

**EXERCISE CLASS: WAVEFRONT SET OF
DISTRIBUTIONS, SYMBOLS AND OSCILLATORY
INTEGRALS**

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DISTRIBUTIONS

Exercise 1. Let F_1, F_2 be two Fréchet spaces. What does it mean for a linear map $u : F_1 \rightarrow F_2$ to be continuous?

Exercise 2. Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Show that there exists $C, M > 0$ such that:

$$\forall \xi \in \mathbb{R}^n, \quad |\widehat{u}(\xi)| \leq C \langle \xi \rangle^M.$$

Exercise 3. The principal value of $1/x$ is defined as the distribution $\text{vp}(1/x) : C_{\text{comp}}^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ such that:

$$\text{vp}(1/x) : \varphi \mapsto \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(x)}{x} dx.$$

- (1) What is $\text{supp}(\text{vp}(1/x))$?
- (2) What is the order of $\text{vp}(1/x)$?
- (3) Compute $\text{WF}(\text{vp}(1/x))$.

Exercise 4. Let $\delta_{\mathbb{R}^k}$ be the Dirac mass on the k -plane $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$, that is

$$(\delta_{\mathbb{R}^k}, \varphi) := \int_{\mathbb{R}^k} \varphi(x, 0) dx.$$

Show that $\text{WF}(\delta_{\mathbb{R}^k}) = N^*\mathbb{R}^k \setminus \{0\}$, the *conormal* to \mathbb{R}^k , that is

$$\{((x, 0), \xi) \mid \forall v \in \mathbb{R}^k \times \{0\}, \langle \xi, v \rangle = 0\}$$

Exercise 5. Define for $z \in \mathbb{C}$ the function x_+^z by:

$$x_+^z = \begin{cases} 0 & \text{on } (-\infty, 0], \\ \exp(z \log(\bullet)) & \text{on } (0, \infty). \end{cases}$$

- (1) Show that x_+^z defines a distribution on \mathbb{R} for $\Re(z) > -1$. Compute its support and its wavefront set.

- (2) Show that $\{\Re(z) > -1\} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ is holomorphic in the sense that for all $\varphi \in C_{\text{comp}}^\infty(\mathbb{R})$, the function $\{\Re(z) > -1\} \ni z \mapsto (x_+^z, \varphi) \in \mathbb{C}$ is holomorphic.

Our goal is to show that $\mathbb{C} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ extends to a meromorphic family of distributions. This means that there exists a maximal countable and isolated subset $\mathcal{P} \subset \mathbb{C}$ and a map $n : \mathcal{P} \rightarrow \mathbb{Z}_+^*$ such that for all $\varphi \in C_{\text{comp}}^\infty(\mathbb{R})$, the function $\mathbb{C} \ni z \mapsto (x_+^z, \varphi) \in \mathbb{C}$ is meromorphic, with poles contained in \mathcal{P} , and of order at most given by n .

For $z \in \mathbb{C} \setminus \mathbb{Z}_-^*$, define for $k > -\Re(z) - 1$:

$$(\text{pf}(x_+^z), \varphi) := (-1)^k \int_0^{+\infty} \frac{x^{z+k}}{(z+1)\dots(z+k)} \partial_x^k \varphi(x) dx.$$

- (3) Show that the definition of $\text{pf}(x_+^z)$ is independent of k as long as $k > -\Re(z) - 1$. Show that it coincides with x_+^z when $\Re(z) > -1$.

Let Γ be the Euler function. Recall that $\Gamma(n+1) = n!$ and that Γ admits a meromorphic extension to \mathbb{C} . We define for $z \in \mathbb{C} \setminus \mathbb{Z}_-^*$:

$$\chi_+^z := \frac{\text{pf}(x_+^z)}{\Gamma(z+1)}.$$

- (4) Show that $\partial_x \chi_+^z = \chi_+^{z-1}$ in $\mathcal{D}'(\mathbb{R})$ for all $z \in \mathbb{C}$ such that $\{\Re(z) > 0\}$.
- (5) Deduce that $\mathbb{C} \ni z \mapsto \chi_+^z \in \mathcal{D}'(\mathbb{R})$ is holomorphic.
- (6) Conclude that $\mathbb{C} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ admits a meromorphic extension from $\{\Re(z) > -1\}$ to \mathbb{C} .

