Geometric inverse problems on Anosov manifolds

Thibault Lefeuvre

April 26, 2021

Abstract

We survey some recent progress on geometric inverse problems on closed Anosov manifolds i.e. compact Riemannian manifolds without boundary for which the geodesic flow is uniformly hyperbolic on the unit tangent bundle, such as negatively-curved manifolds. These geometric inverse problems include:

1. The study of the geodesic X-ray transform which consists in reconstructing a function (or a symmetric tensor) from the knowledge of its integral along closed geodesics,
2. The marked length spectrum conjecture (also known as the Burns-Katok [BK85] conjecture) and related topics which aim to investigate whether the length of closed geodesics (marked by the free homotopy of the manifold) of a Riemannian space determines the metric,
3. The holonomy inverse problem, which investigates whether the holonomy of a connection along closed geodesics determine the connection.

All these questions bring together various fields such as Riemannian geometry, uniformly hyperbolic (or Anosov) dynamical systems, Pollicott-Ruelle theory of resonances and microlocal/semiclassical analysis, and borrow the most recent technologies of these fields. The main ideas of proofs are given and the technical tools are presented in order to make the exposition self-contained.

Contents

1 Introduction

1.1 Historical background

Geometric inverse problems have a long history, going back maybe to the seminal paper of Kac *"Can one hear the shape of a drum"* [Kac66], where the following question is asked: does the spectrum of the Laplacian Δ_g on a smooth Riemannian manifold (M, g) determine its Riemannian[1] structure? Shortly after (and even before!), this question was answered in a negative way as Milnor [Mil64] exhibited pairs of isospectral tori which are not isometric. Nevertheless, it was conjectured for a long time that in negative curvature, due to the chaotic properties of the geodesic flow, the Laplace spectrum should be sufficient to determine the manifold. It was actually proved to be false, as Vigneras [Vig80] exhibited pairs of hyperbolic surfaces that are isospectral but not isometric. Even in the plane, isospectral non-isometric domains were found [GWW92] but these are piecewise smooth and is not known yet if such a spectral rigidity result holds for smooth domains. Nevertheless, spectral rigidity holds for disks and ellipses of small eccentricity by a recent result [HZ19]. We also refer to the survey of Zelditch [Zel04] for further discussions on inverse spectral problems.

Building on these spectral considerations, the general philosophy behind geometric inverse problems is to recover a complete geometric data (such as a metric, a connection, a potential, a Higgs field, ...) from the knowledge of certain partial quantities (also called measurements) such as the Laplace spectrum, the length of closed geodesics, the holonomy of connections along closed orbits, etc. There are many geometric contexts in which one can phrase such problems and this survey is focused on closed Anosov manifolds: these are compact Riemannian manifolds without boundary on which the geodesic flow (on the unit tangent bundle) is *uniformly hyperbolic* (also called *Anosov* in the literature), see (2.2). A typical (and historical!) example is provided by manifolds with negative sectional curvature [Ano67]. These flows have very chaotic properties such as a strong sensitivity to initial conditions. Moreover, closed geodesics (which correspond to closed orbits of the geodesic flow) are dense and one can legitimately expect these to carry important dynamical information. In the inverse problems studied in this survey, the question will be to understand to what extent one can recover a global geometric object (like a metric) from partial information carried by closed geodesics (their length, for example).

This topic has enjoyed considerable progress in the past forty years. In the early 80's, Guillemin-Kazhdan [GK80a, GK80b] initiated a new turn in the field by showing the *infinitesimal spectral rigidity* of negatively-curved surfaces: given a Riemannian manifold (M, g), it is said to be infinitesimally spectrally rigid if any isospectral deformation

[1] Kac's discussion was for smooth domains of $\mathbb{R}^n$ but it is rather natural to formulate it for a general smooth Riemannian manifold.

$(g_\lambda)_{\lambda\in(-1,1)}$ of the metric (such that $g_0 = g$), i.e. such that $\mathrm{spec}(\Delta_{g_\lambda}) = \mathrm{spec}(\Delta_g)$, is isometric to g in the sense that there exists an isotopy $(\phi_\lambda)_{\lambda\in(-1,1)}$ such that $\phi_\lambda^* g_\lambda = g$. The proof (which will be given in §6.2) is somehow exemplary of what we will be concerned about in this survey as it combines elements from three distinct (but of course related!) fields of modern mathematics: microlocal analysis, with the use of the Duistermaat-Guillemin trace formula [DG75] for elliptic operators; hyperbolic dynamical systems, with the use of the Livšic Theorem [Liv72]; and Riemannian geometry, with the use of a crucial energy identity based on the Riemannian structure and called the Pestov identity. This trichotomy will also appear at various places throughout the manuscript.

Following the Guillemin-Kazhdan [GK80a, GK80b] approach, we will study the following questions:

1. Does the integral of a function (or a symmetric tensor field of order $m \in \mathbb{N}$) along closed geodesics determine the function? The underlying operator of integration of symmetric m-tensors along closed geodesics is called *the geodesic X-ray transform*, denoted by I_m, and plays a crucial role in several problems. A vast literature has been devoted to this question in the past twenty years [GK80a, GK80b, CS98, DS03, PSU14, PSU15, Gui17a].

2. Does the length of closed geodesics (marked by the free homotopy of the manifold) determines the Riemannian structure of (M, g)? This is known as the marked length spectrum conjecture (or the Burns-Katok [BK85] conjecture). Despite some partial answers in the 90s [Kat88, BCG95, Ham99] and the proof of the conjecture for negatively-curved surfaces [Ota90, Cro90], this question is still wide open. We will see that the differential of the marked length spectrum is precisely (one half of) the geodesic X-ray transform of symmetric 2-tensors I_2. As a consequence, proving important analytic properties (such as stability estimates) on the X-ray transform will yield interesting (local) *rigidity results* on the marked length spectrum. This analytic approach was recently developed in [GL19d, GKL19, GL19b, GL19c].

3. Given a vector bundle $\mathcal{E} \to M$ equipped with a unitary connection $\nabla^{\mathcal{E}}$ (not necessarily flat!), does the holonomy of the connection along closed geodesics determine the connection? This question reveals unexpected difficulties, especially when $\mathcal{E}$ is not a line bundle or M is not two-dimensional. In the particular case of line bundles, we will see that it is connected to the injectivity (modulo a natural kernel) of the X-ray transform on 1-forms I_1. This question, although less renowned (all the references are essentially contained in the list [Pat09, Pat11, Pat12, Pat13, GPSU16, CL20]), turns out to be as interesting (if not more!) as the two previous.

The study of geometric inverse problems (on closed Anosov manifolds) can now benefit from the recent development of the theory of Pollicott-Ruelle resonances for uniformly hyperbolic flows. This field, going back to earlier work by Ruelle in the 70's, has led to a considerable amount of work over the past twenty years [Liv04, BL07, FS11, FT13, NZ15, DZ16, FT17, GW17, Jé19, GGHW20, TZ20] and is now well-understood: the main idea is to describe the long-time statistical behaviour of a given dynamical system T defined on the phase space X thanks to the spectrum (when it can be defined) of its (unweighted) *transfer operator*, namely:

$$\mathcal{L} : C^0(X) \to C^0(X), \quad \mathcal{L}f(x) = \sum_{Ty=x} f(y).$$

In order to obtain a true spectrum in $\mathbb{C}$, one usually needs to twist the space and work with other regularities than continuous functions $C^0(X)$. We refer to the recent book [Bal18] for further details on this approach. When the system is a uniformly hyperbolic (also called Anosov) flow, as defined in (2.2), the modern way to do this is to work with a scale of *anisotropic Hilbert* (or Banach) *spaces* which are spaces of distributions with low regularity in the expanding direction and high regularity in the contracting direction. The spaces constructed are usually non canonical but the spectrum defined *is* and is called the set of *Pollicott-Ruelle resonances*.

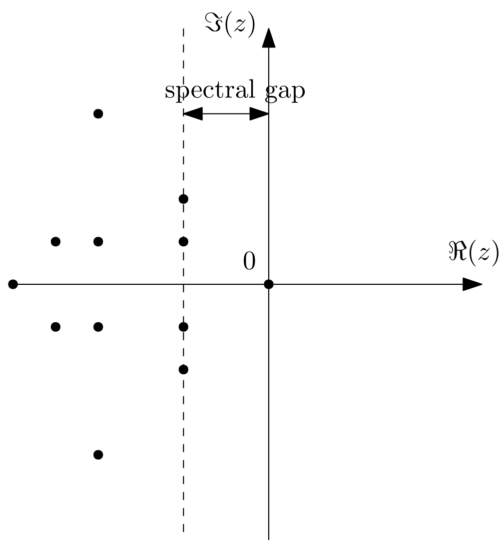


Figure 1: Pollicott-Ruelle resonances of the geodesic flow. The existence of a spectral gap implies that the flow is exponentially mixing with respect to the Liouville measure.

We will apply this toolbox to the specific case of an Anosov geodesic flow, hence the study of *Anosov manifolds*. This situation occurs as long as the Riemannian manifold (M, g) exhibits "enough" areas of negative sectional curvature (see [Ebe73] for further details). As the geodesic flow preserves a canonical contact structure (the *Liouville one-form*), finer microlocal properties can proved. For instance, the Pollicott-Ruelle resonances are all located in a half-space $\{\Re(z) < -\delta\}$ (except 0) which is called a *spectral gap* and implies that the flow is *exponentially mixing* with respect to the Liouville measure [Liv04], see (3.3). They also enjoy the additional (and remarkable!) property to be concentrated in strips [FT13]. As we will see, this microlocal perspective on the dynamical properties of the geodesic flow plays a crucial role in our study, as it allows to describe in a very accurate way the *wavefront set* (namely the singularities) of some important integral operators Π_m acting on symmetric m-tensors, called *generalized X-ray transforms*, which will replace at some point the classical X-ray transforms I_m.

We also point out that much of this theory can be phrased on manifolds with boundary and has also attracted considerable attention. The natural setting is that of manifolds with strictly convex boundary, absence of conjugate points and a hyperbolic trapped set, see [Gui17b, GM18, Lef19, Lef18]. In an even simpler setting, one analogous problem to the Burns-Katok conjecture is Michel's conjecture [Mic82] on simple Riemannian manifolds (topological balls with strictly convex boundary and no conjugate points): it asserts that the *boundary distance function*, namely the Riemannian distance between each pair of points on the boundary, determines the Riemannian structure of the manifold. Partial attempts to solve this conjecture can be found in [Gro83, BCG95, CDS00, SU04, BI10]. Although major breakthroughs have been achieved in the past fifteen years [PU05, UV16, SUV17], it is still open at the moment. It turns out that some recent work [CEG20] has shown that Michel's conjecture would actually be obtained as a corollary of the Burns-Katok conjecture if it were to be proved.

Lastly, let us point out that, although we will adopt a more concise way of writing (especially by avoiding the use of expressions in local coordinates), much of the basic tools of geometric inverse problems (such as symmetric tensor analysis) were already developed in Sharafutdinov's book [Sha94] on integral geometry. Later, Merry-Paternain [MP] published very detailed and accessible lecture notes (with an emphasis on surfaces) to the field which might be useful for the reader to get familiar with elementary notions. A good and broad account can also be found in the survey [IM18]. Eventually, the book [PSU21] which should be available soon will provide a complete introduction to the field in the two-dimensional case. The emphasis of the current survey is on the most recent developments of the field, namely the recurrent use of microlocal analysis, especially through the use of techniques from Pollicott-Ruelle theory. We also avoid to discuss the case $n = 2$ (unless explicitly mentioned), where further results can be obtained.

1.2 Organization of the paper

Part I introduces many notions and standard results of Riemannian geometry, hyperbolic dynamical systems and microlocal analysis, which will be heavily used in Part II. In Section §2, we recall some elements of Riemannian geometry, in particular the horizontal and vertical differentials and discuss the case where the tangent bundle TM is twisted by a Hermitian vector bundle $\mathcal{E}$. We also introduce the notion of *Anosov Riemannian manifolds* and discuss some of their basic properties. In §2.4, we introduce tensor analysis on Riemannian manifolds and explain the links with Fourier analysis in the fibers of the unit tangent bundle SM. In Section §3, we introduce the microlocal framework allowing to study Anosov dynamics from a spectral point of view. In particular, we define the notion of *Pollicott-Ruelle resonances*. For readers who are not familiar with microlocal calculus, we detailed some of the main results involving pseudodifferential operators that are used throughout the manuscript in an Appendix A. Eventually, in Section §4, we explain the Livšic theory of hyperbolic dynamical systems and discuss both some classical and new results in the light of the microlocal framework of Section §3.

In a second Part II, we study the so-called geometric inverse problems in the context of closed Anosov Riemannian manifolds. In Section §5, we study the geodesic X-ray transform from two perspectives: first of all, from a Riemannian viewpoint by means of an L^2-energy identity called the *Pestov identity*; second, from a more modern approach using pseudodifferential operators and Pollicott-Ruelle resonances. In Section §6, we introduce the notion of *marked length spectrum* (i.e. the length of closed geodesics marked by the free homotopy of the manifold) and state the Burns-Katok conjecture; we also present some partial results towards its resolution. Section §7 is devoted to the study of the holonomy problem. In Section §8, we sum up all the open questions.

1.3 Acknowledgement

I warmly thank friends, collaborators and colleagues for several discussions over the past three years which gave birth to this manuscript: Viviane Baladi, Yann Chaubet, Nguyen Viet Dang, Frédéric Faure, Hugo Federico, Livio Flaminio, Sébastien Gouëzel, Malo Jézéquel, Gerhard Knieper, Benjamin Küster, Stéphane Nonnenmacher, Gabriel Rivière, Mikko Salo, Gunther Uhlmann, Andras Vasy, Maciej Zworski. I am particularly grateful to Gabriel Paternain who read an earlier version of this manuscript and made several useful comments. I am also grateful to Yannick Guedes Bonthonneau, Mihajlo Cekić and Colin Guillarmou for extensive discussions over the years which shaped my vision of these problems.

Part I
Preliminary tools

2 Elements of Riemannian geometry

In this first section, we recall some standards elements of Riemannian geometry. We refer to [Pat99] for further details, especially on the geodesic dynamics. We also refer to [PSU15, GPSU16] for the details of the computations.

2.1 Horizontal and vertical differentials

Let (M, g) be a smooth Riemannian manifold of arbitrary dimension $n \geq 2$. Denote by

$$SM = \{(x, v) \in TM \mid |v|_g = 1\},$$

its unit tangent bundle, $\pi : SM \to M$ the projection on the base and ∇ the Levi-Civita connection. The bundle $SM \to M$ is a sphere bundle over M but in general it is not trivial. There is a canonical splitting of the tangent bundle to SM as:

$$T(SM) = \mathbb{H} \oplus \mathbb{V} \oplus \mathbb{R}X,$$

where X is the geodesic vector field, $\mathbb{V} := \ker d\pi$ is the vertical space and $\mathbb{H}$ is the horizontal space[2] defined in the following way. Consider $\mathcal{K} : T(SM) \to TM$; the *connection map* defined as follows: consider $(x, v) \in SM, w \in T_{(x,v)}(SM)$ and a curve $(-\varepsilon, \varepsilon) \ni t \mapsto z(t) \in SM$ such that $z(0) = (x, v), \dot{z}(0) = w$; write $z(t) = (x(t), v(t))$; then $\mathcal{K}_{(x,v)}(w) := \nabla_{\dot{x}(t)} v(t)|_{t=0}$. We denote by g_{Sas} the Sasaki metric on SM, which is the canonical metric on the unit tangent bundle, defined by:

$$g_{\text{Sas}}(w, w') := g(d\pi(w), d\pi(w')) + g(\mathcal{K}(w), \mathcal{K}(w')).$$

Any vector $w \in T(SM)$ can be decomposed according to the splitting

$$w = \alpha(w)X + w_{\mathbb{H}} + w_{\mathbb{V}},$$

where α is the Liouville 1-form[3], $w_{\mathbb{H}} \in \mathbb{H}, w_{\mathbb{V}} \in \mathbb{V}$. The Liouville 1-form is a contact one-form given by $\alpha(w) = g_{\text{Sas}}(X, w)$. It induces a volume form $\alpha \wedge (d\alpha)^{n-1}$ which is called the

[2] We use the convention that $\mathbb{H} := (\mathbb{V} \oplus \mathbb{R}X)^\perp$, and not $\mathbb{V}^\perp$ as usual. In particular, if M is n-dimensional, then $\mathbb{H}$ is $(n-1)$-dimensional.

[3] Also called the contact 1-form. It satisfies $\imath_X \alpha = 1, \imath_X d\alpha = 0$.

Liouville measure (by abuse of notations, the density is identified with the volume form). If $f \in C^\infty(SM)$, its gradient computed with respect to the Sasaki metric can be written as:

$$\nabla_{\mathrm{Sas}} f = (Xf)X + \widetilde{\nabla_{\mathbb{H}}} f + \widetilde{\nabla_{\mathbb{V}}} f,$$

where $\widetilde{\nabla_{\mathbb{H}}} f \in \mathbb{H}$ is the horizontal gradient, $\widetilde{\nabla_{\mathbb{V}}} f \in \mathbb{V}$ is the vertical gradient.

We then consider the vector bundle $\mathcal{N} \to SM$ whose fiber $\mathcal{N}(x,v)$ over $(x,v) \in SM$ is given by $\{v\}^\perp$. For every $(x,v) \in SM$, the maps

$$(\mathbb{H}(x,v), g_{\mathrm{Sas}}) \xrightarrow{\mathrm{d}\pi} (\mathcal{N}(x,v), g), \qquad (\mathbb{V}(x,v), g_{\mathrm{Sas}}) \xrightarrow{\mathcal{K}} (\mathcal{N}(x,v), g)$$

are isometries. These isomorphisms allow to decompose $\mathbb{H} \oplus \mathbb{V} \simeq \mathcal{N} \oplus \mathcal{N}$ by considering the isometry

$$\mathbb{H} \oplus \mathbb{V} \to \mathcal{N} \oplus \mathcal{N}, \quad w \to (\mathrm{d}\pi(w), \mathcal{K}(w)).$$

As a consequence, $\mathbb{H}$ can be identified with $\{(w,0), w \in \mathcal{N}\} \subset \mathcal{N} \oplus \mathcal{N}$, and $\mathbb{V}$ with $\{(0,w), w \in \mathcal{N}\}$. In particular, the operator $\widetilde{\nabla_{\mathbb{H}}}, \widetilde{\nabla_{\mathbb{V}}}$ can be seen to take values in $\mathcal{N}$ by considering $\nabla_{\mathbb{H}} := \mathrm{d}\pi \widetilde{\nabla_{\mathbb{H}}}$ and $\nabla_{\mathbb{V}} := \mathcal{K} \widetilde{\nabla_{\mathbb{V}}}$ instead, which we will do from now on.

The geodesic vector field seen as a differential operator of order 1 induces a differential operator (still denoted by X) $X : C^\infty(SM, \mathcal{N}) \to C^\infty(SM, \mathcal{N})$ defined in the following way: consider a section $w \in C^\infty(SM, \mathcal{N})$, a point (x,v) and denote by $t \mapsto \gamma_{(x,v)}(t) \in M$ the geodesic it generates. Then $t \mapsto (\gamma_{(x,v)}(t), w(t))$ is a well-defined vector field along the geodesic (which is everywhere orthogonal to the direction of the geodesic) and we can consider its covariant derivative

$$\left.\frac{Dw(t)}{dt}\right|_{t=0} =: Xw(x,v).$$

Note that it is a well-defined section of $\mathcal{N}$, i.e. it is everywhere orthogonal to v as the covariant derivative preserves this property. The *propagator* $R(t) : C^\infty(SM, \mathcal{N}) \to C^\infty(SM, \mathcal{N})$ of the operator X is defined to be the (unique) solution of the operator-valued ODE:

$$\dot{R}(t) = -XR(t), \qquad R(0) = \mathbb{1}.$$

It easy to check that given $f \in C^\infty(SM, \mathcal{N})$ and $(x,v) \in SM$, $(R(t)f)(x,v)$ is the parallel transport of the vector $f(\varphi_{-t}(x,v))$ along the geodesic segment $[0,t] \ni s \mapsto \pi(\varphi_{-s}(x,v))$. In particular, this propagator satisfies the obvious bound $\|R(t)\|_{L^2(SM,\mathcal{N}) \to L^2(SM,\mathcal{N})} \leq 1$.

