\theta\, \partial_{x_1}^2 f + \sin\theta\, \partial_{x_1}\partial_{x_2} f\right) \\ &\quad + \sin\theta\left(\cos\theta\, \partial_{x_1}\partial_{x_2} f + \sin\theta\, \partial_{x_2}^2 f\right) \\ &= \cos^2\theta\, \partial_{x_1}^2 f + \sin^2\theta\, \partial_{x_2}^2 f + 2\cos\theta\sin\theta\, \partial_{x_1}\partial_{x_2} f.\end{aligned}$$

$$\partial_\theta f(-) = -r\sin\theta\, \partial_{x_1} f + r\cos\theta\, \partial_{x_2} f$$

$$\begin{aligned}\partial_\theta^2 f &= -r\cos\theta\, \partial_{x_1} f - r\sin\theta\, \partial_{x_2} f \\ &\quad - r\sin\theta\left(-r\sin\theta\, \partial_{x_1}^2 f + r\cos\theta\, \partial_{x_1}\partial_{x_2} f\right) \\ &\quad + r\cos\theta\left(-r\sin\theta\, \partial_{x_1}\partial_{x_2} f + r\cos\theta\, \partial_{x_2}^2 f\right) \\ &= -r\cos\theta\, \partial_{x_1} f - r\sin\theta\, \partial_{x_2} f \\ &\quad + r^2\sin^2\theta\, \partial_{x_1}^2 f + r^2\cos^2\theta\, \partial_{x_2}^2 f \\ &\quad - 2r^2\cos\theta\sin\theta\, \partial_{x_1}\partial_{x_2} f\end{aligned}$$

Puis :

$$\begin{aligned}&\left(\partial_r^2 + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2\right) f \\ &= \cos^2\theta\, \partial_{x_1}^2 f + \sin^2\theta\, \partial_{x_2}^2 f + \cancel{2\cos\theta\sin\theta\, \partial_{x_1}\partial_{x_2} f} \\ &\quad + \cancel{\frac{1}{r}\cos\theta\, \partial_{x_1} f} + \cancel{\frac{1}{r}\sin\theta\, \partial_{x_2} f} \\ &\quad - \cancel{\frac{1}{r}\cos\theta\, \partial_{x_1} f} - \cancel{\frac{1}{r}\sin\theta\, \partial_{x_2} f} + \sin^2\theta\, \partial_{x_1}^2 f + \cos^2\theta\, \partial_{x_2}^2 f \\ &\quad - \cancel{2\cos\theta\sin\theta\, \partial_{x_1}\partial_{x_2} f}. \\ &= \partial_{x_1}^2 f + \partial_{x_2}^2 f.\end{aligned}$$

ouf !

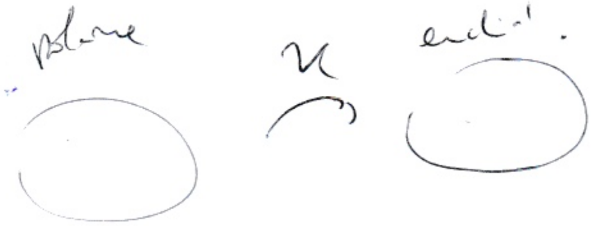
②

2) $\sigma^{tot}_{\Delta_{euc}}(x, \xi) = -|\xi|^2 = -(\xi_x^2 + \xi_y^2)$.

3) On note $\eta = (\eta_r, \eta_\theta)$ le vecteur dual des variables (r, θ) (c-à-d $\eta = \eta_r\, dr + \eta_\theta\, d\theta$). On a

$$\sigma^{tot}_{\Delta_{plane}}((r,\theta), \eta) = -\eta_r^2 + \frac{i\eta_r}{r} - \frac{\eta_\theta^2}{r^2}.$$

4) $\kappa(r,\theta) = (r\cos\theta,\ r\sin\theta)$



$$d\kappa(\partial_r) = \cos\theta\, \partial_{x_1} + \sin\theta\, \partial_{x_2}$$

$$d\kappa(\partial_\theta) = -r\sin\theta\, \partial_{x_1} + r\cos\theta\, \partial_{x_2}.$$

$$d\kappa = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \qquad \det d\kappa = r.$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$d\kappa^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \qquad d\kappa^{-T} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}.$$

Donc si $\eta = \begin{pmatrix} \eta_r \\ \eta_\theta \end{pmatrix}$ est un covecteur, alors il est envoyé sur $\xi = \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = d\kappa^{-T}\begin{pmatrix} \eta_r \\ \eta_\theta \end{pmatrix}$

$$= \underbrace{\frac{1}{r}(r\cos\theta\, \eta_r - \sin\theta\, \eta_\theta)}_{\xi_1} dx_1 + \underbrace{\frac{1}{r}(r\sin\theta\, \eta_r + \cos\theta\, \eta_\theta)}_{\xi_2} dx_2.$$

On calcule $\left(\kappa^* \sigma^{tot}_{\Delta_{euc}}\right)((r,\theta), \eta) = \sigma^{tot}_{\Delta_{euc}}((x_1,x_2), (\xi_1, \xi_2))$

$$= -\frac{1}{r^2}\left[(r\cos\theta\, \eta_r - \sin\theta\, \eta_\theta)^2 + (r\sin\theta\, \eta_r + \cos\theta\, \eta_\theta)^2\right].$$

$$= -\frac{1}{r^2}\Big[r^2\cos^2\theta\,\eta_r^2 - 2r\,\cancel{\cos\theta\sin\theta}\,\eta_r\eta_\theta + \sin^2\theta\,\eta_\theta^2$$
$$+ r^2\sin^2\theta\,\eta_r^2 + 2\cancel{r\cos\theta\sin\theta}\,\eta_r\eta_\theta + \cos^2\theta\,\eta_\theta^2\Big]$$

$$= -\eta_r^2 - \frac{1}{r^2}\eta_\theta^2 = \sigma^{tot}_{\Delta_{plane}} - \frac{i\eta_r}{r}.$$

pas invariant par changement de coordonnée !

5) Immédiat.

Exercice 4 :

1) On a vu que $\sigma_d(x,\xi) = i\xi$ donc $\sigma_D(x,\xi) = i\xi^\#$.

2) Cela signifie que dans ces coord locales $(x_i)_{1\le i\le n}$, si $v = \sum v_i\partial_{x_i}$ est un vecteur, on a :

$$g(v,v) = \langle \underbrace{g(x)}_{\text{matrice sym}} v, v\rangle$$
métrique euclidienne.

Soit $\xi = \sum \xi_i dx_i$ un covecteur. On note $\xi^\#$ le champ de vecteur tel que $(\xi, v) = \langle g(x)\underbrace{\xi^\#}_{z}, v\rangle = g(\xi^\#, v)$.

$$\overset{\shortparallel}{\sum \xi_i v_i} \qquad \overset{\shortparallel}{\sum_i (g(x)z)_i v_i}$$

Donc : $\xi_i = (g(x)z)_i$ càd $\xi = g\xi^\#$ càd $\xi^\# = g^{-1}(x)\xi$.

Puis : $|\xi|^2_{g^{-1}} \overset{\text{déf}}{=} |\xi^\#|^2_g = g(\xi^\#, \xi^\#)$
$$= \langle g(x)\xi^\#, \xi^\#\rangle$$
$$= \langle gg^{-1}\xi, g^{-1}\xi\rangle = \langle g^{-1}\xi, \xi\rangle.$$