

HOMEWORK: FLAT TRACE OF DIFFEOMORPHISMS WITH NON-DEGENERATE FIXED POINTS

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Exercise 1: Preliminaries of linear algebra. Let $A \in \mathcal{M}_n(\mathbb{C})$. The action of A on \mathbb{C}^n extends naturally to $\Lambda^k \mathbb{C}^n$ (for $k = 0, \dots, n$) by setting on pure elements:

$$A(\eta_1 \wedge \dots \wedge \eta_k) := (A\eta_1) \wedge \dots \wedge (A\eta_k),$$

where $\eta_1, \dots, \eta_k \in \mathbb{C}^n$. Show that:

$$\det(\mathbb{1} - A) = \sum_{k=1}^n (-1)^k \operatorname{Tr}(\Lambda^k A).$$

Exercise 2: Flat trace of Morse diffeomorphisms. Let M be a smooth closed manifold and $\Psi : M \rightarrow M$ be a smooth diffeomorphism with non-degenerate fixed points. By this, we mean that for every fixed point x_* of Ψ (i.e. such that $\Psi(x_*) = x_*$), $d\Psi(x_*) - \mathbb{1}$ is invertible. The operator Ψ induces a map $\Psi^* : C^\infty(M) \rightarrow C^\infty(M)$ whose Schwartz kernel is denoted by K .

- (1) Show that, under these assumptions, the number of fixed points is finite.
- (2) Show that $\Psi^* : \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ is bounded.
- (3) What is $\operatorname{supp}(K)$? What is $\operatorname{WF}(K)$?
- (4) Show that $\operatorname{WF}(K) \cap N^*\Delta = \emptyset$. Deduce that $\operatorname{Tr}^b(\Psi^*)$ is well-defined.

We now want to compute $\operatorname{Tr}^b(\Psi^*)$. For that, let $X, Y \subset \mathbb{R}^n$ be open sets containing 0 and let $\psi : X \rightarrow Y$ be a smooth diffeomorphism such that $\psi(0) = 0$, $d\psi(0) - \mathbb{1}$ is invertible and 0 is the only fixed point in X . Let $\chi : \mathbb{R}^n \rightarrow [0, 1]$ be a smooth nonnegative cutoff function with support near 0, such that

$$\int_{\mathbb{R}^n} \chi = 1.$$

We define $\tilde{\chi}_\varepsilon := \varepsilon^{-n} \chi(\bullet/\varepsilon)$ and $\chi_\varepsilon \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\chi_\varepsilon(x, y) := \tilde{\chi}_\varepsilon(x) \tilde{\chi}_\varepsilon(y).$$

We let $K_\psi \in \mathcal{D}'(Y \times X)$ be the Schwartz kernel of ψ^* and let $\iota : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal embedding $x \mapsto (x, x)$. We define

$$K_\varepsilon := \chi_\varepsilon \star K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n),$$

where \star is the usual convolution product. Eventually, we consider $\varphi \in C_{\text{comp}}^\infty(\mathbb{R}^n, [0, 1])$, a smooth cutoff function with support near 0.

- (5) Why is $\text{Tr}^b(\varphi \psi^* \varphi)$ well-defined if the support of φ is close enough to 0?
- (6) Show that $\varphi K_\varepsilon \varphi \rightarrow \varphi K_\psi \varphi$ in $\mathcal{D}'_\Gamma(X \times X)$ as $\varepsilon \rightarrow 0$, where Γ is some well-chosen closed conic subset of $T^*(X \times X)$ that you will introduce.
- (7) Show that $\iota^* : C^\infty(X \times X) \rightarrow C^\infty(X)$ extends uniquely to a continuous map $\iota^* : \mathcal{D}'_\Gamma(X \times X) \rightarrow \mathcal{D}'(X)$.
- (8) Compute $\iota^*(\varphi K_\varepsilon \varphi)$.
- (9) Show that

$$\text{Tr}^b(\varphi \psi^* \varphi) = |\det(d\psi(0) - \mathbb{1})|^{-1}.$$

- (10) Deduce the value of $\text{Tr}^b(\Psi^*)$.