Moreover, X also induces an operator on $C^\infty(SM, \mathrm{End}(\mathcal{N}))$ (once again, still denoted by X) by requiring the following Leibniz rule to be satisfied: for all $w \in C^\infty(SM, \mathcal{N}), U \in$

$C^\infty(SM, \mathrm{End}(\mathcal{N}))$:

$$X(U \cdot w)(x,v) = (XU)(x,v) \cdot w(x,v) + U(x,v) \cdot (Xw(x,v)).$$

The Riemann curvature tensor $\mathbf{R}$ defined as usual for $X, Y \in C^\infty(M, TM)$ by

$$\mathbf{R}(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_Y - \nabla_{[X,Y]}$$

induces a symmetric section $R \in C^\infty(SM, \mathrm{End}(\mathcal{N}))$ defined by

$$\forall (x,v) \in SM, w \in \mathcal{N}(x,v), \qquad R(x,v) \cdot w := \mathbf{R}_x(w,v)v. \tag{2.1}$$

The operators previously introduced satisfy commutation formulas (see also [PSU15, Lemma 2.1]):

Lemma 2.1.

$$[X, \nabla_{\mathbb{V}}] = -\nabla_{\mathbb{H}}, \qquad [X, \nabla_{\mathbb{H}}] = R\nabla_{\mathbb{V}}.$$

The adjoint operators to $\nabla_{\mathbb{V},\mathbb{H}}$ are the respective horizontal and vertical divergence i.e. $\nabla^*_{\mathbb{V},\mathbb{H}} = -\operatorname{div}_{\mathbb{V},\mathbb{H}}$. These operators satisfy the commutation formula (see [PSU15, Lemma 2.1] for instance):

$$\operatorname{div}_{\mathbb{H}} \ \nabla_{\mathbb{V}} - \operatorname{div}_{\mathbb{V}} \ \nabla_{\mathbb{H}} = (n-1)X, \quad [X, \operatorname{div}_{\mathbb{V}}] = -\operatorname{div}_{\mathbb{H}}, \quad [X, \operatorname{div}_{\mathbb{H}}] = -\operatorname{div}_{\mathbb{V}} R.$$

These commutation relations will be used in Section §5 in order to derive the *Pestov identity* (see Lemma 5.7).

Exercise 2.2. Prove the commutation formula for the divergence.

2.2 Anosov Riemannian manifolds

We write $\mathcal{M} := SM$ for the sake of simplicity. We say that the Riemannian manifold (M,g) is *Anosov* if there there exists a continuous flow-invariant splitting of $T\mathcal{M}$ such that:

$$T\mathcal{M} = \mathbb{R} \cdot X \oplus E_s \oplus E_u,$$

where X is the geodesic vector field, E_s and E_u are the stable and unstable vector bundles such that:

$$\begin{aligned} &\forall t \geq 0, \forall w \in E_s, \quad |\mathrm{d}\varphi_t(w)| \leq Ce^{-t\lambda}|w|, \\ &\forall t \leq 0, \forall w \in E_u, \quad |\mathrm{d}\varphi_t(w)| \leq Ce^{-|t|\lambda}|w|, \end{aligned} \tag{2.2}$$

where the constants $C, \lambda > 0$ are uniform and the metric inducing the norm $|\cdot|$ is arbitrary. Moreover, it can be shown that

$$\mathbb{H} \oplus \mathbb{V} = E_s \oplus E_u = \ker \alpha,$$

where we recall that α is the contact 1-form. Examples of Anosov manifolds are provided by manifolds with negative sectional curvature [Ano67].

It is well-known that the identification of $\mathbb{H}$ and $\mathbb{V}$ with $\mathcal{N}$ allows to describe in a nice fashion the differential of the geodesic flow via solutions to the Jacobi equations. More precisely, following the previous paragraph, given $(x, v) \in SM$ and $w \in E_s(x, v) \oplus E_u(x, v)$, we can write $d\varphi_t(w) = (w_{\mathbb{H}}(t), w_{\mathbb{V}}(t))$, where $w_{\mathbb{H},\mathbb{V}}(t) \in \mathcal{N}(\varphi_t(x, v))$. We introduce the *Jacobi equation*

$$\ddot{J}(t) + R(\varphi_t(x, v))J(t) = 0,$$

where $J(t) \in \mathcal{N}(\varphi_t(x, v))$ and R is the operator introduced in (2.1), with initial conditions $J(0) = w_{\mathbb{H}} = w_{\mathbb{H}}(0)$ and $\frac{D}{Dt}J(0) = w_{\mathbb{V}} = w_{\mathbb{V}}(0)$. We have:

$$w_{\mathbb{H}}(t) = J(t), \qquad w_{\mathbb{V}}(t) = \frac{D}{Dt}J(t).$$

Using the standard Rauch lemma for matrix ODEs (see [Kni02, Proposition 2.18]), it is easy to show that the geodesic flow in negative curvature is Anosov [Ano67] (i.e. when the matrix-valued symmetric operator $R \in C^\infty(M, \mathrm{End}(\mathcal{N}))$ satisfies the bounds $-\alpha^2 \leq R \leq -\beta^2 < 0$).

Using the identification with $\mathcal{N} \oplus \mathcal{N}$ one can prove (see [Kni02, pp. 472-473] for instance) the following: there exists $\alpha > 0$ and symmetric operators $U_\pm \in C^\alpha(SM, \mathrm{End}(\mathcal{N}))$ such that for all $(x, v) \in SM$,

$$E_s(x, v) \simeq \{(w, U_+(x, v)w) \mid w \in \mathcal{N}(x, v)\}, E_u(x, v) \simeq \{(w, U_-(x, v)w) \mid w \in \mathcal{N}(x, v)\}.$$

We will write

$$\theta_\pm(x, v) : \mathcal{N}(x, v) \to E_{s/u}(x, v), \quad w \mapsto \theta_\pm(x, v) \cdot w = (w, U_\pm(x, v) \cdot w)$$

The endomorphisms $U_\pm$ are actually differentiable in the flow direction, bounded on $\mathcal{M}$ and solutions to the *Ricatti equation*, namely:

$$XU_\pm + U_\pm^2 + R = 0. \tag{2.3}$$

The satisfy that $U_- - U_+ > 0$ (i.e. it is a symmetric definite positive endomorphism on $\mathcal{M}$).

We now make some further observations on these endomorphisms which will be needed at some stage (in Lemma 5.8). Consider a point $(x, v) \in SM$ and $w \in \mathcal{N}(x, v)$, and write $Z := (w, U_+(x, v)w) \in E_s(x, v)$. We can then write, using the Jacobi vector fields, $\mathrm{d}\varphi_t(Z) = (J(t), \dot{J}(t))$ and since $\mathrm{d}\varphi_t(Z)$ belongs to the stable bundle (which is invariant under the flow), one has

$$\dot{J}(t) = U_+(\varphi_t(x, v))J(t). \tag{2.4}$$

We now consider an orthonormal frame $(E_1(0), ..., E_{n-1}(0))$ of $\{v\}^\perp$ and parallel-transport it along the geodesic $t \mapsto \pi(\varphi_t(x, v))$. We can decompose the Jacobi vector fields as $J(t) = \sum_{i=1}^{n-1} y_i(t)E_i(t)$, where $y_i \in C^\infty(\mathbb{R})$ are smooth functions. Consider $\mathbb{R}^{n-1}$ endowed with its Euclidean structure and denote by $(\mathbf{e}_1, ..., \mathbf{e}_{n-1})$ an orthonormal basis. If we introduce the identification $\rho(t) : \mathbb{R}^{n-1} \to \mathcal{N}(\varphi_t(x, v))$, defined by $\rho(t)\mathbf{e}_i := E_i(t)$, then using that the $E_i(t)$ are parallel transported, we can rewrite (2.4) as:

$$\dot{Y}(t) = U_+(t)Y(t),$$

where $Y(t)^\top = (y_1(t), ..., y_{n-1}(t)) \in \mathbb{R}^{n-1}$ and $U_+(t) := U_+(\varphi_t(x, v))$ is seen as an endomorphism of $\mathbb{R}^{n-1}$. Let $\Phi(t)$ be the resolvent of this equation, i.e. such that $Y(t) = \Phi(t)Y(0)$[4]. In other words, we have: $J(t) = \rho(t)\Phi(t)\rho(0)^{-1}J(0)$. The exponential decay (2.2) then implies that for all $t \geq 0$:

$$\|\Phi(t)\| \leq Ce^{-t\lambda}.$$

One way of rewriting this is the following:

Lemma 2.3. *Consider the propagator $R_{U_+}(t) : C^\infty(SM, \mathcal{N}) \to C^\infty(SM, \mathcal{N})$ defined by:*

$$\dot{R}_{U_+}(t) = (-X + U_+)R_{U_+}(t), \qquad R_{U_+}(0) = \mathbb{1}.$$

Then, there exists $C, \lambda > 0$ such that for all $t \geq 0$:

$$\|R_{U_+}(t)\|_{L^2(SM,\mathcal{N}) \to L^2(SM,\mathcal{N})} \leq Ce^{-\lambda t}.$$

Eventually, let us recall the notion of *conjugate points*:

Definition 2.4. Let $y := \pi(\varphi_t(x, v))$. We say that x and y are *conjugate points* if $\mathrm{d}(\varphi_t)_{(x,v)}(\mathbb{V}) \cap \mathbb{V} \neq \{0\}$. We say that (M, g) has no conjugate points if $\mathrm{d}\varphi_t(\mathbb{V}) \cap \mathbb{V} = \{0\}$ for all $t \in \mathbb{R} \setminus \{0\}$.

The Anosov property has very strong implications on the geometry. In particular, it

[4]Informally, we like to think of it as $\Phi(t)$ " = " $\exp\left(\int_0^t U_+(\varphi_s(x, v))\mathrm{d}s\right)$, just as in the scalar case. Of course, this is completely wrong for matrices. Nevertheless, in the case of a surface, the endomorphisms $U_\pm$ are simply functions $r_\pm$ called the *Ricatti functions* and this is indeed a true equality.

allows to show that

$$E_s \cap \mathbb{V} = E_u \cap \mathbb{V} = \{0\}, \tag{2.5}$$

which is one of the most (if not *the* most) important property of Anosov manifolds. This was originally proved by Klingenberg [Kli74]. Later on, Mañé proved that (2.5) holds under the sole assumption that there exists a Lagrangian fibration[5] in the kernel of the Liouville 1-form α (which is satisfied in particular by Anosov manifolds). In particular, (2.5) prevents the existence of conjugate points.

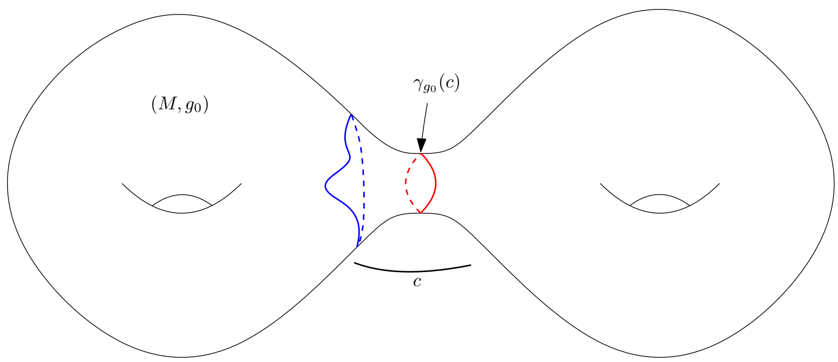


Figure 2: In red, the unique closed geodesic of the free homotopy class $c \in \mathcal{C}$.

We introduce $\mathcal{C}$, the set of *free homotopy classes on the manifold* M; it is well-known that this set is in one-to-one correspondance with conjugacy classes of the fundamental group $\pi_1(M)$. We will use the following:

Lemma 2.5. *For each free homotopy class $c \in \mathcal{C}$, there exists a unique closed geodesic $\gamma_g(c) \in \mathcal{C}$.*

Elements of proofs can be found in the survey of Knieper [Kni02]. In the following, we will consider geometric inverse problems related to *marked quantities* which means that we will be given some data indexed by the set of free homotopy classes.

2.3 Connections on vector bundles

2.3.1 Definition and elementary properties

Let $\mathcal{E} \to M$ be a smooth Hermitian vector bundle. Recall that an affine connection on $\mathcal{E}$ is a first-order linear differential operator acting as

$$\nabla^{\mathcal{E}} : C^\infty(M, \mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E}),$$

[5] The 2-form $d\alpha$ restricted to the vector bundle $E_s \oplus E_u \to SM$ is non-degenerate. In the Anosov case, the vector bundles E_s and E_u are Lagrangian in the sense that $d\alpha|_{E_{s,u}} = 0$.

with the following properties: for all $f \in C^\infty(M), X \in C^\infty(M, TM), u \in C^\infty(M, \mathcal{E})$,

$$\nabla^{\mathcal{E}}_{fX}(u) = f\nabla^{\mathcal{E}}_X u, \qquad \nabla^{\mathcal{E}}_X(fu) = df \otimes u + f\nabla^{\mathcal{E}}_X u.$$

Given a open subset $U \subset M$ and local coordinates $\varphi : U \to V := \varphi(U) \subset \mathbb{R}^n$, the connection can be written in this chart as $\nabla^{\mathcal{E}} = d + \Gamma$, where $\Gamma \in C^\infty(V, T^*V \otimes \mathrm{End}(\mathbb{C}^r))$. We will (improperly) call Γ a *connection* 1-form.

Let $h^{\mathcal{E}}$ be the Hermitian metric in the fibers of $\mathcal{E}$. The connection is said to be *unitary* if it distributes in the following way: for all $X \in C^\infty(M, TM)$ and $u_1, u_2 \in C^\infty(M, \mathcal{E})$,

$$X \cdot \left(h^{\mathcal{E}}(u_1, u_2)\right) = h^{\mathcal{E}}\left(\nabla_X u_1, u_2\right) + h^{\mathcal{E}}\left(u_1, \nabla_X u_2\right). \tag{2.6}$$

In local coordinates, the connection 1-form Γ then takes value in skew-hermitian matrices.

Recall that the curvature tensor $\mathbf{F}_{\nabla^{\mathcal{E}}} \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}(\mathcal{E}))$ is defined for any vector fields $X, Y \in C^\infty(M, TM)$ and $u \in C^\infty(M, \mathcal{E})$ as:

$$\mathbf{F}_{\nabla^{\mathcal{E}}}(X, Y)u = \nabla^{\mathcal{E}}_X \nabla^{\mathcal{E}}_Y u - \nabla^{\mathcal{E}}_Y \nabla^{\mathcal{E}}_X u - \nabla^{\mathcal{E}}_{[X,Y]} u. \tag{2.7}$$

When the connection is unitary, $\mathbf{F}_{\nabla^{\mathcal{E}}}(X, Y)$ takes values in skew-Hermitian endomorphisms.

Given a vector field $X \in C^\infty(M, TM)$ generating a flow $(\varphi_t)_{t\in\mathbb{R}}$, we will denote by C the parallel transport with respect to $\nabla^{\mathcal{E}}$ along the flowlines. More precisely,

$$C(x, t) : \mathcal{E}_x \to \mathcal{E}_{\varphi_t(x)}$$

is the parallel transport with respect to $\nabla^{\mathcal{E}}$ along the segment $[0, t] \ni s \mapsto \varphi_s(x)$.

Any connection $\nabla^{\mathcal{E}}$ induces a canonical connection on the endomorphism bundle $\mathrm{End}(\mathcal{E}) \to M$. It is defined so that it respects the Leibniz rule, namely: for all $u \in C^\infty(M, \mathrm{End}(\mathcal{E})), f \in C^\infty(M, \mathcal{E})$, one has

$$\left[\nabla^{\mathrm{End}(\mathcal{E})} u\right](f) = \nabla^{\mathcal{E}}\left[u(f)\right] - u\left(\nabla^{\mathcal{E}} f\right).$$

In local coordinates, one can easily verify that $\nabla^{\mathrm{End}(\mathcal{E})} = d + [\Gamma, \bullet]$. Moreover, if $\nabla^{\mathcal{E}}$ is unitary, then so is $\nabla^{\mathrm{End}(\mathcal{E})}$, i.e. it distributes as in (2.6) with respect to the natural metric $h^{\mathrm{End}(\mathcal{E})}$ on $\mathrm{End}(\mathcal{E})$. The latter defined as follows: for all $u_1, u_2 \in C^\infty(M, \mathrm{End}(\mathcal{E}))$,

$$h^{\mathrm{End}(\mathcal{E})}(u_1, u_2) = \mathrm{Tr}(u_1 u_2^{*_h}),$$

where ${}^{*_h} : \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E})$ denotes the linear operator of taking the adjoint (with respect to $h^{\mathcal{E}}$), namely: for all $x \in M,\ f_1, f_2 \in \mathcal{E}_x, u \in \mathrm{End}(\mathcal{E}_x)$:

$$h^{\mathcal{E}}(uf_1, f_2) = h^{\mathcal{E}}(f_1, u^{*_h} f_2).$$

Note that, in the following, we will sometimes drop the notation $h^{\mathcal{E}}$ and use $\langle \bullet, \bullet \rangle$ instead.

2.3.2 The mixed connection

We now consider two Hermitian vector bundles $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$ equipped with respective connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$; they can be written in some local patch of coordinates $V \subset \mathbb{R}^n$ as $\nabla^{\mathcal{E}_i} = d + \Gamma_i$, for some $\Gamma_i \in C^\infty(V, T^*V \otimes \mathrm{End}(\mathcal{E}_i))$. Let $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ be the vector bundle of homomorphisms from $\mathcal{E}_1$ to $\mathcal{E}_2$. Observe that $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \simeq \mathcal{E}_1 \otimes \mathcal{E}_2^*$ and the *mixed connection* is then the natural tensor connection obtained on this product. In local coordinates, it is defined in the following way:

Definition 2.6. We define the *mixed connection* ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}$ on $\mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1)$, induced by $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$, in coordinates by:

$$ {}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_1,\mathcal{E}_2)} u := du + \Gamma_1(\cdot)u - u\Gamma_2(\cdot). $$

When the connections are unitary, one can check that the mixed connection is also unitary. Even though this is not clearly indicated in the notation, we insist on the fact that the mixed connection ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}$ *depends on a choice* of connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. Note that in the particular case where $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, and we have two connections $\nabla^{\mathcal{E}}_{1,2}$ on $\mathcal{E}$, we obtain:

$$ \begin{aligned} {}^{\mathrm{M}}\nabla^{\mathrm{End}(\mathcal{E})} u &:= du + \Gamma_1(\bullet)u - u\Gamma_2(\bullet) \\ &= du + [\Gamma_1(\bullet), u] - u(\Gamma_2(\bullet) - \Gamma_1(\bullet)) = \nabla_1^{\mathrm{End}(\mathcal{E})} u - u(\Gamma_2(\bullet) - \Gamma_1(\bullet)), \end{aligned} $$

and in the very specific case where $\nabla^{\mathcal{E}}_1 = \nabla^{\mathcal{E}}_2$, we retrieve the usual connection on the endomorphism bundle. We will denote by $P(x,t) : \mathrm{End}(\mathcal{E}_x) \to \mathrm{End}(\mathcal{E}_{\varphi_t} x)$ the parallel transport with respect to the mixed connection. Observe that for $u \in \mathrm{Hom}(\mathcal{E}_{2x}, \mathcal{E}_{1x})$, we have:

$$ P(x,t)u = C_1(x,t)uC_2(x,t)^{-1}. \tag{2.8} $$

If $\mathbf{F}_{\nabla^{\mathcal{E}_i}} \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}(\mathcal{E}_i))$ denotes the curvature tensor of the connection, then a quick computation shows:

$$ \mathbf{F}_{{}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}} u = \mathbf{F}_{\nabla^{\mathcal{E}_1}} \cdot u - u \cdot \mathbf{F}_{\nabla^{\mathcal{E}_2}}. \tag{2.9} $$

Eventually, observe that if $p \in C^\infty(\mathcal{M}, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$ is an isomorphism and ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}$ is the mixed connection induced by $\nabla^{\mathcal{E}_{1,2}}$, and ${}^{\mathrm{M}}\nabla^{\mathrm{End}(\mathcal{E}_1)}$ is the mixed connection induced by $\nabla^{\mathcal{E}_1}$ and $p_*\nabla^{\mathcal{E}_2}$, then:

$$ {}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}(up) = \left({}^{\mathrm{M}}\nabla^{\mathrm{End}(\mathcal{E}_1)} u\right) p. \tag{2.10} $$

A similar formula holds for the multiplication by p on the left.