Exercise 6. Let $f \in C^\infty(X)$, $\Im(f) \geq 0$, where $X \subset \mathbb{R}^n$ an open subset. Fix $\varepsilon > 0$.

- (1) Show that

$$\frac{1}{f(x) + i\varepsilon} = \frac{1}{i} \int_0^{+\infty} e^{i(f(x) + i\varepsilon)\tau} d\tau.$$

- (2) We assume that $df(x) \neq 0$ when $f(x) = 0$. Show that the limit

$$\frac{1}{f(x) + i0} := \lim_{\varepsilon \rightarrow 0} \frac{1}{f(x) + i\varepsilon}$$

exists in $\mathcal{D}'(X)$.

- (3) What is $\text{singsupp}((f(x) + i0)^{-1})$?
- (4) Compute $\text{WF}((f(x) + i0)^{-1})$.

(5) For $n = 1$, show that

$$\frac{1}{x \pm i0} = \mp i\pi\delta_0 + \text{vp}(1/x), \quad \delta_0 = \frac{1}{2i\pi} \left(\frac{1}{x - i0} - \frac{1}{x + i0} \right).$$

WAVEFRONT SET

Exercise 7. Let X be an open subset of \mathbb{R}^n and let $u \in \mathcal{D}'(X)$.

- (1) What does it mean for a distribution to be real?
- (2) Show that if u is real, then $\text{WF}(u)$ is invariant by the action of the fiberwise antipodal map $(x, \xi) \mapsto (x, -\xi)$.
- (3) Assume X is such that $R(X) = X$, where $R(x) = -x$. What does it mean for u to be even or odd?
- (4) Show that if u is even or odd, then: $(x, \xi) \in \text{WF}(u)$ iff $(-x, -\xi) \in \text{WF}(u)$.

Exercise 8: tensor product and wavefront set. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be open subsets and $\Gamma_1 \subset T_0^*X, \Gamma_2 \subset T_0^*Y$ be two closed cones. Show that the map

$$C^\infty(X) \otimes C^\infty(Y) \ni (u, v) \mapsto u \otimes v \in C^\infty(X \times Y),$$

extends uniquely to a continuous distribution

$$\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(Y) \ni (u, v) \mapsto u \otimes v \in \mathcal{D}'_{\Gamma_3}(X \times Y)$$

where:

$$\Gamma_3 = (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times O_Y) \cup (O_X \times \Gamma_2).$$

Exercise 9: multiplication of distributions. Let $X \subset \mathbb{R}^n$ be an open subset and let $u_1, u_2 \in \mathcal{D}'(X)$.

- (1) Define, when possible, the product

$$\mathcal{D}'(X) \times \mathcal{D}'(X) \ni (u_1, u_2) \mapsto u_1 \times u_2 \in \mathcal{D}'(X),$$

as the unique continuous extension of the product $C^\infty(X) \times C^\infty(X) \rightarrow C^\infty(X)$. *Hint: You can either mimic the proof of Lemma 2.1.18 in the course, or use the previous exercise by considering the embedding map of the diagonal $\iota : X \rightarrow X \times X, x \mapsto (x, x)$.*

- (2) Show that

$$\text{WF}(u_1 \times u_2) \subset \text{WF}(u_1) \cup \text{WF}(u_2) \cup (\text{WF}(u_1) \oplus \text{WF}(u_2)). \quad (0.1)$$

- (3) For $x_1, x_2 \in \mathbb{R}^n, x_1 \neq x_2$, compute $\delta_{x_1} \times \delta_{x_2}$. Why is $\delta_{x_1}^2$ not (a priori) well-defined?

(4) In \mathbb{R}^2 , define

$$(\delta_x, \varphi) := \int_{\mathbb{R}^2} \varphi(x, 0) dx, \quad (\delta_y, \varphi) := \int_{\mathbb{R}^2} \varphi(0, y) dy.$$

Show that $\delta_x \times \delta_y$ is well-defined and compute it.

(5) Find an example where the inclusion (0.1) is an equality and an example where it is not.