Exercise 3: Flat trace and fixed points. The action of Ψ on functions/distributions by pullback can be naturally extended to k -forms. More precisely, if $f \in C^\infty(M, \Lambda^k T^* M)$, $x \in M$, $v_1, \dots, v_k \in T_x M$, then we can define:

$$[\Psi_{(k)}^* f]_x(v_1, \dots, v_k) := f_{\Psi(x)}(d\Psi(x)(v_1), \dots, d\Psi(x)(v_k)).$$

Similarly to the pullback of functions, this action naturally extends by continuity as a map

$$\Psi_{(k)}^* : \mathcal{D}'(M, \Lambda^k T^* M) \rightarrow \mathcal{D}'(M, \Lambda^k T^* M).$$

We let $K_{(k)}$ be the Schwartz kernel of the operator acting on these bundles.

- (1) Given $k = 0, \dots, n$, what is $\text{supp}(K_{(k)})$? $\text{WF}(K_{(k)})$?
- (2) Show that the flat trace $\text{Tr}^b(\Psi_{(k)}^*)$ is well-defined and that

$$\text{Tr}^b(\Psi_{(k)}^*) = \sum_{j=1}^N \frac{\text{Tr}(\Lambda^k d\Psi(x_j)^\top)}{|\det(\mathbb{1} - d\Psi(x_j))|},$$

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where x_1, \dots, x_N are the fixed points and $\Lambda^k d\Psi(x_j)^\top$ denotes the linear operator induced by $d\Psi(x_j) : T_{x_j}M \rightarrow T_{x_j}M$ on $\Lambda^k T_{x_j}^*M$.

(3) Let

$$\Lambda^* T^* M := \bigoplus_{k=0}^n \Lambda^k T^* M.$$

Let $L_\Psi : C^\infty(M, \Lambda^* T^* M) \rightarrow C^\infty(M, \Lambda^* T^* M)$ be the operator defined as

$$L_\Psi \left(\sum_{k=0}^n f_k \right) = \sum_{k=0}^n (-1)^k \Psi_{(k)}^* f_k,$$

where $f_k \in C^\infty(M, \Lambda^k T^* M)$. We denote by the same letter L its continuous extension as a map

$$L_\Psi : \mathcal{D}'(M, \Lambda^* T^* M) \rightarrow \mathcal{D}'(M, \Lambda^* T^* M),$$

Show that $\text{Tr}^b(L_\Psi)$ is well-defined and that

$$\text{Tr}^b(L_\Psi) = \sum_{j=1}^N \text{sgn} \det(\mathbb{1} - d\Psi(x_j)),$$

that is the number of fixed points of Ψ counted with signs.

Exercise 4: Examples. Consider the sphere

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$

- (1) Let $R : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the antipodal map, given by $R(v) := -v$. Show that $\text{Tr}^b(L_R)$ is well-defined and compute it.
- (2) Let $N := (0, 0, 1)$ be the North pole. Consider the exponential map

$$\mathbb{S}^1 \times [0, \pi] \ni (u, r) \mapsto \exp_N(ru) \in \mathbb{S}^2,$$

where \mathbb{S}^1 is identified here with $\{v \in T_N \mathbb{S}^2 \mid |v| = 1\}$. Show that the vector field defined in coordinates by

$$X(u, r) := -r(\pi - r)\partial_r$$

is well-defined on \mathbb{S}^2 and smooth.

- (3) Let $(\varphi_t)_{t \in \mathbb{R}}$ be the flow generated by X . Describe (and draw on a picture) its flowlines.
- (4) Define $\Psi := \varphi_1$. Show that $\text{Tr}^b(L_\Psi) = 2$. What does 2 represent for the sphere?

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