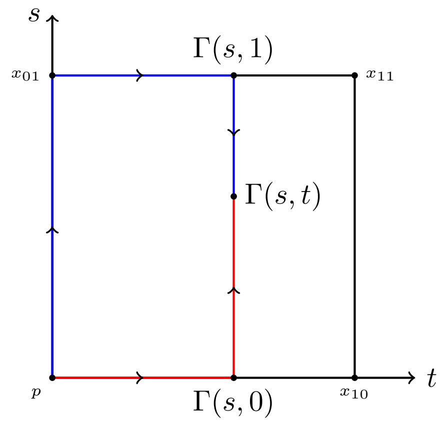


Figure 3: The homotopy Γ in Lemma 2.7 with the corresponding points in $\mathcal{M}$: in blue and red are the trajectories along which the parallel transport maps $C_\uparrow$ (vertical) and $C_\to$ (horizontal) are taken, respectively.

2.3.3 Ambrose-Singer formula

The celebrated Ambrose-Singer formula (see eg. [KN69, Theorem 8.1]) relates the holonomy of a connection with the curvature; more precisely, it asserts that along a closed (homotopically trivial) loop, the holonomy is equal to the integral of the curvature in a disk bounded by the loop.

Given a Hermitian vector bundle $\mathcal{E} \to M$ equipped with unitary connection $\nabla^{\mathcal{E}}$, we consider points $p = x_{00}, x_{01}, x_{10}, x_{11}$ and curves $\gamma_i, \eta_i : [0,1] \to M$ for $i = 1, 2$, such that

$$\gamma_1(0) = \eta_1(0) = p, \quad \gamma_1(1) = \eta_2(0) = x_{10}, \quad \eta_1(1) = \gamma_2(0) = x_{01}, \quad \gamma_2(1) = \eta_2(1) = x_{11}.$$

Assume that we have a smooth homotopy $\Gamma : [0,1]^2 \to M$ such that for all $s, t \in [0,1]$:

$$\Gamma(0,t) = \eta_1(t), \quad \Gamma(s,1) = \gamma_2(s), \quad \Gamma(s,0) = \gamma_1(s), \quad \Gamma(1,t) = \eta_2(t).$$

We define the vertical map $C_\uparrow(s,t) : \mathcal{E}_p \to \mathcal{E}_{\Gamma(s,t)}$ as the parallel transport with respect to $\nabla^{\mathcal{E}}$ from $\mathcal{E}_p$ to $\mathcal{E}_{\Gamma(0,1)}$, then $\mathcal{E}_{\Gamma(0,1)}$ to $\mathcal{E}_{\Gamma(s,1)}$ and $\mathcal{E}_{\Gamma(s,1)}$ to $\mathcal{E}_{\Gamma(s,t)}$, along $\Gamma(0,\bullet)$, $\Gamma(\bullet,1)$ and $\Gamma(s,\bullet)$, respectively. We define the horizontal map $C_\to(s,t) : \mathcal{E}_p \to \mathcal{E}_{\Gamma(s,t)}$ as the parallel transport with respect to $\nabla^{\mathcal{E}}$ from $\mathcal{E}_p$ to $\mathcal{E}_{\Gamma(s,0)}$ and $\mathcal{E}_{\Gamma(s,0)}$ to $\mathcal{E}_{\Gamma(s,t)}$, along $\Gamma(\bullet,0)$ and $\Gamma(s,\bullet)$, respectively. We refer to Figure 3 for further details.

We are ready to prove the formula:

Lemma 2.7. *We have:*

$$C_\uparrow^{-1}(1,1)C_\to(1,1) - \mathbb{1}_{\mathcal{E}_p} = \int_0^1 \int_0^1 C_\uparrow(s,t)^{-1} \mathbf{F}_{\nabla^{\mathcal{E}}}(\partial_t, \partial_s) C_\to(s,t)\, dt\, ds. \tag{2.11}$$

Proof. Let $w_1, w_2 \in \mathcal{E}_p$. For the sake of simplicity, we write $\nabla = \nabla^{\mathcal{E}}$. We have:

$$\begin{aligned}
&h^{\mathcal{E}}\left(w_1, (C_\uparrow(1,1)^{-1}C_\rightarrow(1,1) - \mathbb{1}_{\mathcal{E}_p})w_2\right) = h^{\mathcal{E}}\left(C_\uparrow(1,1)w_1, C_\rightarrow(1,1)w_2\right) - h^{\mathcal{E}}\left(C_\uparrow(0,1)w_1, C_\rightarrow(0,1)w_2\right)\\
&= \int_0^1 \partial_s h^{\mathcal{E}}\left(C_\uparrow(s,1)w_1, C_\rightarrow(s,1)w_2\right) ds\\
&= \int_0^1 h^{\mathcal{E}}\left(C_\uparrow(s,1)w_1, \nabla_{\partial_s} C_\rightarrow(s,1)w_2\right) ds\\
&= \int_0^1 \Big[\int_0^1 \Big(\partial_t h^{\mathcal{E}}\left(C_\uparrow(s,t)w_1, \nabla_{\partial_s} C_\rightarrow(s,1)w_2\right) + h^{\mathcal{E}}\left(C_\uparrow(s,0)w_1, \nabla_{\partial_s} C_\rightarrow(s,0)w_2\right)\Big) dt\Big] ds\\
&= \int_0^1 \int_0^1 h^{\mathcal{E}}\left(C_\uparrow(s,t)w_1, \nabla^{\mathcal{E}}_{\partial_t}\nabla^{\mathcal{E}}_{\partial_s} C_\rightarrow(s,t)w_2\right) ds\, dt\\
&= \int_0^1 \int_0^1 h^{\mathcal{E}}\left(w_1, C_\uparrow(s,t)^{-1} \underbrace{(\nabla^{\mathcal{E}}_{\partial_t}\nabla^{\mathcal{E}}_{\partial_s} - \nabla^{\mathcal{E}}_{\partial_s}\nabla^{\mathcal{E}}_{\partial_t})}_{=\mathbf{F}_{\nabla^{\mathcal{E}}}(\partial_t,\partial_s)} C_\rightarrow(s,t)w_2\right) ds\, dt,
\end{aligned}$$

as the Lie bracket $[\partial_s, \partial_t] = 0$. This completes the proof, since w_1 and w_2 were arbitrary. □

2.3.4 Twisted differentials

We now complete the discussion of §2.1 by adding the twist by an arbitrary Hermitian vector bundle $\mathcal{E} \to M$ equipped with a unitary connection $\nabla^{\mathcal{E}}$. Using the fibration $\pi : SM \to M$, we can pullback the pair $(\mathcal{E}, \nabla^{\mathcal{E}})$ over SM and consider the bundle $\pi^*\mathcal{E} \to SM$ equipped with the pullback connection $\pi^*\nabla^{\mathcal{E}}$. If $(e_1, ..., e_r)$ is a smooth local orthonormal basis of $\mathcal{E}$ (in a neighborhood of a point $x_0 \in M$), then smooth sections $f \in C^\infty(SM, \pi^*\mathcal{E})$ can be written in a neighborhood of x_0 as:

$$f(x,v) = \sum_{k=1}^r f_k(x,v) e_k(x) \in \mathcal{E}_x,$$

where $f_k \in C^\infty(SM)$ is only locally defined.

The geodesic vector field X induces an operator

$$\mathbf{X} := (\pi^*\nabla^{\mathcal{E}})_X : C^\infty(SM, \pi^*\mathcal{E}) \to C^\infty(SM, \pi^*\mathcal{E}).$$

As before, this operator gives rise in turn to an operator (still denoted by $\mathbf{X}$)

$$\mathbf{X} : C^\infty(SM, \mathcal{N} \otimes \pi^*\mathcal{E}) \to C^\infty(SM, \mathcal{N} \otimes \pi^*\mathcal{E}),$$

which acts in the following way: given local sections $w \in C^\infty(SM, \mathcal{N})$ and $f \in C^\infty(SM, \pi^*\mathcal{E})$:

$$\mathbf{X}(w \otimes f) := (Xw) \otimes f + w \otimes (\mathbf{X}f).$$

The connection $\pi^*\nabla^{\mathcal{E}}$ gives rise as before to differential operators:

$$\nabla^{\mathcal{E}}_{\mathbb{H},\mathbb{V}} : C^\infty(SM, \pi^*\mathcal{E}) \to C^\infty(SM, \pi^*\mathcal{E} \otimes \mathcal{N}),$$

defined in the following way: given $f \in C^\infty(SM, \pi^*\mathcal{E})$, the covariant derivative $(\pi^*\nabla^{\mathcal{E}})f \in C^\infty(SM, \pi^*\mathcal{E} \otimes T^*(SM))$ can be identified with an element of $C^\infty(SM, \pi^*\mathcal{E} \otimes T(SM))$ by using the musical isomorphism $T^*(SM) \to T(SM)$ induced by the Sasaki metric. Using the maps $\mathrm{d}\pi$ and $\mathcal{K}$, one can then consider the projections:

$$\nabla^{\mathcal{E}}_{\mathbb{H}} f := \mathrm{d}\pi(\pi^*\nabla^{\mathcal{E}} f), \quad \nabla^{\mathcal{E}}_{\mathbb{V}} f := \mathcal{K}(\pi^*\nabla^{\mathcal{E}} f),$$

as elements taking values in $\pi^*\mathcal{E} \otimes \mathcal{N}$. In local coordinates, these operators have explicit expressions in terms of the connection 1-form A and we refer to [GPSU16, Lemma 3.2] for further details.

We introduce the following operator $F^{\mathcal{E}} \in C^\infty(SM, \mathcal{N} \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$ defined by:

$$\langle f^{\mathcal{E}}_x(v, w)e, e' \rangle = \langle F^{\mathcal{E}}(x, v)e, w \otimes e' \rangle,$$

where $(x, v) \in SM, w \in \mathcal{N}(x, v) = \{v\}^\perp$ and $e, e' \in \mathcal{E}_x$. The twisted operators $\nabla^{\mathcal{E}}_{\mathbb{H},\mathbb{V}}$ also enjoy commuting properties which involve this operator $F^{\mathcal{E}}$. More precisely, we have (see [GPSU16, Lemma 3.2]):

Lemma 2.8.
$$[X, \nabla^{\mathcal{E}}_{\mathbb{V}}] = -\nabla^{\mathcal{E}}_{\mathbb{H}}, \qquad [X, \nabla^{\mathcal{E}}_{\mathbb{H}}] = R\nabla^{\mathcal{E}}_{\mathbb{V}} + F^{\mathcal{E}}.$$

The adjoint operators to $\nabla^{\mathcal{E}}_{\mathbb{V},\mathbb{H}}$ are the respective twisted horizontal and vertical divergence i.e. $(\nabla^{\mathcal{E}}_{\mathbb{V},\mathbb{H}})^* = -\operatorname{div}^{\mathcal{E}}_{\mathbb{V},\mathbb{H}}$, which satisfy:

$$\operatorname{div}^{\mathcal{E}}_{\mathbb{H}} \nabla_{\mathbb{V}} - \operatorname{div}^{\mathcal{E}}_{\mathbb{V}} \nabla_{\mathbb{H}} = (n-1)X, \quad [X, \operatorname{div}^{\mathcal{E}}_{\mathbb{V}}] = -\operatorname{div}^{\mathcal{E}}_{\mathbb{H}}, \quad [X, \operatorname{div}^{\mathcal{E}}_{\mathbb{H}}] = -\operatorname{div}_{\mathbb{V}} R + (F^{\mathcal{E}})^*.$$

Exercise 2.9. Prove the dual formulas.

2.4 Symmetric tensors

2.4.1 Symmetric tensors in a Euclidean vector space

We recall some elementary properties of symmetric tensors on Riemannian manifolds. The reader is referred to [DS10] for an extensive discussion. We consider an n-dimensional

Euclidean vector space (E, g_E) with orthonormal frame $(\mathbf{e}_1, ..., \mathbf{e}_n)$. We denote by $\otimes^m E^*$ the m-th tensor power of E^* and by $\otimes^m_S E^*$ the symmetric tensors of order m, namely the tensors $u \in \otimes^m E^*$ satisfying:

$$u(v_1, ..., v_m) = u(v_{\sigma(1)}, ..., v_{\sigma(m)}),$$

for all $v_1, ..., v_m \in E$ and $\sigma \in \mathfrak{S}_m$, the permutation group of $\{1, ..., m\}$. If $K = (k_1, ..., k_m) \in \{1, ..., n\}^m$, we define $\mathbf{e}^*_K = \mathbf{e}^*_{k_1} \otimes ... \otimes \mathbf{e}^*_{k_m}$, where $\mathbf{e}^*_i(\mathbf{e}_j) := \delta_{ij}$. We introduce the symmetrization operator $\mathcal{S} : \otimes^m E^* \to \otimes^m_S E^*$ defined by:

$$\mathcal{S}(\eta_1 \otimes ... \otimes \eta_m) := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \eta_{\sigma(1)} \otimes ... \otimes \eta_{\sigma(m)},$$

where $\eta_1, ..., \eta_m \in E^*$. Given $v \in E$, we define $v^\flat \in E^*$ by $v^\flat(w) := g_E(v, w)$ and call $\flat : E \to E^*$ the musical isomorphism, following the usual terminology. Its inverse is denoted by $\sharp : E^* \to E$. The scalar product g_E naturally extends to $\otimes^m E^*$ (and thus to $\otimes^m_S E^*$) using the following formula:

$$g_{\otimes^m E^*}(v_1^\flat \otimes ... \otimes v_m^\flat, w_1^\flat \otimes ... \otimes w_m^\flat) := \prod_{j=1}^m g_E(v_j, w_j),$$

where $v_i, w_i \in E$. In particular, if $u = \sum_{i_1,...,i_m=1}^n u_{i_1...i_m} \mathbf{e}^*_{i_1} \otimes ... \otimes \mathbf{e}^*_{i_m}$, then $\|u\|^2_{\otimes^m E^*} = \sum_{i_1,...,i_m=1}^n |u_{i_1...i_m}|^2$. For the sake of simplicity, we will still write g_E instead of $g_{\otimes^m E^*}$. The operator $\mathcal{S}$ is an orthogonal projection with respect to this scalar product.

There is a natural trace operator $\mathcal{T} : \otimes^m E^* \to \otimes^{m-2} E^*$ (it is formally defined to be 0 for $m = 0, 1$) given by:

$$\mathcal{T} u := \sum_{i=1}^n u(\mathbf{e}_i, \mathbf{e}_i, \cdot, ..., \cdot), \tag{2.12}$$

and it also maps $\mathcal{T} : \otimes^m_S E^* \to \otimes^{m-2}_S E^*$. Its adjoint (with respect to the metric $g_{\otimes^m E^*}$) on symmetric tensors is the map $\mathcal{J} : \otimes^m_S E^* \to \otimes^{m+2}_S E^*$ given by $\mathcal{J} u := \mathcal{S}(g_E \otimes u)$. It is easy to check that the map $\mathcal{J}$ is injective. This implies by standard linear algebra that one has the decomposition, where $\otimes^m_S E^*|_{0-\mathrm{Tr}} = \ker \mathcal{T} \cap \otimes^m_S E^*$ denotes the trace-free symmetric m-tensors:

$$\otimes^m_S E^* = \otimes^m_S E^*|_{0-\mathrm{Tr}} \oplus^\perp \mathcal{J} \otimes^{m-2}_S E^* = \oplus_{k \geq 0} \mathcal{J}^k \otimes^{m-2k}_S E^*|_{0-\mathrm{Tr}}. \tag{2.13}$$

Let $\mathbb{S}_E$ be the unit sphere of E and define the pullback operator $\pi^*_m : \otimes^m_S E^* \to L^2(\mathbb{S}_E)$ by the formula

$$\pi^*_m f(v) := f(v, ..., v).$$

We introduce $\Omega_m := \ker(\Delta_{\mathbb{S}_E} + m(m+n-2))$ where $\Delta_{\mathbb{S}_E}$ denotes the Laplacian on the unit sphere of E. The space $L^2(\mathbb{S}_E)$ is endowed with the natural scalar product:

$$\langle u, u' \rangle_{L^2(\mathbb{S}_E)} \int_{\mathbb{S}_E} u(v)\overline{u'(v)}\mathrm{d}v,$$

where $\mathrm{d}v$ denotes the Riemannian volume form induced by the metric $g_E|_{\mathbb{S}_E}$ on the sphere. We will denote by π_{m*} the adjoint of π_m^* with respect to this scalar product. The following mapping property is important:

Lemma 2.10. *The map*

$$\pi_m^* : \otimes_S^m E^*|_{0-\mathrm{Tr}} \to \Omega_m,$$

is an isomorphism. More precisely, $\pi_{m*}\pi_m^* = c(n,m)\mathbb{1}$, *where*

$$c(n,m) = \frac{m!\pi^{n/2}}{2^{m-1}\Gamma(n/2+m)}.$$

In particular, this implies the following graded mapping property:

$$\pi_m^* : \otimes_S^m E^* = \oplus_{k\geq 0} \mathcal{J}^k \otimes_S^{m-2k} E^*|_{0-\mathrm{Tr}} \to \oplus_{k\geq 0}\Omega_{m-2k}.$$

Proof. First of all, one introduces the space $\mathbf{P}_m(E)$ of homogeneous polynomials of degree $m \in \mathbb{N}$ on E (i.e. satisfying $p(\lambda v) = \lambda^m v$ for all $\lambda > 0$) and $\mathbf{H}_m(E)$ the set of harmonics polynomials of degree m i.e. satisfying $\Delta_E p = 0$, where Δ_E is the Laplacian on E induced by g_E. For $u \in \otimes_S^m E^*$, writing $\lambda_m(u)(v) := u(v, ..., v)$, it is clear that $\lambda_m : \otimes_S^m E^* \to \mathbf{P}_m(E)$. Moreover, it is immediate that $\lambda_m : \otimes_S^m E^*|_{0-\mathrm{Tr}} \to \mathbf{H}_m(E)$ by using the formula, for $u \in \otimes_S^m E^*$ (see [DS10, Lemma 2.4] for instance):

$$m(m-1)\pi_{m-2}^* \operatorname{Tr}(u) = \Delta_E \pi_m^* u.$$

Then, introducing the restriction operator $r_m : \mathbf{P}_m(E) \to C^\infty(\mathbb{S}_E)$ defined by $r_m(u) := u|_{\mathbb{S}_E}$ (hence $\pi_m^* = r_m\lambda_m$), we see that $r_m : \mathbf{H}_m(E) \to \Omega_m$ as follows from the following formula (see [GHL04, Proposition 4.48] for instance):

$$\Delta_E(u)|_{\mathbb{S}_E} = \Delta_{\mathbb{S}_E}(u|_{\mathbb{S}_E}) + \left.\frac{\partial^2 u}{\partial r^2}\right|_{\mathbb{S}_E} + (n-1) \left.\frac{\partial u}{\partial r}\right|_{\mathbb{S}_E},$$

where r is the radial coordinate, using the homogeneity of u. This proves the announced mapping properties. As to the equality $\pi_{m*}\pi_m^* = c(n,m)\mathbb{1}$, it relies on Schur's lemma and requires some extra work, especially for the computation of the value of $c(n,m)$ (we refer to [DS10, Lemma 2.4] for further details). □