Exercise 10: pullback and pushforward of distributions. Let $X \subset \mathbb{R}^n, F \subset \mathbb{R}^m$ be two open subsets. Let $\pi : X \times F \rightarrow X$ be given by $\pi(x, y) = x$. For $u \in C_{\text{comp}}^\infty(X \times F), f \in C_{\text{comp}}^\infty(X)$, define:

$$\pi^* f(x, y) := f(x), \quad \pi_* u(x) := \int_F u(x, y) dy.$$

- (1) Show that $\pi^* : \mathcal{E}'(X) \rightarrow \mathcal{D}'(X \times F)$ extends continuously and bound $\text{WF}(\pi^* f)$ in terms of $\text{WF}(f)$.
- (2) Deduce that $\pi^* : \mathcal{E}'(X) \rightarrow \mathcal{D}'_\Gamma(X \times F)$ extends continuously for some well-chosen conic subset $\Gamma \subset T_0^*(X \times F)$.
- (3) Show that $\pi_* : \mathcal{E}'(X \times F) \rightarrow \mathcal{E}'(X)$ extends continuously and bound $\text{WF}(\pi_* u)$ in terms of $\text{WF}(u)$.

Exercise 11. Let M^n be a smooth closed oriented n -dimensional manifold and $\pi : E \rightarrow M$ be an oriented fiber bundle, with fiber diffeomorphic to F^k , a closed oriented k -dimensional manifold. Let ω_E be a smooth volume form on E and ω_M be a smooth volume form on M .

- (1) Recall the definition of a fiber bundle.
- (2) Show the existence of $\nu \in C^\infty(E, \Lambda^k T^* E)$ such that $\omega_E = \nu \wedge \pi^* \omega_M$. Show that the restriction of ν to each fiber $E_x \hookrightarrow E$ is a (positive) volume form.
- (3) Consider the pullback operator $\pi^* : C^\infty(M) \rightarrow C^\infty(E)$, defined by $\pi^* f(x, v) := f(x)$. Show that it extends uniquely to a continuous map $\pi^* : L^2(M, \omega_M) \rightarrow L^2(E, \omega_E)$.
- (4) We let $\pi_* : L^2(E, \omega_E) \rightarrow L^2(M, \omega_M)$ be the adjoint of π^* . Compute π_* .
- (5) Show that $\pi^* : \mathcal{D}'(M) \rightarrow \mathcal{D}'(E)$ extends continuously. Bound $\text{WF}(\pi^* f)$ in terms of $\text{WF}(f)$.
- (6) Show that $\pi_* : \mathcal{D}'(E) \rightarrow \mathcal{D}'(M)$ extends continuously. Bound $\text{WF}(\pi_* u)$ in terms of $\text{WF}(u)$.

Exercise 12. Let M^n be a smooth closed manifold and let $X \in C^\infty(M, TM)$ be a smooth vector field. Let $(\varphi_t)_{t \in \mathbb{R}}$ be the flow generated by X . It acts by pullback on smooth functions as $\varphi_t^* : C^\infty(M) \rightarrow C^\infty(M)$, $\varphi_t^* f(x) := f(\varphi_t(x))$.

- (1) Show that $\varphi_t^* : \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ extends continuously.
- (2) Compute $\text{WF}(\varphi_t^* f)$ in terms of $\text{WF}(f)$. Explain this heuristically.

Let $\chi \in C^\infty(\mathbb{R})_{\text{comp}}$ be a smooth cutoff function. Define the operator

$$E := \int_{-\infty}^{+\infty} \chi(t) \varphi_t^* dt.$$

- (3) Compute $\text{WF}(Eu)$ in terms of $\text{WF}(u)$. *Hint: Consider the projection $\pi : M \times \mathbb{R} \rightarrow M$, $\pi(x, t) = x$.*

Exercise 13. Let $X \subset \mathbb{R}^n$ be an open subset and $S \subset T_0^*X$ be a closed conic subset. Show that there exists $u \in \mathcal{D}'(X)$ such that $\text{WF}(u) = S$.

SYMBOLS

Exercise 14. Define $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, $\xi'^2 = \sum_{j=1}^{n-1} \xi_j^2$, $\xi^2 = \xi'^2 + \xi_n^2$. To which symbol space do the following belong?