2.4.2 Symmetric tensors on a Riemannian manifold

We now consider the Riemannian manifold (M, g) and denote by $\mathrm{d}\mu$ the Liouville measure on the unit tangent bundle SM. All the previous definitions naturally extend to the vector bundle $TM \to M$ that is for $f, f' \in C^\infty(M, \otimes^m T^*M)$, we define the L^2-scalar product

$$\langle f, f' \rangle = \int_M \langle f_x, f'_x \rangle_x \mathrm{d}\,\mathrm{vol}(x),$$

where $\langle \cdot, \cdot \rangle_x$ is the scalar product on T_xM introduced in the previous paragraph and $\mathrm{d}\,\mathrm{vol}(x)$ is the Riemannian measure induced by g. The map $\pi_m^* : C^\infty(M, \otimes^m T^*M) \to C^\infty(SM)$ is the canonical morphism given by $\pi_m^* f(x, v) = f_x(v, ..., v)$, whose formal adjoint with respect to the two L^2-inner products (that is to say on $L^2(SM, d\mu)$ and $L^2(\otimes^m T^*M, d\,\mathrm{vol})$) is π_{m*}, i.e.

$$\langle \pi_m^* f, h \rangle_{L^2(SM, \mathrm{d}\mu)} = \langle f, \pi_{m*} h \rangle_{L^2(\otimes^m T^*M, d\,\mathrm{vol})}.$$

If ∇ denotes the Levi-Civita connection, we set $D := \mathcal{S} \circ \nabla : C^\infty(M, \otimes^m T^*M) \to C^\infty(M, \otimes^{m+1} T^*M)$ to be the symmetrized covariant derivative. Its formal adjoint with respect to the L^2-scalar product is $D^* = -\operatorname{Tr}(\nabla \cdot)$ where the trace is taken with respect to the two first indices, as in the previous paragraph. One has the following well-known relation between the geodesic vector field X on SM and the operator D:

Lemma 2.11. $X\pi_m^* = \pi_{m+1}^* D$

Proof. First of all, one observes that $\pi_{m+1}^* D = \pi_{m+1}^* \mathcal{S} \nabla = \pi_{m+1}^* \nabla$ as the antisymmetric part of the tensor is going to vanish by applying π_{m+1}^*. We fix a point $x_0 \in M$ and consider normal coordinates centered at x_0. In these coordinates, if $f = f_I dx_I$, then:

$$X(x_0, v) = \sum_{i=1}^n v_i \partial_{x_i}, \quad \nabla f(x_0) = \sum_{i=1}^n \partial_{x_i} f_I(x_0) dx_i \otimes dx_I$$

Thus:

$$(X\pi_m^* f)(x_0, v) = \sum_{i=1}^n v_i \partial_{x_i}(f_I v_I) = \sum_{i=1}^n (\partial_{x_i} f_I) v_i v_I = \pi_{m+1}^* (\nabla f)(x_0, v)$$

Since x_0 was arbitrary, this completes the proof. □

The operator D is a differential operator of order 1 with principal symbol given by $\sigma(D)(x, \xi) : f \mapsto i\mathcal{S}(\xi \otimes f) = ij_\xi f$, where j_ξ is the symmetric multiplication by ξ. Its adjoint has principal symbol $\sigma(D^*)(x, \xi) : f \mapsto -i\imath_{\xi^\sharp}$, where $\xi^\sharp$ denotes the vector naturally associated to the covector ξ via the metric g and $\imath_{\xi^\sharp}$ is the contraction.

Lemma 2.12. *D is elliptic. It is injective on tensors of odd order, and its kernel is reduced to $\mathbb{R}g^{\otimes m/2}$ on even tensors.*

When m is even, we will denote by $K_m = c_m\mathcal{S}(g^{\otimes m/2})$, with $c_m > 0$, a unitary vector in the kernel of D.

Proof. We fix $(x,\xi) \in T^*M$. We consider a symmetric tensor $f = \sum_{i_1,\ldots i_m=1}^n f_{i_1\ldots i_m} dx_{i_1} \otimes \ldots \otimes dx_{i_m}$ of order m. We then have:

$$j_\xi f = \frac{1}{m+1}\sum_{l=0}^{m+1}\sum_{i_1,\ldots i_m=1}^{n} f_{i_1\ldots i_m} dx_{i_1}\otimes \ldots \otimes dx_{i_{l-1}} \otimes \xi \otimes dx_{i_{l+1}} \otimes \ldots \otimes dx_{i_m}$$

Thus, separating the case $l = 0$ and $l \neq 0$ in the previous sum, we obtain:

$$\imath_{\eta^\sharp} j_\xi f = \frac{1}{m+1}\langle \xi, \eta^\sharp\rangle f + \frac{m}{m+1} j_\xi \imath_{\eta^\sharp}$$

In particular, for $\eta = \xi$, using the non-negativity of the operator $j_\xi \imath_{\xi^\sharp}$, we obtain for $f \in \otimes_S^m T_x^*M$:

$$|\sigma(D)(x,\xi)f|^2 = \langle \imath_{\xi^\sharp} j_\xi f, f\rangle \geq \frac{|\xi|^2|f|^2}{m+1},$$

i.e. $\|\sigma(x,\xi)\| \geq C|\xi|$, so the operator is uniformly elliptic and can be inverted (on the left) modulo a compact remainder, see Proposition A.5: there exists pseudodifferential operators Q, R of respective order $-1, -\infty$ such that $QD = 1 + R$.

We now investigate $\ker(D)$: if $Df = 0$ for some tensor $f \in C^{-\infty}(M, \otimes_S^m T^*M)$, then f is smooth (see Proposition A.5) and $\pi_{m+1}^* Df = X\pi_m^* f = 0$. By ergodicity of the geodesic flow, $\pi_m^* f = c \in \Omega_0$ is constant. If m is odd, then $\pi_m^* f(x,v) = -\pi_m^* f(x,-v)$ so $f \equiv 0$. If m is even, then $f = \mathcal{J}^{m/2}(u_{m/2})$ where $u_{m/2} \in \otimes_S^0 E^* \simeq \mathbb{R}$ so $f = c'\sigma(g^{\otimes m/2})$. □

By classical elliptic theory, the ellipticity and injectivity of D imply that for all $s \in \mathbb{R}$:

$$H^s(M, \otimes_S^m T^*M) = D(H^{s+1}(M, \otimes_S^{m-1} T^*M)) \oplus \ker D^*|_{H^s(M,\otimes_S^m T^*M)}, \tag{2.14}$$

and the decomposition still holds in the smooth category and in the $C^{k,\alpha}$-topology for $k \in \mathbb{N}, \alpha \in (0,1)$. This is the content of the following theorem:

Theorem 2.13 (Tensor decomposition)**.** *Let $s \in \mathbb{R}$ and $f \in H^s(M, \otimes_S^m T^*M)$. Then, there exists a unique pair of symmetric tensors*

$$(p,h) \in H^{s+1}(M, \otimes_S^{m-1} T^*M) \times H^s(M, \otimes_S^m T^*M),$$

such that $f = Dp + h$ and $D^*h = 0$. Moreover, if $m = 2l+1$ is odd, $\langle p, K_{2l}\rangle = 0$.

The proof will be an immediate consequence of the following dicussion. When m is even, we denote by $\Pi_{K_m} := \langle K_m, \cdot\rangle K_m$ the orthogonal projection onto $\ker(D)$. We define

$\Delta_m := D^*D + \varepsilon(m)\Pi_{K_m}$, where $\varepsilon(m) = 1$ for m even, $\varepsilon(m) = 0$ for m odd. The operator Δ_m is an elliptic differential operator of order 2 which is invertible: as a consequence, its inverse is also pseudodifferential of order -2 (see [Shu01, Theorem 8.2]). We can thus define the operator

$$\pi_{\ker D^*} := \mathbb{1} - D\Delta_m^{-1}D^*, \tag{2.15}$$

so that $h = \pi_{\ker D^*} f$. One can check that this is indeed exactly the L^2-orthogonal projection on solenoidal tensors, it is a pseudodifferential operator of order 0 (as a composition of pseudodifferential operators).

Since $\sigma(D)(x, \xi) = ij_\xi$ is injective, we know that given $(x, \xi) \in T^*M$, the space $\otimes_S^m T_x^*M$ breaks up as the direct sum

$$\begin{aligned} \otimes_S^m T_x^*M &= \operatorname{ran}\left(i\sigma(D)(x,\xi)|_{\otimes_S^{m-1}T_x^*M}\right) \oplus \ker\left(i\sigma(D^*)(x,\xi)|_{\otimes_S^m T_x^*M}\right) \\ &= \operatorname{ran}\left(j_\xi|_{\otimes_S^{m-1}T_x^*M}\right) \oplus \ker\left(\imath_{\xi^\sharp}|_{\otimes_S^m T_x^*M}\right) \end{aligned}$$

We denote by $\pi_{\ker \imath_{\xi^\sharp}}$ the projection on $\ker\left(\imath_{\xi^\sharp}|_{\otimes_S^m T_x^*M}\right)$ parallel to $\operatorname{ran}\left(j_\xi|_{\otimes_S^{m-1}T_x^*M}\right)$. It is then straightforward to check that:

Lemma 2.14. *The operator $\pi_{\ker D^*}$ is pseudodifferential of order 0 with principal symbol $\sigma_{\pi_{\ker D^*}} = \pi_{\ker i_\xi}$.*

2.5 Fourier analysis in the fibers

2.5.1 Trivial line bundle

For every $x \in M$, the unit sphere

$$S_xM = \left\{v \in T_xM \mid |v|_x^2 = 1\right\} \subset SM$$

(endowed with the Sasaki metric introduced earlier) is isometric to the canonical sphere $(\mathbb{S}^{n-1}, g_{\mathrm{can}})$. Denote by $\Delta_{\mathbb{V}}$ the vertical Laplacian obtained for $f \in C^\infty(SM)$ as $\Delta_{\mathbb{V}} f(x, v) = \Delta_{g_{\mathrm{can}}}(f|_{S_xM})(v)$, where $\Delta_{g_{\mathrm{can}}}$ is the spherical Laplacian. For $m \geq 0$, we denote as in the previous paragraph

$$\Omega_m(x) = \ker(\Delta_{\mathbb{V}}(x) + m(m + n - 2)),$$

the vector space of spherical harmonics of degree m for the spherical Laplacian $\Delta_{\mathbb{V}}$. We will use the convention that $\Omega_m = \{0\}$ if $m < 0$. If $f \in C^\infty(SM)$, it can then be decomposed as $f = \sum_{m\geq 0} \widehat{f}_m$, where $\widehat{f}_m \in C^\infty(M, \Omega_m)$ is the L^2-orthogonal projection of f onto the spherical harmonic of degree m. We will say that f has *finite degree* if its expansion in spherical harmonics is finite, and we call *degree* of f (denoted by $\deg(f)$) the highest degree of its non vanishing spherical harmonics. The following mapping property is crucial:

Lemma 2.15. *The geodesic vector field acts as*

$$X : C^\infty(M, \Omega_m) \to C^\infty(M, \Omega_{m-1}) \oplus C^\infty(M, \Omega_{m+1}).$$

Proof. Consider $f \in C^\infty(M, \Omega_m)$, fix an arbitrary point $x_0 \in M$ and take normal coordinates at $x_0 \in M$. Then $X(x_0, v) = \sum_{i=1}^n v_i \partial_{x_i}$ and thus $Xf(x_0, v) = \sum_{i=1}^n v_i (\partial_{x_i} f)(x_0, v)$. But it is clear that $\partial_{x_i} f$ is still a spherical harmonics of degree m as the operator does not affect the v-variable and then the lemma boils down to proving that the product of a degree 1 spherical harmonics with a degree m is the sum of two spherical harmonics of degree $m - 1$ and $m + 1$. Since this is a well-known fact, we leave that as an exercise for the reader. □

Exercise 2.16. Prove the following:

1. If $f \in \Omega_1, u \in \Omega_m$, then $fu \in \Omega_{m-1} \oplus \Omega_{m+1}$.
2. More generally, show that if $f \in \Omega_k, u \in \Omega_m$, then $fu \in \oplus_{\ell=0}^k \Omega_{m+k-2\ell}$.

We define X_+ as the L^2-orthogonal projection of X on the higher modes Ω_{m+1}, namely if $u \in C^\infty(M, \Omega_m)$, then $X_+ u := \widehat{(Xu)}_{m+1}$ and X_- as the L^2-orthogonal projection of X on the lower modes Ω_{m-1}. For $m \geq 0$, the operator $X_+ : C^\infty(M, \Omega_m) \to C^\infty(M, \Omega_{m+1})$ is elliptic and thus has a finite dimensional kernel by Proposition A.5 (see [DS10]). The operator $X_- : C^\infty(M, \Omega_m) \to C^\infty(M, \Omega_{m-1})$ is of divergence type. It can be checked that $X_+^* = -X_-$: this is a direct consequence of the fact that X is formally skew-adjoint on $L^2(SM)$ as it preserves the Liouville measure. It is worth introducing the following terminology as these elements will play an important role in the following:

Definition 2.17. Elements in the kernel of X_+ are called *Conformal Killing Tensors (CKTs)*, associated to the trivial line bundle.

For $m = 0$, the kernel of X_+ on $C^\infty(M, \Omega_0)$ always contains the constant functions. We call *non trivial CKTs* elements in $\ker X_+$ which are not constant functions on SM. The kernel of X_+ is invariant by changing the metric by a conformal factor (see [GPSU16, Section 3.6]).

As mentioned in Lemma 2.10, there is a one-to-one correspondance between trace-free symmetric tensors of degree m and spherical harmonics of degree m, namely the map

$$\pi_m^* : C^\infty(M, \otimes_S^m T^*M|_{0-\mathrm{Tr}}) \to C^\infty(M, \Omega_m)$$

is (up to a constant) an isometry (see Lemma 2.10). We now introduce the (pointwise in $x \in M$) orthogonal projection $\mathcal{P} : \otimes_S^m T_x^*M \to \otimes_S^m T_x^*M|_{0-\mathrm{Tr}}$ onto trace-free symmetric tensors. We have the following identification of $\mathcal{P}D$ with X_+ and D^* with X_-:

$$X_+\pi_m^* = \pi_{m+1}^*\mathcal{P}D, \quad X_-\pi_m^* = -\frac{m}{n+2(m-2)}\pi_{m-1}^*D^* \tag{2.16}$$

The following decay property will be needed:

Lemma 2.18. *Let $u \in C^\infty(SM, \pi^*\mathcal{E})$ and write $u = \sum_{m\geq 0} \widehat{u}_m$, where $\widehat{u}_m \in C^\infty(M, \Omega_m \otimes \mathcal{E})$. Then there exists $\beta > 0$ such that, for any even $\alpha \in \mathbb{N}$, there exists a constant $C_\alpha > 0$ such that:*

$$\sup_{x\in M} \|\widehat{u}_m(x,\cdot)\|_{L^2(S_xM)} \leq \frac{C_\alpha \|u\|_{C^\alpha(SM,\pi^*\mathcal{E})}}{m^{\alpha-\beta}}$$

Proof. Fix a point $p \in M$, consider $(e_1, ..., e_r)$ a local orthonormal basis of $\mathcal{E}$ around p. We can write $u(x, v) = \sum_{k=1}^r u_k(x, v) \otimes e_k(x)$, where $u_k \in C^\infty(SM)$ and each u_k can be decomposed into Fourier modes $u_k = \sum_{m\geq 0}(\widehat{u_k})_m$ where $(\widehat{u_k})_m \in C^\infty(M, \Omega_m)$. We then have

$$(\widehat{u})_m(x, v) = \sum_{k=1}^r (\widehat{u_k})_m(x, v)e_k(x).$$

Then:

$$\|(\widehat{u})_m(x,\cdot)\|^2_{L^2(S_xM)} = \int_{S_xM} \sum_{k=1}^r |(\widehat{u_k})_m(x, v)|^2 \mathrm{d}v = \sum_{k=1}^r \|(\widehat{u_k})_m(x,\cdot)\|^2_{L^2(S_xM)},$$

so the lemma actually boils down to the trivial case $\mathcal{E} = \mathbb{C}$ i.e. it suffices to show

$$\|\widehat{f}_m(x,\cdot)\|^2_{L^2(S_xM)} \leq \frac{C_\alpha \|f\|^2_{C^\alpha(SM)}}{m^{\alpha-\beta}},$$

for any smooth function $f \in C^\infty(SM)$.

We fix a point $x \in M$ is fixed in a neighborhood of p. We identify $S_xM \simeq \mathbb{S}^{n-1}$. We write $f = \sum_{m\geq 0} \widehat{f}_m$, where $\widehat{f}_m \in \Omega_m(x)$. Let $\omega_1, ..., \omega_{j(m)}$ be an L^2-orthonormal basis of spherical harmonics of degree m, i.e. for all $i = 1, ..., j(m)$, we have:

$$-\Delta_{\mathbb{V}}\omega_i = \lambda_m \omega_i,$$

where $\lambda_m = m(m + n - 2)$. Note that we have

$$j(m) = \binom{n-1+m}{m} - \binom{n+m-3}{m-2},$$

and the important observation is that $j(m) \lesssim m^\beta$, for some exponent $\beta > 0$. Indeed,

$$j(m+1) = \frac{n+m}{m+1}\binom{n-1+m}{m} - \frac{n+m-2}{m-1}\binom{n+m-3}{m-2} \leq \frac{n+m}{m+1} j(m),$$

and the bound follows easily.

We can further decompose

$$\widehat{f}_m = \sum_{i=1}^{j(m)} \alpha_i \omega_i,$$

where $\alpha_i = \langle f, \omega_i \rangle_{L^2(\mathbb{S}^{n-1})}$. This implies that for any $\alpha \in \mathbb{N}$:

$$|\alpha_i| = \frac{|\langle -\Delta_{\mathbb{V}}^\alpha f, \omega_i \rangle_{L^2}|}{\lambda_m^\alpha} \leq \frac{\|\Delta_{\mathbb{V}}^\alpha f(x,\cdot)\|_{L^\infty(S_xM)} \|\omega_i\|_{L^2}}{\lambda_m^\alpha} \leq C \frac{\|f\|_{C^{2\alpha}(SM)}}{m^{2\alpha}},$$

where C only depends on the dimension and some potential choices made in the definition of $C^{2\alpha}(SM)$. Hence:

$$\|\widehat{f}_m\|^2_{L^2(S_xM)} = \sum_{i=1}^{j(m)} |\alpha_i|^2 \lesssim j(m) \|f\|^2_{C^{2\alpha}(SM)} m^{-4\alpha} \lesssim m^{\beta - 4\alpha} \|f\|^2_{C^{2\alpha}(SM)}.$$

This proves the claim. □

In particular, it will be convenient to have the following result at hand:

Lemma 2.19. *For $u = \sum_{m \geq 0} \widehat{u}_m \in C^\infty(SM)$, one has:*

$$\|X_+ \widehat{u}_m(x,\cdot)\|_{L^2(S_xM)} \lesssim \frac{\|u\|_{C^{\alpha+1}}}{m^{\alpha-\beta}}.$$

Proof. This is a straightforward consequence of the previous Lemma as X_+u is smooth if u is smooth. □

2.5.2 Twisted Fourier analysis in the fibers

Consider $(\mathcal{E}, \nabla^{\mathcal{E}})$ a Hermitian vector bundle of rank r equipped with a unitary connection over the smooth Riemannian n-manifold (M, g) with $n \geq 2$. Let SM be the unit sphere bundle and $\pi : SM \to M$ be the projection. We consider the pullback bundle $(\pi^*\mathcal{E}, \pi^*\nabla^{\mathcal{E}})$ over SM. The geodesic vector field X induces the operator $\mathbf{X} := (\pi^*\nabla^{\mathcal{E}})_X$, acting on sections of $C^\infty(SM, \pi^*\mathcal{E})$. As before, by standard Fourier analysis in the sphere fibers, we can write $f \in C^\infty(SM, \mathcal{E})$ as $f = \sum_{m \geq 0} f_m$, where $f_m \in C^\infty(M, \Omega_m \otimes \mathcal{E})$ and pointwise in $x \in M$:

$$\Omega_m(x) \otimes \mathcal{E}(x) := \ker(\Delta_{\mathbb{V}}^{\mathcal{E}} + m(m+n-2)),$$

is the kernel of the vertical Laplacian $\Delta_{\mathbb{V}}^{\mathcal{E}}$. Note that this Laplacian is independent of the connection $\nabla^{\mathcal{E}}$, it only depends on $\mathcal{E}$ and g, as can be seen from the expression

$$\Delta_{\mathbb{V}}^{\mathcal{E}}(\sum_{k=1}^{r} f_k e_k) = \sum_{k=1}^{r} (\Delta_{\mathbb{V}} f_k) e_k,$$

if $(e_1, ..., e_r)$ denotes a local orthonormal basis of $\mathcal{E}$ around a point x_0 (the e_i's are only x-dependent). Elements in this kernel are called the *twisted spherical harmonics of degree* m. As in the non-twisted case, we will say that $f \in C^\infty(SM, \mathcal{E})$ has *finite Fourier content* if its expansion in spherical harmonics only contains a finite number of terms and we denote by $\deg(f)$ its degree. It is easy to check that the operator $\mathbf{X}$ still maps

$$\mathbf{X} : C^\infty(M, \Omega_m \otimes \mathcal{E}) \to C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E}) \tag{2.17}$$

and can thus be decomposed as $\mathbf{X} = \mathbf{X}_+ + \mathbf{X}_-$, where, if $u \in C^\infty(M, \Omega_m \otimes \mathcal{E})$, $\mathbf{X}_+ u \in C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})$ denotes the orthogonal projection on the twisted spherical harmonics of degree $m+1$. The operator $\mathbf{X}_+$ is elliptic and thus has finite-dimensional kernel whereas $\mathbf{X}_-$ is of divergence type. Moreover, $\mathbf{X}_+^* = -\mathbf{X}_-$, where the adjoint is computed with respect to the canonical L^2 scalar product on SM induced by the Sasaki metric.

Definition 2.20. We call *twisted Conformal Killing Tensors* (CKTs) elements in the kernel of $\mathbf{X}_+|_{C^\infty(M, \Omega_m \otimes \mathcal{E})}$.

The twisted CKTs are always invariant by conformal change of the metric (see [GPSU16,]). We say that the twisted CKTs are trivial when the kernel is reduced to $\{0\}$ and this is known to be a generic property of connections:

Theorem 2.21 (Cekic-L. '20)**.** *The set of unitary connections without CKTs is residual*[6].

We now explain the link with (twisted) symmetric tensors. Given a section $u \in C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$, the connection $\nabla^{\mathcal{E}}$ produces an element $\nabla^{\mathcal{E}} u \in C^\infty(M, T^*M \otimes (\otimes_S^m T^*M) \otimes \mathcal{E})$. In coordinates, if $(e_1, ..., e_r)$ is a local orthonormal frame for $\mathcal{E}$ and $\nabla^{\mathcal{E}} = d + \Gamma$, for some one-form with values in skew-hermitian matrices Γ, we have:

$$\begin{aligned} \nabla^{\mathcal{E}}(\sum_{k=1}^{r} u_k \otimes e_k) &= \sum_{k=1}^{r} \nabla u_k \otimes e_k + u_k \otimes \nabla^{\mathcal{E}} e_k \\ &= \sum_{k=1}^{r} \left(\nabla u_k + \sum_{l=1}^{r} \sum_{i=1}^{n} \Gamma_{il}^k u_l \otimes dx_i \right) \otimes e_k, \end{aligned} \tag{2.18}$$

[6]In the sense that for all $k \geq 2$, this set is an intersection of dense open subsets of connections with regularity C^k.

where $u_k \in C^\infty(M, \otimes_S^m T^*M)$ and ∇ is the Levi-Civita connection. The symmetrization operator $\mathcal{S}_\mathcal{E} : C^\infty(M, \otimes^m T^*M \otimes \mathcal{E}) \to C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$ is defined by:

$$\mathcal{S}_\mathcal{E}\left(\sum_{k=1}^r u_k \otimes e_k\right) = \sum_{k=1}^r \mathcal{S}(u_k) \otimes e_k,$$

where $u_k \in C^\infty(M, \otimes_S^m T^*M)$ and $\mathcal{S}$ is the symmetrization operators of tensors previously introduced. We can symmetrize (2.18) to produce an element $D_\mathcal{E} := \mathcal{S}_\mathcal{E}\nabla^\mathcal{E} u \in C^\infty(M, \otimes_S^{m+1} T^*M \otimes \mathcal{E})$ given in coordinates by:

$$D_\mathcal{E}\left(\sum_{k=1}^r u_k \otimes e_k\right) = \sum_{k=1}^r \left(Du_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k \mathcal{S}(u_l \otimes dx_i)\right) \otimes e_k, \tag{2.19}$$

where $D = \mathcal{S}\nabla$ (∇ being the Levi-Civita connection) is the usual symmetric derivative of symmetric tensors introduced in the previous paragraph. The operator $D_\mathcal{E}$ is a first order differential operator and the expression of its principal symbol

$$\sigma_{\text{princ}}(D_\mathcal{E}) \in C^\infty(T^*M, \text{Hom}(\otimes_S^m T^*M \otimes \mathcal{E}, \otimes_S^{m+1} T^*M \otimes \mathcal{E}))$$

can be read off from (2.19), namely $\sigma_{\text{princ}}(D_\mathcal{E}) = \sigma_{\text{princ}}(D) \otimes \text{id}_\mathcal{E}$:

$$\begin{aligned}\sigma_{\text{princ}}(D_\mathcal{E})(x,\xi) \cdot \left(\sum_{k=1}^r u_k(x) \otimes e_k(x)\right) &= \sum_{k=1}^r (\sigma_{\text{princ}}(D)(x,\xi) \cdot u_k(x)) \otimes e_k(x) \\ &= i \sum_{k=1}^r \mathcal{S}(\xi \otimes u_k(x)) \otimes e_k(x),\end{aligned}$$

where $e_k(x) \in \mathcal{E}_x, u_k(x) \in \otimes_S^m T_x^*M$ and the basis $(e_1(x), ..., e_r(x))$ is assumed to be orthonormal. As a consequence, it is an injective map and $D_\mathcal{E}$ acting on twisted symmetric tensors of order m is a left-elliptic operator and can be inverted on the left modulo a smoothing remainder; its kernel is finite-dimensional (see Proposition A.5) and consists of elements called *twisted Killing Tensors*. We also record the same relation as in Lemma 2.11.

Lemma 2.22. $\pi_{m+1}^* D_\mathcal{E} = \mathbf{X}\pi_m^*$

The adjoint

$$D_\mathcal{E}^* : C^\infty(M, \otimes_S^{m+1} T^*M \otimes \mathcal{E}) \to C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$$

has a surjective principal symbol given by $\sigma_{D_\mathcal{E}^*}(x,\xi) = -i\imath_{\xi^\sharp} \otimes \text{id}_\mathcal{E}$. As before, there is an explicit link between $\mathbf{X}_-/D_\mathcal{E}^*$ and $\mathbf{X}_+/D_\mathcal{E}$. We have the following equalities (see [GPSU16,

p. 22]) on $C^\infty(M, \otimes_S^m T^*M|_{0-\mathrm{Tr}} \otimes \mathcal{E})$:

$$\mathbf{X}_+\pi_m^* = \pi_{m+1}^*\mathcal{P}D_{\mathcal{E}}, \quad \mathbf{X}_-\pi_m^* = -\frac{m}{n-2+2m}\pi_{m-1}^*D_{\mathcal{E}}^*. \tag{2.20}$$

3 Microlocal framework

Throughout this section, we consider the case of a smooth closed manifold $\mathcal{M}$ endowed with an Anosov vector field X preserving a smooth measure $\mathrm{d}\mu$ and generating a flow $(\varphi_t)_{t\in\mathbb{R}}$. Here, Anosov is understood in the sense of (2.2). (It will be applied with $\mathcal{M} = SM$ and the geodesic vector field X.)

3.1 Rough description of the L^2-spectrum

In this paragraph, we study the L^2-spectrum of the operator X and show the need to introduce other functional spaces in order to obtain a good spectral theory. Since X preserves the smooth measure $\mathrm{d}\mu$, it is skew-adjoint on $L^2(SM, \mathrm{d}\mu)$, with dense domain

$$\mathcal{D}_{L^2} := \left\{u \in L^2(\mathcal{M}, \mathrm{d}\mu) \mid Xu \in L^2(\mathcal{M}, \mathrm{d}\mu)\right\}.$$

Equivalently, $-iX$ is self-adjoint. As we will see, its L^2-spectrum consists of absolutely continuous spectrum on $\mathbb{R}$ and of embedded eigenvalues. We first prove that the L^2-spectrum of iX is $\mathbb{R}$.

Lemma 3.1. $\sigma_{L^2}(iX) = \mathbb{R}$

The proof actually works for more general operators like $\nabla_X^{\mathcal{E}}$, where $\nabla^{\mathcal{E}}$ is a unitary connection on a Hermitian vector bundle $\mathcal{E} \to \mathcal{M}$. The proof we give is that of Guillemin [Gui77, Lemma 3], following Helton.

Proof. We argue by contradiction. Assume $\sigma(-iX) \neq \mathbb{R}$, then since $\sigma(-iX)$ is closed, there exists an interval I of $\mathbb{R}$ such that $I \cap \sigma(-iX) = \emptyset$. Let $f \in C^\infty_{\mathrm{comp}}(I)$, $f \neq 0$. Then $f(-iX) = 0$ and this operator is given by[7]

$$f(-iX) = \int_{-\infty}^{+\infty} \hat{f}(t)e^{tX}\mathrm{d}t$$

[7]Formally, this follows from the following computation, where $dP(\lambda)$ is the spectral measure of $-iX$:

$$f(-iX) = \int_{-\infty}^{+\infty} f(\lambda)dP(\lambda) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\lambda t}\hat{f}(t)dP(\lambda)dt = \int_{-\infty}^{+\infty}\hat{f}(t)e^{tX}dt$$

The justification of the permutation is not difficult since f has compact support.

Given $a \in C^\infty(\mathcal{M})$, $f(-iX)a$ is continuous. Moreover, it is given at $x_0 \in \mathcal{M}$ by:

$$f(-iX)a(x_0) = \int_{-\infty}^{+\infty} \hat{f}(t)a(\varphi_t x_0)\mathrm{d}t$$

We now consider g, a smooth function on $\mathbb{R}$ with compact support and a constant $A > 0$. If $x_0 \in \mathcal{M}$ is not periodic, then we can construct $a \in C^\infty(\mathcal{M}), h \in C^\infty(\mathbb{R})$ such that $a(\varphi_t x_0) = g(t) + h(t)$ for all $t \in \mathbb{R}$, where $\|h\|_\infty \leq \|g\|_\infty$ and $\mathrm{supp}(h) \cap [-A, A] = \emptyset$ (define a by $a(\varphi_t x_0)$ on a sufficiently large segment of the orbit of x_0 and then extend to a sufficiently small tubular neighborhood in order to obtain a smooth function). Then:

$$f(-iX)a(x_0) = 0 = \int_{-\infty}^{+\infty} \hat{f}(t)g(t)\mathrm{d}t + \int_{-\infty}^{+\infty} \hat{f}(t)h(t)\mathrm{d}t$$

As $A \to +\infty$, the second integral converges to 0 since $\hat{f}$ is Schwartz. We thus obtain that $\int_{-\infty}^{+\infty} \hat{f}(t)g(t)dt = 0$ for any smooth function g with compact support, thus $\hat{f} \equiv 0$ and $f \equiv 0$. □

The goal of this Section is to go *beyond* the L^2-spectrum and to reveal *resonances* which are true eigenvalues in the half-space $\{\Re(z) \leq 0\}$. This is the content of the Pollicott-Ruelle theory.

3.2 Pollicott-Ruelle resonances

3.2.1 Description of the resonances

As it is harmless, we can consider a more general case than in the previous paragraph. We assume that $\mathcal{E} \to \mathcal{M}$ is a Hermitian vector bundle over $\mathcal{M}$. Let $\nabla^{\mathcal{E}}$ be a unitary connection on $\mathcal{E}$ and set $\mathbf{X} := \nabla^{\mathcal{E}}_X$. Since X preserves $\mathrm{d}\mu$ and $\nabla^{\mathcal{E}}$ is unitary, the operator $\mathbf{X}$ is skew-adjoint on $L^2(SM, \mathcal{E}; \mathrm{d}\mu)$, with dense domain

$$\mathcal{D}_{L^2} := \left\{u \in L^2(\mathcal{M}, \mathcal{E}; \mathrm{d}\mu) \mid \mathbf{X}u \in L^2(\mathcal{M}, \mathcal{E}; \mathrm{d}\mu)\right\}. \tag{3.1}$$

As we will see, L^2-spectrum consists of absolutely continuous spectrum on $i\mathbb{R}$ and of embedded eigenvalues. We introduce $e^{-t\mathbf{X}}$, the *propagator* of $\mathbf{X}$, namely the parallel transport by $\nabla^{\mathcal{E}}$ along the flowlines of X. Recall that for $x \in \mathcal{M}, t \in \mathbb{R}$, $C(x,t) : \mathcal{E}_x \to \mathcal{E}_{\varphi_t(x)}$ denotes the parallel transport (with respect to the connection $\nabla^{\mathcal{E}}$) along the flowline $(\varphi_s(x))_{s\in[0,t]}$. If $f \in C^\infty(\mathcal{M}, \mathcal{E})$, then $(e^{-t\mathbf{X}}f)(x) = C(\varphi_{-t}(x), t)(f(\varphi_{-t}(x)))$. If $\mathbf{X} = X$ is simply the vector field acting on functions (i.e. $\mathcal{E}$ is the trivial line bundle), then $e^{-tX}f = f(\varphi_{-t}(\cdot))$ is nothing but the composition with the flow.

We introduce the resolvents

$$\begin{aligned}\mathbf{R}_+(z) &:= (-\mathbf{X}-z)^{-1} = -\int_0^{+\infty} e^{-tz}e^{-t\mathbf{X}}\mathrm{d}t,\\ \mathbf{R}_-(z) &:= (\mathbf{X}-z)^{-1} = -\int_{-\infty}^{0} e^{zt}e^{-t\mathbf{X}}\mathrm{d}t,\end{aligned} \tag{3.2}$$

initially defined for $\Re(z) > 0$ since

$$\| \mathbf{R}_+(z)\|_{L^2\to L^2} \leq \int_0^{+\infty} e^{-\Re(z)t}\|e^{-t\mathbf{X}}\|_{L^2 \to L^2}\mathrm{d}t \leq \int_0^{+\infty} e^{-\Re(z)t}\mathrm{d}t = \Re(z)^{-1}.$$

(Let us stress on the conventions here: $-\mathbf{X}$ is associated to the positive resolvent $\mathbf{R}_+(z)$ whereas $\mathbf{X}$ is associated to the negative one $\mathbf{R}_-(z)$.) We are going to show that the resolvents can be meromorphically extended to the whole complex plane by making $\mathbf{X}$ act one anisotropic Sobolev spaces $\mathcal{H}^s_\pm$, that is we can *beyond* the L^2-spectrum axis.

Theorem 3.2 (Faure-Sjöstrand '11)**.** *There exists a constant $c > 0$ such that for any $s > 0$, there exists a Hilbert space $\mathcal{H}^s_+$, such that on the half-space $\{\Re(z) > -cs\}$,*

$$(-\mathbf{X}-z)^{-1} : \mathcal{D}_{\mathcal{H}^s_+} \to \mathcal{H}^s_+$$

is a meromorphic family of unbounded operators with domain $\mathcal{D}_{\mathcal{H}^s_+} = \left\{u \in \mathcal{H}^s_+, \mathbf{X}u \in \mathcal{H}^s_+\right\}$ which are Fredholm of index 0.

The poles of the resolvents are called the *Pollicott-Ruelle resonances* and have been widely studied in the aforementioned literature [Liv04, GL06, BL07, FRS08, FS11, FT13, DZ16]. These resonances (and the resonant states associated to them) are intrinsic to the flow and do not depend on any choice of construction of the anisotropic Sobolev spaces. They carry important dynamical information on the flow. In particular, it can be shown in the simplest case where $\mathcal{E} = \mathbb{C}$ and $\mathbf{X} = X$ is the geodesic vector field acting on functions, that there is a single pole on the imaginary axis at 0 and this is actually equivalent to the fact that the flow is *mixing*, i.e. given $f_{1,2} \in C^\infty(SM)$ with 0-average (with respect to the Liouville measure $\mathrm{d}\mu$):

$$\int_{SM} f_1(\varphi_t(x,v))f_2(x,v)\mathrm{d}\mu(x,v) \to_{t\to 0} 0. \tag{3.3}$$

This will be proved in Lemma 3.10. It can even be shown that for contact Anosov flows, there exists a *spectral gap*, namely a small resonance-free strip on the left of the imaginary axis and this implies that the flow is actually *exponentially mixing* (with respect to the Liouville measure $\mathrm{d}\mu$) i.e. the converge to 0 in (3.3) is exponentially fast, see [Liv04, FT13, NZ15, GC20]. Such a behaviour for a dynamical system is a prototype of a chaotic behaviour.

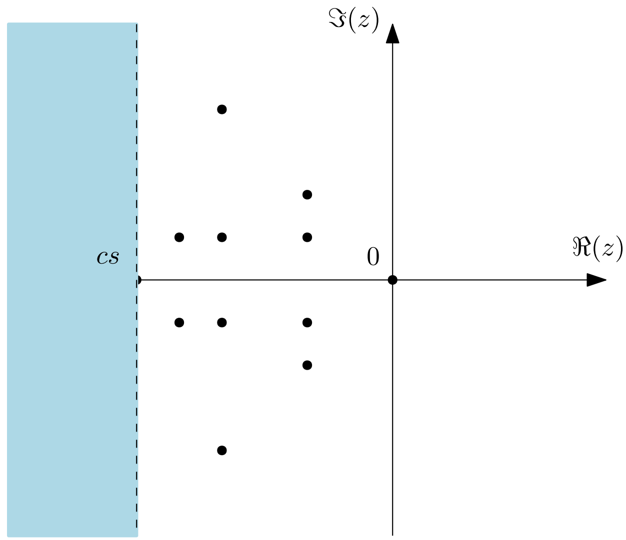


Figure 4: Resonances of the operator X acting on functions. It can be shown that these are symmetric with respect to the real axis, see [FS11].