- (1) $(\xi'^2 + i\xi_n)^{-1}$,
- (2) $\chi(|\xi|)(\xi'^2 + i\xi_n)^{-1}$, where $\chi \in C^\infty(\mathbb{R})$ is such that $\chi = 1$ for $|x| \geq 1$ and $\chi = 0$ for $|x| \leq 1/2$,
- (3) $(\xi^2 + 1)^{-1}$,
- (4) $(\xi'^2 + 1)^1$,
- (5) $e^{i\xi^2}$,
- (6) $e^{ix \cdot \xi}$.

Exercise 15.

- (1) Let $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all $x \in X$, $a(x, \bullet)$ has compact support in \mathbb{R}^N . Show that $a \in S^{-\infty}(X \times \mathbb{R}^N)$.
- (2) Let $a \in C^\infty(X \times \mathbb{R}^N)$ be positively homogeneous of order m for $|\theta| \geq 1$, namely $a(x, \lambda\theta) = \lambda^m a(x, \theta)$ for all $\lambda \geq 1$, $|\theta| \geq 1$. Show that $a \in S_{1,0}^m(X \times \mathbb{R}^N)$.

Exercise 16. Assume that $a \in S_{\rho,0}^m(X \times \mathbb{R}^N)$ with $m < 0$ and $\rho > 1$. Show that $a \in S^{-\infty}(X \times \mathbb{R}^N)$.

Hint: Apply $|\theta| \partial_\theta$ many times and then integrate by parts to recover the expression of a .

Exercise 17: Borel's theorem. The goal of this exercise is to show the following:

Theorem 0.1 (Borel). *For every sequence $(a_\alpha)_{\alpha \in \mathbb{N}^n}$ of complex numbers $a_\alpha \in \mathbb{C}$, there exists $u \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha u(0) = a_\alpha$.*

Let $\chi \in C^\infty(\mathbb{R}^n, [0, 1])$ be such that $\chi = 0$ for $|x| \geq 1$ and $\chi = 1$ for $|x| \leq 1$. Define for $\lambda > 0$:

$$u_N(x, \lambda) := \chi(\lambda x) \sum_{|\alpha|=N} \frac{a_\alpha}{\alpha!} x^\alpha.$$

- (1) Compute $\partial_x^\beta u_N(0)$.
- (2) Show that $\|u_N(x, \lambda)\|_{C^{N-1}} \leq 2^{-N}$ if $\lambda \geq \lambda_N$, where λ_N is chosen large enough.
- (3) Show that $u(x) := \sum_{N \geq 0} u_N(x, \lambda_N)$ solves the problem.

OSCILLATORY INTEGRALS, STATIONARY PHASE

Exercise 18: The Cauchy problem for the wave equation. We consider the following Cauchy problem for the wave equation:

$$\begin{cases} \partial_t^2 f - \Delta f = 0 \\ f(t=0) = 0, \partial_t f(t=0) = u, \end{cases} \quad (0.2)$$

where $u \in C_{\text{comp}}^\infty(\mathbb{R}^n)$.

- (1) Show existence and uniqueness of a solution $f \in C^\infty([0, \infty), \mathcal{S}(\mathbb{R}^n))$ for (0.2) if $u \in C_{\text{comp}}^\infty(\mathbb{R}^n)$.
- (2) Show that for all $t \geq 0, x \in \mathbb{R}^n$:

$$f(t, x) = \int_{\mathbb{R}_\xi^n} \int_{\mathbb{R}_y^n} e^{i(x-y) \cdot \xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|}) u(y) dy d\xi$$

We let $\chi \in C^\infty(\mathbb{R}^n)$ be a smooth cutoff function such that $\chi = 0$ near $\xi = 0$ and $\chi = 1$ for $|\xi| \geq 1$. We decompose the solution as

$$f(t, x) = f_+(t, x) + f_-(t, x) + k(t, x),$$

where

$$f_{\pm}(t, x) = \pm \int_{\mathbb{R}_{\xi}^n} \int_{\mathbb{R}_y^n} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} e^{\pm it|\xi|} \chi(\xi) u(y) dy d\xi =: F_{\pm}(t)u$$

$$k(t, x) = \int_{\mathbb{R}_{\xi}^n} \int_{\mathbb{R}_y^n} e^{i(x-y)\cdot\xi} (2i|\xi|)^{-1} (e^{it|\xi|} - e^{-it|\xi|}) (1 - \chi(\xi)) u(y) dy d\xi =: K(t)u$$