We introduce the dual decomposition

$$T^*\mathcal{M} = \mathbb{R}E_0^* \oplus E_s^* \oplus E_u^*,$$

where $E_0^*(E_s \oplus E_u) = 0, E_s^*(E_s \oplus \mathbb{R}X) = 0, E_u^*(E_u \oplus \mathbb{R}X) = 0$. As indicated before, we will show that there exists a constant $c > 0$ such that $\mathbf{R}_\pm(z) \in \mathcal{L}(\mathcal{H}^s_\pm)$ are meromorphic in $\{\Re(z) > -cs\}$. For $\mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$), the space $\mathcal{H}^s_+$ (resp. $\mathcal{H}^s_-$) consists of distributions which are microlocally H^s in a neighborhood of E_s^* (resp. H^{-s} in a neighborhood of E_s^*) and microlocally H^{-s} in a neighborhood of E_u^* (resp. H^s in a neighborhood of E_u^*), see [FS11, DZ16]. These spaces also satisfy $(\mathcal{H}^s_+)' = \mathcal{H}^s_-$ (where one identifies the spaces using the L^2-pairing). These resolvents satisfy the following equalities on $\mathcal{H}^s_\pm$, for z not a resonance:

$$\mathbf{R}_\pm(z)(\mp\mathbf{X} - z) = (\mp\mathbf{X} - z)^{-1}\,\mathbf{R}_\pm(z) = \mathbb{1}_\mathcal{E} \tag{3.4}$$

Given $z \in \mathbb{C}$, not a resonance, we have:

$$\mathbf{R}_+(z)^* = \mathbf{R}_-(\overline{z}),$$

where this is understood in the following way: given $f_1, f_2 \in C^\infty(\mathcal{M}, \mathcal{E})$, we have

$$\langle \mathbf{R}_+(z)f_1, f_2\rangle_{L^2} = \langle f_1, \mathbf{R}_-(\overline{z})f_2\rangle_{L^2}.$$

(We will always use this convention for the definition of the adjoint.) Since the operators are skew-adjoint on L^2, all the resonances (for both the positive and the negative resolvents $\mathbf{R}_\pm$) are contained in $\{\Re(z) \leq 0\}$, see [Gui17a, Lemma 2.5] for instance. A point $z_0 \in \mathbb{C}$ is a resonance for $-\mathbf{X}$ (resp. $\mathbf{X}$) i.e. is a pole of $z \mapsto \mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$) if and only if there exists a non-zero $u \in \mathcal{H}^s_+$ (resp. $\mathcal{H}^s_-$) for some $s > 0$ such that $-\mathbf{X}u = z_0 u$ (resp. $\mathbf{X}u = z_0 u$). If γ is a small counter clock-wise oriented circle around z_0, then the spectral projector onto the resonant states is

$$\Pi^\pm_{z_0} = -\frac{1}{2\pi i}\int_\gamma \mathbf{R}_\pm(z)\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma (z \pm \mathbf{X})^{-1}\mathrm{d}z,$$

where we use the abuse of notation that $-(\mathbf{X}+z)^{-1}$ (resp. $(\mathbf{X}-z)^{-1}$) to denote the meromorphic extension of $\mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$).

The fact that resonances are independent of the construction of the anisotropic Sobolev space can also be seen from the following caracterization lemma. Here $\mathcal{D}'_{E^*_{s,u}}$ denotes the space of distributions with wavefront set contained in $E^*_{s,u}$.

Lemma 3.3. *A complex number $z_0 \in \mathbb{C}$ is a pole of the meromorphic extension of $z \mapsto (-\mathbf{X}-z)^{-1}$ from $\{\Re(z) > 0\}$ to $\mathbb{C}$ if and only if there exists a distribution $u \in \mathcal{D}'_{E^*_u}$ such that $(-\mathbf{X}-z_0)u = 0$.*

We leave the proof as an exercise for the reader.

3.2.2 Proof of Theorem 3.2

We will consider the simple case where $\mathcal{E} = \mathbb{C}$ i.e. there is no twist, as this does not make a real difference. We denote by $\mathbf{H}$ the Hamiltonian vector field on the symplectic manifold $T^*\mathcal{M}$ induced by the Hamiltonian $\sigma_P(x,\xi) = \langle \xi, X(x)\rangle$ (the principal symbol of $P := \frac{1}{i}X$) and by $(\Phi_t)_{t\in\mathbb{R}}$ the symplectic flow generated. A quick computation shows that $\Phi_t = (\varphi_t, d\varphi_t^{-\top})$ and the dual spaces $E^*_{s,u}$ previously introduced play a similar role as $E_{s,u}$ in the Anosov definition (2.2), namely:

$$|\Phi_t(x,\xi)| \leq Ce^{-\lambda t}|\xi|, \forall t \geq 0, \xi \in E^*_s,$$
$$|\Phi_t(x,\xi)| \leq Ce^{-\lambda |t|}|\xi|, \forall t \leq 0, \xi \in E^*_u.$$

Alors note that since $(\Phi_t)_{t\in\mathbb{R}}$ is 1-homogeneous in the ξ variable, it induces a flow $(\Phi^{(1)}_t)_{t\in\mathbb{R}}$ on the unit sphere $S^*\mathcal{M}$. If $\kappa : T^*\mathcal{M} \to S^*\mathcal{M}$ denotes the canonical projection, then $\kappa(E^*_s)$ is a hyperbolic repeller/source and $\kappa(E^*_u)$ is a hyperbolic attractor/sink for the dynamics of $(\Phi^{(1)}_t)_{t\in\mathbb{R}}$ (see Figure 5). The following lemma asserts the existence of an *escape function* which is a crucial tool in the proof of the meromorphic extension of the resolvent $(-\mathbf{X}-z)^{-1}$.

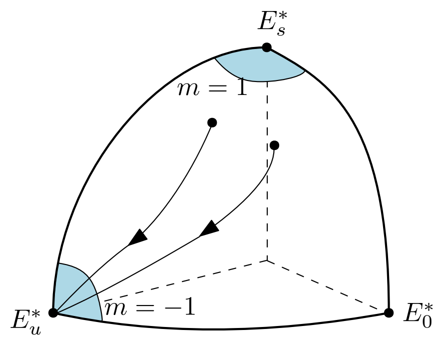


Figure 5: The projective flow induced by $\mathbf{H}$ on the unit cosphere $S^*\mathcal{M}$.

Lemma 3.4 (Faure-Sjöstrand)**.** *There exists a* 0*-homogenous* order function $m \in C^\infty(T^*\mathcal{M}\setminus \{0\}, [-1,1])$ *such that* $\mathbf{H}\cdot m \leq 0$*,* $m \equiv 1$ *in a conic neighborhood of* E_s^**,* $m \equiv -1$ *in a conic neighborhood of* E_u^* *and there exists an* escape function $G_m \in S^0_{\rho,1-\rho}(T^*\mathcal{M})$*, for all* $\rho < 1$*, constructed from* m*, such that:*

- *There exist constants* $C_1, R > 0$ *such that on* $|\xi| \geq R$ *intersected with a conic neighborhood of* $\Sigma := E_s^* \oplus E_u^*$*, one has* $\mathbf{H} \cdot G_m \leq -C_1 < 0$*.*
- *For* $|\xi| \geq R$*,* $\mathbf{H} \cdot G_m \leq C_2$ *for some constant* $C_2 > 0$*.*

An important remark is that $G_m \in S^0_{\rho,1-\rho}$ and $e^{G_m} \in S^m_{\rho,1-\rho}$ for any $\rho < 1$ (these are the anisotropic classes introduced in Appendix A) and we will sometimes write this as S^{m+}. In other words, G_m narrowly misses the usual class $S^0_{1,0}$. This will not be a problem when working in Sobolev regularity (that is when working with spaces from L^2) but may (and actually will) induce complications when using other spaces like Hölder-Zygmund spaces. More precisely, e^{G_m} satisfies the following symbolic estimates in coordinates:

$$\forall (x,\xi) \in T^*\mathcal{M}, \qquad |\partial_\xi^\alpha \partial_x^\beta e^{G_m}(x,\xi)| \leq C_{\alpha,\beta} (\log\langle\xi\rangle)^{|\alpha|+|\beta|} \langle\xi\rangle^{m(x,\xi)-|\alpha|},$$

where $\alpha, \beta \in \mathbb{N}^{n+1}$.

The anisotropic Sobolev spaces are then defined thanks to the operator $A_s := \mathrm{Op}(e^{sG_m}) \in \Psi_h^{sm+}(M)$ by:

$$\mathcal{H}^s_+(\mathcal{M}) := A_s^{-1}(L^2(\mathcal{M})), (\mathcal{H}^s_+)' := \mathcal{H}^s_-(\mathcal{M}) = A_s(L^2(\mathcal{M})) \tag{3.5}$$

They satisfy some elementary but important properties such that $C^\infty(\mathcal{M})$ is dense in $\mathcal{H}^s_+(\mathcal{M})$ and that $\mathcal{H}^s_+(\mathcal{M})$ is stable by multiplication by smooth functions. We can now go for the proof of Theorem 3.2:

Proof of Theorem 3.2. The computation rules of symbols in anisotropic classes enjoy the same properties (composition rules, ellipticity, etc.) as symbols in the usual classes (see

[FRS08]). We leave it as an exercise to the reader to check that all the symbols and pseudodifferential operators are in the right anisotropic classes.

We consider a cutoff function $\chi \in C_c^\infty([0,+\infty))$ such that $\chi \equiv 1$ on $[0,1/2]$ and $\chi \equiv 0$ outside $[0,1]$. We then define for $T>0$ the function $\chi_T(t) := \chi(t/T)$. We have:

$$(X+\lambda)\int_0^{+\infty} \chi_T(t)e^{-t(X+\lambda)}\mathrm{d}t = \mathbb{1} + \int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t$$

Note that the integral on the right-hand side is actually performed for $t \in [0,T]$, that is on a finite time interval, as will be all the integrals in the following. Let $P := \mathrm{Op}(p)$, where $p \in S^0(T^*M)$ and $p \equiv 1$ in a conic neighborhood of $\Sigma := E_s^* \oplus E_u^*$ and $p \equiv 0$ outside this conic neighborhood. We define $A_s := \mathrm{Op}(e^{sG_m}) \in \Psi_h^{sm+}(M)$, where $s>0$ is some fixed number. Up to a lower order modification, we can assume that A_s is invertible. We introduce $H+\lambda := A_s(X+\lambda){A_s}^{-1}$. Then:

$$(H+\lambda)\underbrace{A_s\int_0^{+\infty} \chi_T(t)e^{-t(X+\lambda)}{A_s}^{-1}\mathrm{d}t}_{:=Q(\lambda)} = \mathbb{1} + \underbrace{A_s\int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t{A_s}^{-1}}_{:=R(\lambda)} \tag{3.6}$$

Note that $\|R(\lambda)\|_{\mathcal{L}(L^2,L^2)} = O(\langle\Re(\lambda)\rangle^{-\infty})$ for $\Re(\lambda) \gg 0$. In particular, for $\Re(\lambda) \gg 0$, $\mathbb{1}+R(\lambda)$ is invertible on L^2.

Then, we write:

$$R(\lambda) = A_s\int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}tP{A_s}^{-1} + A_s\int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t(\mathbb{1}-P){A_s}^{-1} \tag{3.7}$$

By elementary wavefront set arguments (see Example A.18) we have that

$$\int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t(\mathbb{1}-P) \in \Psi^{-\infty}$$

As a consequence

$$\mathbb{C} \ni \lambda \mapsto A_s\int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t(\mathbb{1}-P){A_s}^{-1} \in \Psi^{-\infty}$$

is a holomorphic family of compact operators on L^2. Then, we deal with the first term in (3.7). First, notice that by Egorov's Theorem (see Lemma A.7 or [Zwo12, Theorem 11.1] for further details)

$$e^{tX}A_se^{-tX} = e^{tX}\,\mathrm{Op}(e^{sG_m})e^{-tX} = \mathrm{Op}(e^{sG_m\circ\Phi_t}) + K_t,$$

where $e^{sG_m\circ\Phi_t} \in S^{sm\circ\Phi_t +}$ and thus

$$\mathrm{Op}(e^{sG_m\circ\Phi_t}) \in \Psi^{sm\circ\Phi_t +}, \qquad K_t \in \Psi^{sm\circ\Phi_t - 1+}$$

Thus:

$$\begin{aligned} A_s \int_0^{+\infty} \chi_T'(t) e^{-t(X+\lambda)} \mathrm{d}t P {A_s}^{-1} &= \int_0^{+\infty} \chi_T'(t) e^{-t\lambda} A_s e^{-tX} P {A_s}^{-1} \mathrm{d}t \\ &= \int_0^{+\infty} \chi_T'(t) e^{-t\lambda} e^{-tX} e^{tX} A_s e^{-tX} P {A_s}^{-1} \mathrm{d}t \\ &= \int_0^{+\infty} \chi_T'(t) e^{-t\lambda} e^{-tX} \left(\mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} p) + K_t' P {A_s}^{-1} \right) \mathrm{d}t \end{aligned}$$

But on the support of p, we have $\mathbf{H}\cdot m \leq 0$, so

$$e^{G_m\circ\Phi_t - G_m} p \in S^{m\circ\Phi_t - m}_{\rho,1-\rho} \subset S^0_{\rho,1-\rho},$$

for all $\rho < 1$. Thus $\mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} p) \in \Psi^0_{\rho,1-\rho}(M)$ for all $\rho < 1$ and this is bounded on L^2. Moreover, $K_t' P {A_s}^{-1} \in \Psi^{-1+}(M)$ and is thus compact on L^2. Since e^{-tX} is bounded on L^2, we deduce that

$$\int_0^{+\infty} \chi'(t) e^{-t\lambda} e^{-tX} K_t' P {A_s}^{-1} \mathrm{d}t$$

is compact on L^2. We now need to study the norm of the operator in $\Psi^0_{\rho,1-\rho}$. Let $q \in C^\infty(T^*M)$ be a smooth cutoff function such that $q(x,\xi) \equiv 0$ for $|\xi| \leq R$ and $q(x,\xi) = 1$ for $|\xi| \geq R+1$. We write

$$\mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} p) = \mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} pq) + \mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} p(1-q))$$

The last operator is in $\Psi^{-\infty}$ and is thus compact on L^2. We are left with the operator $\mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} pq)$. Note that

$$\limsup_{|\xi| \to\infty} e^{s(G_m\circ\Phi_t(x,\xi) - G_m(x,\xi))} pq(x,\xi) \leq e^{-C_1 sT/2},$$

since $\mathbf{H}\cdot G_m \leq -C_1 < 0$ on the support of pq. By the Calderon-Vaillancourt Theorem (see [Shu01, Theorem 6.4] for instance), for $t \in [0,T]$, we can write $\mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)} pq) = A_t + L_t$, where $A_t \in \Psi^0_{\rho,1-\rho}$, $L_t \in \Psi^{-\infty}$ and $\|A_t\|_{\mathcal{L}(L^2,L^2)} \leq e^{-C_1 st/2}$. Since the operator L_t contributes to a compact operator in (3.6), we can forget it.

In (3.6), we thus obtain that

$$\mathbb{1} + R(\lambda) = \mathbb{1} + B(\lambda) + K(\lambda),$$

with $K(\lambda)$ holomorphic (on $\mathbb{C}$) family of compact operators on L^2 and using $\|e^{-tX}\|_{\mathcal{L}(L^2,L^2)} \leq C_0 e^{\omega t}$:

$$\begin{aligned} \|B(\lambda)\|_{\mathcal{L}(L^2,L^2)} &= \| \int_0^T \chi_T'(t) e^{-t\lambda} e^{-tX} A_t \mathrm{d}t \|_{\mathcal{L}(L^2,L^2)} \\ &\leq C_0 \int_0^T |\chi_T'(t)| e^{-t\Re(\lambda)} e^{-C_1 st/2} e^{\omega t} \mathrm{d}t \\ &\leq \frac{C_0 \|\chi'\|_{L^\infty}}{T} \int_0^T e^{-(C_1 s/2 + \Re(\lambda) - \omega)t} \mathrm{d}t \leq \frac{C_0 \|\chi'\|_{L^\infty}}{T(C_1 s/2 + \Re(\lambda) - \omega)} \end{aligned} \tag{3.8}$$

This can be made smaller than 1 for some well-chosen constants. Indeed, choose $T > 0$ large enough so that $C_0 \|\chi'\|_{L^\infty}/T < C_1 s/8$. Then, for $\Re(\lambda) > \omega - C_1 s/4$, one obtains:

$$\frac{\|\chi'\|_{L^\infty}}{T(C_1 s/2 + \Re(\lambda) - \omega)} < \frac{\|\chi'\|_{L^\infty}}{TC_1 s/4} < 1/2$$

Therefore, by (3.8), $\|B(\lambda)\|_{\mathcal{L}(L^2,L^2)} < 1$. In fine, we obtain that $\mathbb{1} + B(\lambda)$ is invertible by Neumann series and thus in (3.6), we obtain that $\mathbb{1} + B(\lambda) + K(\lambda)$ is a holomorphic family of Fredholm operators on $\Re(\lambda) > \omega - cs$ (where $c := C_1/4$) with index 0. We then conclude by the analytic Fredholm Theorem. The space we are looking for is $\mathcal{H}^s_+(\mathcal{M}) := A_s^{-1}(L^2(\mathcal{M}))$. □

3.3 Description of the L^2-spectrum

In this paragraph, we complete the description of the L^2-spectrum for the operator $\mathbf{X}$ initiated in §3.1, using Theorem 3.2.

3.3.1 Spectral measure

First of all, we need the:

Lemma 3.5. *The poles of the resolvent on $i\mathbb{R}$ are of rank* 1.

Proof. This is a mere consequence of skew-adjointness of the operator $\mathbf{X}$ which implies that $\|\mathbf{R}_+(z)\|_{L^2 \to L^2} \leq 1/\Re(z)$, as we saw. □

However, the residudes (which are spectral projectors onto the resonant states) may have an arbitrary multiplicity. We can now complete the description of the L^2-spectrum:

Lemma 3.6. *We have:*

1. *$\sigma_{L^2}(i\mathbf{X})$ consists of absolutely continuous spectrum and pure point spectrum,*
2. *λ_0 is in the pure point spectrum of $i\mathbf{X}$ if and only if $i\lambda_0$ is a pole of the resolvent,*

3. $\sigma_{\mathrm{ac}}(i\mathbf{X}) = \mathbb{R}$. *Moreover, the absolutely continuous spectral measure is given by*

$$\mathrm{d}P(\lambda) = -\frac{1}{2\pi}(\mathbf{R}_+(-i\lambda) + \mathbf{R}_-(i\lambda)).$$

Proof. Fix $\lambda_0 \in \mathbb{R}$ and assume that $i\lambda_0$ is not a resonance for $-\mathbf{X}$, that is $\mathbf{R}_+(i\lambda_0)$ is well-defined. Then, so is $\mathbf{R}_-(-i\lambda_0) = \mathbf{R}_+(i\lambda_0)^*$. Then, Stone's formula gives that for $\delta > 0$ small enough:

$$\begin{aligned}\frac{1}{2}(\mathbf{1}_{[\lambda_0-\delta,\lambda_0+\delta]}(i\mathbf{X}) + \mathbf{1}_{(\lambda_0-\delta,\lambda_0+\delta)}(i\mathbf{X})) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\lambda_0-\delta}^{\lambda_0+\delta}\left((i\mathbf{X}-(\lambda+i\varepsilon))^{-1} - (i\mathbf{X}-(\lambda-i\varepsilon))^{-1}\right)\mathrm{d}\lambda \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\lambda_0-\delta}^{\lambda_0+\delta}(-\mathbf{R}_-(-i\lambda+\varepsilon) - \mathbf{R}_+(i\lambda+\varepsilon))\,\mathrm{d}\lambda \\ &= -\frac{1}{2\pi}\int_{\lambda_0-\delta}^{\lambda_0+\delta}(\mathbf{R}_-(-i\lambda) + \mathbf{R}_+(i\lambda))\,\mathrm{d}\lambda,\end{aligned} \tag{3.9}$$

where the convergence is in the weak sense[8], that is by applying the expression to $f_1 \in C^\infty(\mathcal{M})$ and testing against $f_2 \in C^\infty(\mathcal{M})$ — the permutation of the limit and the integral being guaranteed by the holomorphy of the integrand. Taking the limit $\delta \to 0$ in (3.9), we see that the right-hand side converges to 0. Hence λ_0 cannot be in the pure point spectrum, otherwise the left-hand side would converge to $\Pi_{\lambda_0}^{L_2}$.

Now, assume that $i\lambda_0$ is a resonance for $-\mathbf{X}$ (that is there exists a distribution $u \in \mathcal{D}'_{E_u^*}$ such that $(-\mathbf{X} - i\lambda_0)u = 0$) and write, for z near $i\lambda_0$:

$$\mathbf{R}_+(z) = \mathbf{R}_+^{\mathrm{hol}}(z) - \frac{\Pi_{i\lambda_0}^+}{z - i\lambda_0},$$

where $\mathbf{R}_+^{\mathrm{hol}}(z)$ is holomorphic in z. (Note that the resolvent has this form as the poles are of order 1, see Lemma 3.5.) Inserting this into Stone's formula (3.9), we then obtain (in the weak sense):

$$\lim_{\delta\to 0}\frac{1}{2}(\mathbf{1}_{[\lambda_0-\delta,\lambda_0+\delta]}(i\mathbf{X}) + \mathbf{1}_{(\lambda_0-\delta,\lambda_0+\delta)}(i\mathbf{X})) = \Pi_{\lambda_0}^{L_2} = \frac{\Pi_{i\lambda_0}^+ + (\Pi_{i\lambda_0}^+)^*}{2},$$

that is λ_0 is in the pure point spectrum.

Formula (3.9) also allows to show that there is no singular continuous spectrum, as the

[8] The limit in Stone's formula is in the strong sense but we here want to inverse limit and integration.

spectral measure is given by

$$\mathrm{d}P(\lambda) = -\frac{1}{2\pi}(\mathbf{R}_+(-i\lambda) + \mathbf{R}_-(i\lambda))\mathrm{d}\lambda$$

orthogonally to the L^2-eigenstates associated to the discrete pure point spectrum. Eventually, since $\sigma(-iX) = \mathbb{R}$ and the only discrete eigenvalue is 0 and the absolutely continuous spectrum is closed, $\sigma_{\mathrm{ac}}(-iX) = \mathbb{R}$. □

It turns out that one can even prove the following remarkable property:

Lemma 3.7. *Resonant states associated to resonances on $i\mathbb{R}$ are smooth. In other words, if $(-\mathbf{X} - i\lambda_0)u = 0$ and $u \in \mathcal{D}'_{E^*_{s,u}}$, then u is smooth.*

We refer to [DZ17, Lemma 2.3] for a proof. This can be obtained as a consequence of *radial source/sink estimates* (with some extra work, though), see [DZ16] for instance. As this is a bit out of scope of the present survey, we do not detail these estimates. This has the following consequence:

Lemma 3.8. *The L^2-eigenstates corresponding to the pure point spectrum are smooth. In other words, if $(-\mathbf{X} - i\lambda_0)u = 0$ and $u \in L^2(\mathcal{M}, \mathcal{E})$, then $u \in C^\infty(\mathcal{M}, \mathcal{E})$.*

Proof. We know by the proof of Lemma 3.6 that

$$\Pi^{L_2}_{\lambda_0} = \frac{1}{2}(\Pi^+_{i\lambda_0} + (\Pi^+_{i\lambda_0})^*),$$

where $(\Pi^+_{i\lambda_0})^* = \Pi^-_{-i\lambda_0}$. These projectors take value in $C^\infty(\mathcal{M}, \mathcal{E})$ and therefore so does $\Pi^{L_2}_{\lambda_0}$. □

3.3.2 Dynamical properties of the flow and resonances

We now go back more specifically to the spectral theory of the vector field X:

Lemma 3.9. *Assume X generates an Anosov flow preserving the smooth volume $\mathrm{d}\mu$. Then it is ergodic.*

First of all, observe that the constant function $\mathbf{1}$ is always a resonant state at 0.

Proof. As X preserves a smooth measure, the previous paragraph applies. By definition, the flow is ergodic with respect to $\mathrm{d}\mu$ if and only if for $u \in L^2(\mathcal{M}, \mathrm{d}\mu)$, $Xu = 0$ implies that u is constant. Now, if $Xu = 0$, and u is in L^2, then u is smooth (by Lemma 3.8) and it is a resonant state at 0. It is then immediate that u is constant. □

Recall that a flow is said to be mixing (with respect to the probability measure $\mathrm{d}\mu$) if, given $f_1, f_2 \in C^\infty(\mathcal{M})$, one has:

$$C_t(f_1,f_2) := \int_{SM} f_1(\varphi_t(x,v)) f_2(x,v) \mathrm{d}\mu(x,v) - \int_{\mathcal{M}} f_1 \mathrm{d}\mu \times \int_{\mathcal{M}} f_2 \mathrm{d}\mu \to_{t\to 0} 0.$$

Lemma 3.10. *The flow is mixing if and only if* 0 *is the only resonance on the real axis.*

Proof. We fix $\varepsilon > 0$. If the flow is mixing, there exists a time T_ε such that for all $T > T_\varepsilon$, $|C_t(f_1,f_2)| < \varepsilon$. Moreover, for $\Re(\lambda) > 0$, using the integral formula (3.2):

$$-\lambda \langle R_+(\lambda) f_1, f_2\rangle = \underbrace{\int_0^{T_\varepsilon} \lambda e^{-\lambda t} \langle f_1 \circ \varphi_{-t}, f_2\rangle_{L^2(\mathcal{M})} \, \mathrm{d}t}_{\leq (1-e^{-\lambda T_\varepsilon})\|f_1\|_{L^2}\|f_2\|_{L^2}} + \underbrace{\int_{T_\varepsilon}^{+\infty} \lambda e^{-\lambda t} \langle f_1, \mathbf{1}\rangle\langle f_2, \mathbf{1}\rangle \mathrm{d}t}_{=e^{-\lambda T_\varepsilon}\langle f_1,\mathbf{1}\rangle\langle f_2,\mathbf{1}\rangle}$$
$$+ \underbrace{\int_{T_\varepsilon}^{+\infty} \lambda e^{-\lambda t} C_t(f_1,f_2) \mathrm{d}t}_{\leq \varepsilon e^{-\lambda T_\varepsilon}}$$

As $\lambda \to 0$, we obtain that

$$\lim_{\lambda\to 0^+} \lambda \langle R_+(\lambda) f_1, f_2\rangle = \langle f_1, \mathbf{1}\rangle\langle f_2, \mathbf{1}\rangle + \mathcal{O}(\varepsilon)$$

and since $\varepsilon > 0$ was chosen arbitrarily small, we obtain that 0 is a pole of order 1 of $R_+(\lambda)$ with residue $-\mathbf{1} \otimes \mathbf{1}$, the projection on the constants. The same arguments also immediately show that for $\lambda_0 \in \mathbb{R} \setminus \{0\}$,

$$\lim_{\lambda \to i\lambda_0^+} (\lambda - i\lambda_0) \langle R_+(\lambda) f_1, f_2\rangle = 0$$

As to R_-, the same arguments apply and the residue at 0 is $-\mathbf{1} \otimes \mathbf{1}$.

The converse is obtained from the fact that the spectrum on $(\mathbb{C} \cdot \mathbf{1})^\perp$ is absolutely continuous. Indeed, for $f_1, f_2 \in C^\infty(\mathcal{M})$, orthogonal to the constants, one has:

$$\begin{aligned} \langle e^{tX} f_1, f_2\rangle_{L^2} &= \int_{-\infty}^{+\infty} e^{it\lambda} \langle \mathrm{d}P(\lambda) f_1, f_2\rangle_{L^2} \\ &= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it\lambda} \langle (R_+(-i\lambda) + R_-(i\lambda)) f_1, f_2\rangle_{L^2} \mathrm{d}\lambda \\ &= \frac{1}{2\pi} \widehat{T}(-t), \end{aligned}$$

where $T(\lambda) := -\langle (R_+(-i\lambda) + R_-(i\lambda)) f_1, f_2\rangle_{L^2}$. By the spectral theorem, $T \in L^1(\mathbb{R})$ (and $-\int \langle (R_+(-i\lambda) + R_-(i\lambda)) f_1, f_2\rangle_{L^2} \mathrm{d}\lambda = \langle f_1, f_2\rangle_{L^2}$) so by the Riemann-Lebesgue theorem, one has

$$\lim_{t\to+\infty} \langle e^{tX} f_1, f_2\rangle_{L^2} = \lim_{t\to+\infty} \frac{1}{2\pi}\widehat{T}(-t) = 0,$$

that is the flow is mixing. □

In order to prove *exponential mixing* (i.e. $C_t(f_1, f_2)$ converges exponentially fast to 0 if the f_i's have 0 average) and not only mixing, one needs to prove the existence of a resonance-free strip $\{\Re(z) > -\delta\}$ for some $\delta > 0$, see [Liv04, FT13, NZ15, GC20]. This is a much more difficult question and will not be treated in the present survey.

3.4 Resonances at $z = 0$

The description of the resolvent at $z = 0$ will play an important role in the following. By the previous paragraph, we can write in a neighborhood of $z = 0$ the following Laurent expansion (beware the sign conventions):

$$\mathbf{R}_+(z) = -\mathbf{R}_0^+ - \frac{\Pi_0^+}{z} + \mathcal{O}(z).$$

(Or in other words, using our abuse of notations, $(\mathbf{X} + z)^{-1} = \mathbf{R}_0^+ + \Pi_0^+/z + \mathcal{O}(z)$.) And:

$$\mathbf{R}_-(z) = -\mathbf{R}_0^- - \frac{\Pi_0^-}{z} + \mathcal{O}(z).$$

(Or in other words, using our abuse of notations, $(z - \mathbf{X})^{-1} = \mathbf{R}_0^- + \Pi_0^-/z + \mathcal{O}(z)$.) As a consequence, these equalities define the two operators $\mathbf{R}_0^\pm$ as the holomorphic part (at $z = 0$) of the resolvents $-\mathbf{R}_\pm(z)$. We introduce:

$$\Pi := \mathbf{R}_0^+ + \mathbf{R}_0^- . \tag{3.10}$$

Note that, due to the embedding properties $H^s \hookrightarrow \mathcal{H}_\pm^s \hookrightarrow H^{-s}$, we can *a priori* only say that these operators are bounded as maps $H^s \to H^{-s}$. We have the:

Lemma 3.11. *The operator* Π *satisfies the following properties:*

- $\Pi : H^s(\mathcal{M}, \mathcal{E}) \to H^{-s}(\mathcal{M}, \mathcal{E})$ *is bounded for any* $s > 0$*;*
- *We have* $(\mathbf{R}_0^+)^* = \mathbf{R}_0^-$, $(\Pi_0^+)^* = \Pi_0^- = \Pi_0^+$*;*
- Π *is formally self-adjoint;*
- *It is nonnegative in the sense that for all* $f \in C^\infty(\mathcal{M}, \mathcal{E})$, $\langle \Pi f, f\rangle_{L^2} = \langle f, \Pi f\rangle_{L^2} \geq 0$*;*
- *Eventually, the following statements are equivalent:* $\langle \Pi f, f\rangle_{L^2} = 0$ *if and only if* $\Pi f = 0$ *if and only if* $f = \mathbf{X}u + v$ *for some* $u \in C^\infty(\mathcal{M}, \mathcal{E})$ *and* $v \in \ker(\mathbf{X})$.

Proof. First of all, for z near 0:

$$\begin{aligned}\mathbf{R}_+(z)^* = \mathbf{R}_-(\overline{z}) &= -\mathbf{R}_0^- - \Pi_0^-/\overline{z} + \mathcal{O}(\overline{z}) \\ &= -(\mathbf{R}_0^+)^* - (\Pi_0^+)^*/\overline{z} + \mathcal{O}(\overline{z}),\end{aligned}$$

which proves $(\mathbf{R}_0^+)^* = \mathbf{R}_0^-, (\Pi_0^+)^* = \Pi_0^-$.

We now show that $\Pi_0^+ = \Pi_0^-$. Since $\mathbf{X}$ is skew-adjoint, we know by [DZ17, Lemma 2.3] that resonant states at 0 are smooth. Therefore, for any $s > 0$

$$\ker(-\mathbf{X}|_{\mathcal{H}_+^s}) = \ker(\mathbf{X}|_{\mathcal{H}_-^s}) = \operatorname{ran}(\Pi_0^-|_{C^\infty(\mathcal{M},\mathcal{E})}) = \operatorname{ran}(\Pi_0^+|_{C^\infty(\mathcal{M},\mathcal{E})})$$

(since $C^\infty(\mathcal{M},\mathcal{E})$ is dense in anisotropic Sobolev spaces). Moreover, $\ker(\Pi_0^-|_{C^\infty(\mathcal{M},\mathcal{E})}) = \ker(\Pi_0^+|_{C^\infty(\mathcal{M},\mathcal{E})})$. Indeed, if $f_1 \in C^\infty(\mathcal{M},\mathcal{E}) \cap \ker(\Pi_0^-)$, then for any $f_2 \in C^\infty(\mathcal{M},\mathcal{E})$, one has $0 = \langle \Pi_0^- f_1, f_2\rangle_{L^2} = \langle f_1, \Pi_0^+ f_2\rangle_{L^2}$, that is f_1 is orthogonal to $\operatorname{ran}(\Pi_0^+) = \operatorname{ran}(\Pi_0^-)$ and thus for any f_2, $0 = \langle f_1, \Pi_0^- f_2\rangle_{L^2} = \langle \Pi_0^+ f_1, f_2\rangle_{L^2}$, so $f_1 \in C^\infty(\mathcal{M},\mathcal{E}) \cap \ker(\Pi_0^+)$. As a consequence, the two projections agree on smooth sections.

To show the nonnegativity, we apply Stone's formula to the self-adjoint operator $i\mathbf{X}$ (with dense domain $\mathcal{D}_{L^2}$ previously defined in (3.1)). More precisely, taking $\mathcal{H} := L^2(\mathcal{M},\mathcal{E};\mathrm{d}\mu) \cap \ker \Pi_0^+$, the spectrum of $i\mathbf{X}$ on $\mathcal{H}$ (near the spectral value 0) is only absolutely continuous and if $\pi_{[a,b]}$ denotes the spectral projection onto the energies $[a,b]$, we obtain:

$$\begin{aligned}\pi_{[a,b]} &= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_a^b \left((i\mathbf{X} - (\lambda + i\varepsilon))^{-1} - (i\mathbf{X} - (\lambda - i\varepsilon))^{-1}\right)\mathrm{d}\lambda \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_a^b (-\mathbf{R}_-(-i\lambda+\varepsilon) - \mathbf{R}_+(i\lambda+\varepsilon))\,\mathrm{d}\lambda \\ &= -\frac{1}{2\pi}\int_a^b (\mathbf{R}_-(-i\lambda) + \mathbf{R}_+(i\lambda))\,\mathrm{d}\lambda,\end{aligned}$$

where the limit is understood in the weak sense (by applying it to $f \in C^\infty(SM,\mathcal{E}) \cap \ker \Pi_0^+$ and pairing it to f). We then obtain:

$$\partial_\lambda \pi_{(-\infty,\lambda)}|_{\lambda=0} = \frac{1}{2\pi}(\mathbf{R}_0^- + \mathbf{R}_0^+) = \frac{\Pi}{2\pi} \geq 0.$$

Assume $\langle \Pi f, f\rangle_{L^2} = 0$ for some $f \in C^\infty(\mathcal{M},\mathcal{E})$; as $\mathbf{R}_0^- = (\mathbf{R}_0^+)^*$, equivalently we have $\Re(\langle \mathbf{R}_0^+ f, f\rangle_{L^2}) = 0$. Using the fact that $\mathcal{H}_+^s = \ker \Pi_0^+ \oplus \operatorname{ran} \Pi_0^+$ for any $s > 0$, as well as the relation $\mathbf{X}\,\mathbf{R}_0^+ = \mathbb{1} - \Pi_0^+$ given in (3.12) below, we have that $\mathbf{X} : \ker \Pi_0^+ \xrightarrow{\cong} \ker \Pi_0^+$ is an isomorphism with inverse $\pm\mathbf{R}_0^+$. Thus setting $u := \pm\mathbf{R}_0^+ f$ and $v := \Pi_0^+ f$, we may write $f = \mathbf{X}u + v$. We compute

$$0 = \Re(\langle \mathbf{R}_0^+ f, f\rangle_{L^2}) = \Re(\langle u, \Pi_0^+ f + \mathbf{X}u\rangle_{L^2}) = \Re(\langle u, \mathbf{X}u\rangle_{L^2}) = -\Im(\langle -i\mathbf{X}u, u\rangle_{L^2}), \quad (3.11)$$

using that Π_0^+ is formally self-adjoint and $u \in \ker \Pi_0^+$. Since $f \in C^\infty(\mathcal{M}, \mathcal{E}) \subset \mathcal{H}_+^s$ for any $s > 0$, we have $u \in \mathcal{H}_+^s$ for any $s > 0$, and so the wavefront set of u satisfies $\mathrm{WF}(u) \subset E_u^*$. Thus again an application of [DZ17, Lemma 2.3] gives $u \in C^\infty$. It is then immediate that $\Pi f = 0$, thus completing the proof.[9] □

In the following, we will write $\ker \mathbf{X}$ instead of $\ker \mathbf{X}|_{\mathcal{H}_\pm^s}$ in order not to burden the notations, but be careful that we are always referring to elements in anisotropic spaces (otherwise, $\ker \mathbf{X}|_{H^{-s}}$ is infinite dimensional for any $s > 0$). We also record here for the sake of clarity the following identities:

$$\begin{aligned} &\Pi_0^+ \mathbf{R}_0^+ = \mathbf{R}_0^+ \Pi_0^+ = 0, \ \Pi_0^- \mathbf{R}_0^- = \mathbf{R}_0^- \Pi_0^- = 0, \\ &\mathbf{X}\Pi_0^\pm = \Pi_0^\pm \mathbf{X} = 0, \ \mathbf{X}\,\mathbf{R}_0^+ = \mathbf{R}_0^+ \mathbf{X} = \mathbb{1} - \Pi_0^+, \ -\mathbf{X}\,\mathbf{R}_0^- = -\mathbf{R}_0^- \mathbf{X} = \mathbb{1} - \Pi_0^-. \end{aligned} \tag{3.12}$$

We also have:

Lemma 3.12. *We have:*

1. *If $u \in \ker(\mathbf{X})$, then $u \in C^\infty(\mathcal{M}, \mathcal{E})$ and u does not vanish unless $u \equiv 0$,*
2. *There exists a basis $u_1, ..., u_p$ of $\ker(\mathbf{X})$ such that*
$$\Pi_0^\pm = \sum_{i=1}^p \langle \cdot, u_i \rangle_{L^2} u_i.$$
3. *Let $u_1, ..., u_p$ be a basis of $\ker(\mathbf{X})$. Then for all $x \in \mathcal{M}$, the vectors $(u_1(x), ..., u_p(x))$ are independent as elements of $\mathcal{E}_x$. We can thus always assume that $(u_1(x), ..., u_p(x))$ are orthonormal.*
4. *In particular, $\dim(\ker(\mathbf{X})) \leq \mathrm{rank}(\mathcal{E})$.*

This Lemma is a simple consequence of the previous discussion and we leave it for the reader as an exercise.