- (3) Show that the operator $K(t)$ is smoothing.
- (4) Show that $F_{\pm}(t)$ is an operator whose Schwartz kernel $K_{\pm}(t)$ is given by an oscillatory integral. Compute $\text{WF}(K_{\pm}(t))$.
- (5) Show that $F_{\pm}(t) : \mathcal{E}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ is continuous. Given $u \in \mathcal{E}'(\mathbb{R}^n)$, compute $\text{WF}(F_{\pm}(t)u)$ in terms of $\text{WF}(u)$.
- (6) Show existence and uniqueness of a solution $f \in C^{\infty}([0, \infty), \mathcal{S}'(\mathbb{R}^n))$ if $u \in \mathcal{E}'(\mathbb{R}^n)$.
- (7) Take $u := \delta_0$. Compute $\text{WF}(f(t, \bullet))$. Can you explain this from a physical perspective?

Exercise 19: The Morse lemma. Let $X \subset \mathbb{R}^n$ and $\phi \in C^{\infty}(X)$. Assume that $\nabla\phi(x_0) = 0$ and that the Hessian $\nabla^2\phi(x_0)$ is non-degenerate (i.e. invertible). Show that there exists a diffeomorphism $\kappa : U \rightarrow V$, where U is a small neighborhood of x_0 and V is a small neighborhood of $0 \in \mathbb{R}^n$ such that for all $y \in V$:

$$\kappa^*\phi(y) = \phi(x_0) + \frac{1}{2} (y_1^2 + \dots + y_r^2 - (y_{r+1}^2 + \dots + y_n^2)).$$

The quantity $\text{sgn}(\nabla^2\phi(x_0)) := r - (n - r)$ is called the signature of the Hessian.

Exercise 20. Let $Q \in \mathcal{M}_n(\mathbb{R})$ be non-degenerate symmetric. Show that the following identities holds for all $\xi \in \mathbb{R}^n$:

- (1) Further assuming that Q is definite positive:

$$\mathcal{F} \left(e^{-\frac{1}{2}\langle Q\bullet, \bullet \rangle} \right) (\xi) = (2\pi)^{n/2} |\det Q|^{-1/2} e^{-\frac{1}{2}\langle Q^{-1}\xi, \xi \rangle}.$$

(Fourier transform of a Gaussian.)

- (2)

$$\mathcal{F} \left(e^{\frac{i}{2}\langle Q\bullet, \bullet \rangle} \right) (\xi) = (2\pi)^{n/2} e^{i\frac{\pi}{4}\text{sgn}(Q)} |\det Q|^{-1/2} e^{-\frac{i}{2}\langle Q^{-1}\xi, \xi \rangle}.$$

(Fourier transform of an imaginary quadratic phase function.)

Exercise 21. Study the convergence in $\mathcal{D}'(\mathbb{R}^n)$ as $h \rightarrow 0$ of:

- (1) $u_h(x) := h^{-N} e^{-\frac{i}{h}x}$, $v_h(x) := h^{-1/2} e^{-\frac{i}{h}x^2/2}$, $w_h(x) := h^{-1/2} e^{+\frac{i}{h}x^2/2}$,
- (2) $u_h(x) := h^{-N} e^{-\frac{i}{h}f(x)}$, $v_h(x) := h^{-1/2} e^{-\frac{i}{h}(f(x))^2/2}$, where $f \in C^\infty(\mathbb{R})$ and $f' \neq 0$.

Exercise 22: Stirling's formula. Define

$$F(\lambda) := \Gamma(\lambda + 1) = \int_0^{+\infty} e^{-t} t^\lambda dt,$$

for $\lambda \geq 0$. Recall that $F(n) = \Gamma(n + 1) = n!$ for all $n \geq 0$. We want to find an asymptotic of F as $\lambda \rightarrow \infty$.

- (1) Rewrite this integral by means of the change of variable $t = \lambda(1 + s)$.
- (2) Show that

$$F(\lambda) = (\lambda e^{-1})^\lambda \sqrt{2\pi\lambda} (1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \dots),$$

and give a precise meaning to "...".

- (3) Deduce Stirling's formula.
- (4) Compute a_1, a_2 .

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