4 Livšic theory

As in the previous section, we consider the case of a smooth manifold $\mathcal{M}$ endowed with an Anosov vector field X, and denote by $\mathcal{G}$ the set of periodic orbits. We will also always assume that the flow is *transitive* i.e. there is a dense orbit. For such an Anosov flow, periodic orbits are dense and one can expect that the knowledge of the behaviour of a function (or a more general object) along closed geodesics allows to reconstruct the function on the whole of $\mathcal{M}$ up to some natural obstructions. This is the content of the Livšic theory.

[9] Note that the positivity of Π alternatively follows from (3.11) and Lemma [DZ17, Lemma 2.3].

4.1 Elementary properties of Anosov flows

4.1.1 Shadowing lemmas

We first need to recall some results on periodic orbits for Anosov flows. An integral version of the Anosov property (2.2) is the existence of *strong stable* and *strong unstable* manifolds $W^{s,u}$: given $x \in \mathcal{M}$, there exists two (smooth) immersed submanifolds

$$W^{s,u}(x) := \{y \in \mathcal{M} \mid d(\varphi_t x, \varphi_t y) \to_{t\to\pm+\infty} 0\},$$

whose tangent space at $y \in W^{s,u}(x)$ is given by $E_{s,u}(y)$. We will denote by $W^{s,u}_\varepsilon(x)$ the set of points

$$W^{s,u}_\varepsilon(x) := \{y \in \mathcal{M} \mid \forall \pm t \geq 0, d(\varphi_t x, \varphi_t y) \leq \varepsilon, d(\varphi_t x, \varphi_t y) \to_{t\to\pm+\infty} 0\}.$$

The following Proposition is known as the *Anosov closing lemma*.

Proposition 4.1 (Anosov closing lemma)**.** *There exists constants $C, \theta, T_0 > 0$ such that for $\varepsilon > 0$ small enough, if $x \in \mathcal{M}$ satisfies $d(\varphi_T x, x) < \varepsilon$ for some $T > T_0$, then there exists a periodic point $x_0 \in \mathcal{M}$ of period $T + \tau$, with $\tau \leq C\varepsilon$, such that*

$$\max\left(d(x, x_0), d(\varphi_T x, x_0)\right) < \varepsilon.$$

Moreover, for all $t \in [0, T]$:

$$d(\varphi_t x, \varphi_t p) \leq C\varepsilon e^{-\theta \min(t, T-t)}.$$

Although we isolated it, this closing lemma follows from a more general shadowing lemma which is the content of the following Theorem. We will write $\gamma = [xy]$ if γ is an orbit segment with endpoints x and y.

Theorem 4.2 (Specification Theorem)**.** *There exist $\varepsilon_0, T_*, C, \theta > 0$ with the following property. Consider $\varepsilon < \varepsilon_0$, and a (possibly infinite) sequence of orbit segments $\gamma_i = [x_i y_i]$ of length T_i greater than T_* such that for any n, $d(y_n, x_{n+1}) \leq \varepsilon$. Then there exists a true orbit γ of the flow and times τ_i such that γ restricted to $[\tau_i, \tau_i + T_i]$* shadows *γ_i up to $C\varepsilon$. More precisely, for all $t \in [0, T_i]$, one has*

$$d(\gamma(\tau_i + t), \gamma_i(t)) \leq C\varepsilon e^{-\theta \min(t, T_i - t)}.$$

Moreover:

$$|\tau_{i+1} - (\tau_i + T_i)| \leq C\varepsilon.$$

Eventually, if the sequence of segments γ_i is periodic, then the orbit γ is periodic.

We refer to [KH95, Corollary 18.1.8], [HF, Theorem 5.3.2] and [HF, Proposition 6.2.4] for proofs.

In particular, if γ_0 is an orbit segment $[xy]$ with $d(y,x) \leq \varepsilon_0$, then applying the above theorem to $\gamma_i = \gamma_0$ for all $i \in \mathbb{Z}$, one obtains a periodic orbit that shadows γ_0: this is nothing the *Anosov closing lemma*, see Proposition 4.1.

4.1.2 Homoclinic orbits

We now fix an arbitrary periodic point $x_* \in \mathcal{M}$ of period $T_* > 0$ and denote by γ_* its orbit (primitive) orbit.

Definition 4.3 (Homoclinic orbits)**.** A point $p \in \mathcal{M}$ is said to be *homoclinic* to x_* if $p \in W^{ws}(x_*) \cap W^{wu}(x_*)$ (in other words, there exists $t_0^{\pm} \in \mathbb{R}$ such that $d(\varphi_{t+t_0^{\pm}}p, \varphi_t x_*) \to_{t\to\pm\infty} 0$). We say that an orbit γ is homoclinic to x_* if it contains a point $p \in \gamma$ that is homoclinic to x_* and we denote by $\mathcal{H} \subset \mathcal{M}$ the set of homoclinic orbits to x_*.

Note that due to the hyperbolicity, the convergence of the point p to x_* is exponentially fast. More precisely, let γ be the orbit of p and let $\mathbb{R} \ni t \mapsto \gamma(t)$ be a parametrization of γ by unit speed. Then, there exists uniform constants $C, \lambda > 0$ (independent of γ) and $A_\pm \in \mathbb{R}$ (depending on γ) such that the following holds:

$$d(\gamma(A_\pm \pm nT_*), x_*) \leq Ce^{-\lambda n}. \tag{4.1}$$

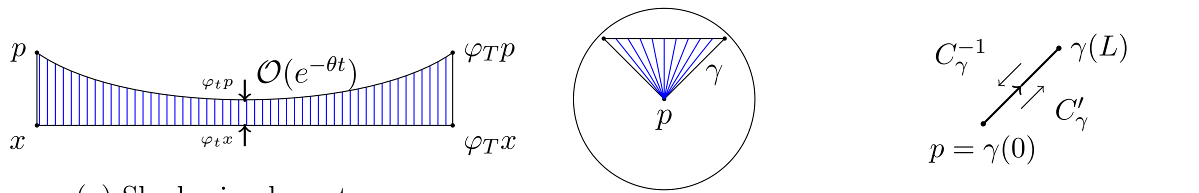
We introduce $x_n^\pm := \gamma(A_\pm \pm nT_*)$. Note that for each $\gamma \in \mathcal{H}$, a non-canonical choice is made in order to define the points $x_n^\pm$, that is $x_0^\pm$ corresponds to an (aritrary) choice of points in $W^{s,u}_{\mathrm{loc}}(x_*) \cap \gamma$. Homoclinic orbits will play a very important role in the following as we shall see in due course.

Lemma 4.4. *Assume that the flow is transitive. Then the set $\mathcal{W}$ of points belonging to a homoclinic orbit in $\mathcal{H}$ is dense in $\mathcal{M}$.*

Proof. This is a straightforward consequence of the shadowing Theorem 4.2 as one can concatenate a long segment S of a transitive orbit with γ_*, namely one applies Theorem 4.2 with $...\gamma_*\gamma_* S \gamma_*\gamma_*....$ □

It can be shown that in the case of an Anosov geodesic vector field, the set of homoclinic orbits is in one-to-one correspondance with $\pi_1(M)/\langle\widetilde{\gamma}_*\rangle$, where $\widetilde{\gamma}_* \in \pi_1(M)$ is any element whose conjugacy class corresponds to the free homotopy class of γ_*.

Exercise 4.5. Prove this claim.



(a) Shadowing homotopy. (b) Radial homotopy. (c) Straight-line homotopy.

Figure 6: Diagrammatic presentation of the geometries considered in Lemma 4.6.

4.1.3 Applications of the Ambrose-Singer formula

If $x, y \in \mathcal{M}$ are at a distance strictly less than the injectivity radius of $\mathcal{M}$ divided by 2, we denote by $C_{x\to y} : \mathcal{E}_x \to \mathcal{E}_y$ the parallel transport with respect to $\nabla^{\mathcal{E}}$ along the shortest geodesic from x to y, by $C(x,t) : \mathcal{E}_x \to \mathcal{E}_{\varphi_t x}$ the parallel transport along the flow and by C_γ the parallel transport along a curve γ.

Lemma 4.6. *We record the following consequences of the Ambrose-Singer formula:*

1. *Assume we are in the setting of Theorem 4.2: for some $C, \varepsilon, T > 0$, let $x, p \in \mathcal{M}$ satisfy $d(\varphi_t x, \varphi_t p) \leq C\varepsilon e^{-\theta \min(t, T-t)}$ for all $t \in [0,T]$. Then for any $0 \leq T_1 \leq T$, we have:*

$$\|C(\varphi_{T_1} x, -T_1) C_{\varphi_{T_1} p \to \varphi_{T_1} x} C(p, T_1) C_{x\to p} - \mathbb{1}_{\mathcal{E}_x}\|_x \leq \frac{c_0 C\varepsilon}{\theta} \times \|F_\nabla\|_{C^0},$$

where $c_0 = c_0(X, g) > 0$ depends only on the flow X and the metric.

2. *Assume $\gamma \subset B(p, \imath/2)$ is a closed piecewise smooth curve at p of length L. Then for some $C = C(g) > 0$ depending on the metric:*

$$\|C_\gamma - \mathrm{id}_p\|_p \leq CL \times \sup_{x\in\gamma} d(p,y) \times \|F_\nabla\|_{C^0}.$$

3. *Let $\gamma : [0, L] \to M$ be a unit speed curve based at p, and ∇' be a second unitary connection on $\mathcal{E}$, whose parallel transport along γ we denote by C'_γ. Then:*

$$\|C_\gamma^{-1} C'_\gamma - \mathrm{id}_p\|_p \leq L \times \|\nabla - \nabla'\|_{C^0}.$$

The geometries appearing in (1), (2) and (3) are depicted in Figure 6 (A), (B) and (C), respectively. The proof is left as an exercise for the reader and is a straightforward consequence of the Ambrose-Singer formula, see Lemma 2.7.

Exercise 4.7. Prove Lemma 4.6.

We also have the following result to which we will refer to as the *spiral Lemma*:

Lemma 4.8. *Let $x_* \in \mathcal{M}$ be a periodic point of period T_* and let $x_0 \in W^s_{\mathrm{loc}}(x)$. Define $x_n := \varphi_{nT_*}x_0$. Then:*

$$\lim_{n\to+\infty} C(x_*, nT_*)^{-1} C_{x_n\to x_*} C(x_0, nT_*) C_{x_*\to x_0} \in \mathrm{U}(\mathcal{E}_{x_*})$$

exists.

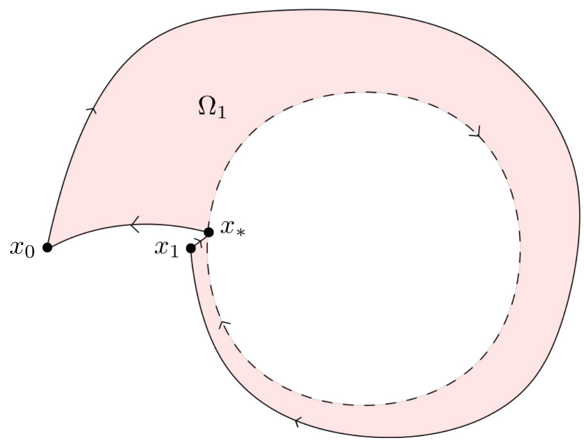


Figure 7: The spiral Lemma: the set Ω_1 corresponds to the area over which the integral in the Ambrose-Singer formula is computed for $n = 1$.

Proof. We write $q_n := C(x_*, nT_*)^{-1} C_{x_n\to x_*} C(x_0, nT_*) C_{x_*\to x_0}$ and apply the Ambrose-Singer formula (Lemma 2.7) as in the first item of the previous Lemma:

$$q_n - \mathbb{1}_{\mathcal{E}_{x_0}} = \int_0^{nT_*} \int_0^{\ell(t)} C_\uparrow^{-1}(s,t) F_\nabla(\partial_t \tau_t(s), \partial_s \tau_t(s)) C_\to(s,t)\, ds\, dt,$$

where τ_t is the unit speed shortest geodesic of length $\ell(t)$ from $\varphi_t x_0$ and $\varphi_t x_*$. Observe that this integral converges absolutely as:

$$\int_0^{nT_*} \| \int_0^{\ell(t)} C_\uparrow^{-1}(s,t) F_\nabla(\partial_t \tau_t(s), \partial_s \tau_t(s)) C_\to(s,t)\, \mathrm{d}s \,\| \mathrm{d}t \leq \int_0^{nT_*} C \|F_\nabla\|_{C^0} e^{-\lambda t} \mathrm{d}t < \infty,$$

and thus the limit exists. □

4.2 Abelian and non-Abelian X-ray transform

4.2.1 A preliminary regularity result

As usual in Livšic theory, one is usually able to construct solutions to some transport equations (called *cohomological equations*) with a certain Hölder or Lipschitz-continuous regularity. It is then quite difficult to bootstrap the regularity of the solution and to show that it is actually smooth. The historical argument for that is based on the Journé Lemma [Jou86]; a more analytic approach can be found in [dlL01] and a microlocal approach based on *radial source estimates* was recently developed in [GL]. We will use the following:

Theorem 4.9 (Bonthonneau-L. '21). *Assume X is an Anosov flow and let $\nabla^{\mathcal{E}}$ be a unitary connection on the smooth vector bundle $\mathcal{E} \to \mathcal{M}$. If $\nabla^{\mathcal{E}}_X u = f$ with $f \in C^\infty(\mathcal{M}, \mathcal{E})$ and $u \in C^\alpha(\mathcal{M}, \mathcal{E})$ for some exponent $\alpha > 0$, then u is smooth.*

We say that the L^∞-threshold associated to the operator $\nabla^{\mathcal{E}}_X$ is equal to 0. The statement actually works more generally for *Lie derivatives* acting on vector bundles, namely $\mathbf{X} : C^\infty(\mathcal{M}, \mathcal{E}) \to C^\infty(\mathcal{M}, \mathcal{E})$ is said to be a Lie derivative if it satisfies: for all $f \in C^\infty(\mathcal{M}), u \in C^\infty(\mathcal{M}, \mathcal{E})$,

$$\mathbf{X}(fu) = (Xf)u + f\mathbf{X}u.$$

If the norm of the propagator is uniformly bounded $\|e^{t\mathbf{X}}\|_{L^\infty \to L^\infty} \leq 1$, the same regularity statement as Theorem 4.9 holds *verbatim* and the L^∞-threshold is 0. More general results can be obtained even when the operator has exponentially increasing propagator norm (such as the Lie derivative $\mathcal{L}_X$ acting on the bundle $T^*\mathcal{M} \to \mathcal{M}$ for instance) but it then makes a positive threshold appear, see [GL, Theorem 1.1].

4.2.2 Exact Abelian Livšic Theorem

The usual Abelian X-ray transform consists in integrating continuous (or Hölder-continuous) functions along closed geodesics.

Definition 4.10. We define the X-ray transform $I : C^0(\mathcal{M}) \to \ell^\infty(\mathcal{G})$ by:

$$If : \mathcal{G} \ni \gamma \mapsto \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t(x)) \mathrm{d}t,$$

where $\ell(\gamma)$ is the period of $\gamma \in \mathcal{G}$ and $x \in \gamma$ is an arbitrary point.

It is straightforward that any function of the form $f = Xu$, for u sufficiently regular is in the kernel of I. The celebrated Livšic Theorem caracterizes the kernel of the X-ray transform:

Theorem 4.11 (Livšic '72, De La Llave-Marco-Moriyon '86). *If $f \in C^\alpha(\mathcal{M})$ for some $\alpha \in (0,1) \cup \mathbb{N} \cup \{+\infty\}$ and $If = 0$, then there exists $u \in C^\alpha(\mathcal{M})$ such that $f = Xu$.*

For $\alpha \in (0,1)$ (i.e. in Hölder regularity), the original proof can be found in the paper of Livšic [Liv72]. We also refer to the proof of Guillemin-Kazhdan [GK80a, Appendix] and to

[KH95, Theorem 19.2.4]. The idea is to define u as the integral of f over a dense orbit in the manifold and then to compute the Hölder regularity. The hardest part of the previous statement is to prove that u is more regular than Hölder continuous when f is smoother: this was originally proved in [dlLMM86] and we will conclude thanks to Theorem 4.9.

Proof. We first deal with the Hölder case, that is we assume $f \in C^\alpha(\mathcal{M})$, $\alpha \in (0,1)$. We consider a point x_0 whose orbit $\mathcal{O}(x_0)$ is dense in $\mathcal{M}$ and we define

$$u(\varphi_t x_0) := \int_0^t f(\varphi_s x_0) ds,$$

(remark that $Xu = f$ on $\mathcal{O}(x_0)$ by construction). Let us prove that u is C^α on $\mathcal{O}(x_0)$. We pick $x, y \in \mathcal{O}(x_0)$ such that $d(x,y) < \varepsilon_0$ (in particular, the Anosov closing lemma of Proposition 4.1 is satisfied at this scale). We write $x = \varphi_t x_0, y = \varphi_{t+T} x_0$ and we assume that $T \geq T_*$ which is always possible since the orbit is dense. Let p be the periodic point of period $T + \tau$ (with $|\tau| \leq C d(x,y)$) closing the segment of orbit $[xy]$. We have:

$$\begin{aligned} u(x) - u(y) &= \int_0^T f(\varphi_s x) ds \\ &= \underbrace{\int_0^T f(\varphi_s x) - f(\varphi_s p) ds}_{=(\mathrm{I})} + \underbrace{\int_0^{T+\tau} f(\varphi_s p) ds}_{=(\mathrm{II})} - \underbrace{\int_T^{T+\tau} f(\varphi_s p) ds}_{=(\mathrm{III})} \end{aligned}$$

And:

$$|(\mathrm{I})| \leq \int_0^T \|f\|_{C^\alpha} d(\varphi_s x, \varphi_s p)^\alpha ds \leq C \|f\|_{C^\alpha} d(x,y)^\alpha \int_0^T e^{-\alpha\theta \min(s, T-s)} ds \lesssim d(x,y)^\alpha$$

By hypothesis, we know that $(\mathrm{II}) = 0$. And $|(\mathrm{III})| \leq \|f\|_\infty |\tau| \lesssim d(x,y)$. As a consequence, u is C^α on $\mathcal{O}(x_0)$ (and its C^α norm is controlled by that of f). Since $\mathcal{O}(x_0)$ is dense in M, u admits a unique C^α-extension to M and it satisfies $Xu = f$. Now if we further assume that f is smooth, we then obtain by Theorem 4.9 that u is smooth. □

4.2.3 Approximate Abelian Livšic Theorem

It is also possible to prove a positive version of the Livšic theorem (see [LT05]) i.e. if $If \geq 0$, then f is *cohomologous* to a positive function i.e. there exists a function u and h such that $f = Xu + h$ and $h \geq 0$. In the following, we will rather need an *approximate version of the Livšic theorem* proved in [GL19a]:

Theorem 4.12 (Gouëzel-L. '19)**.** *There exists $C, \tau, \alpha > 0$ such that the following holds:*

assume that $f \in C^1(\mathcal{M})$ *and* $\|f\|_{C^1} \leq 1$ *and*

$$\sup_{\gamma \in \mathcal{G}} |If(\gamma)| < \varepsilon,$$

for some $\varepsilon > 0$ *small enough. Then, there exists* $u, h \in C^\alpha(\mathcal{M})$ *such that* $f = Xu + h$ *and* $\|h\|_{C^\alpha} \leq \varepsilon^\tau$*.*

The idea of proof goes as follows: first of all, one constructs a specific orbit, of controlled length $\mathcal{O}(\varepsilon^{-1/2})$ which is sufficiently dense in the manifold (i.e. ε^{β_1} dense) and sufficiently separated (i.e. a transverse disk to the orbit of size $\sim \varepsilon^{\beta_2}$ does not hit another portion of the orbit). Once one has this good orbit, one can more or less follow the proof of the exact Livšic Theorem. Note that, in contrast to the exact Livšic Theorem, it is not clear yet if a smoother version of the approximate Livšic Theorem exists:

Question 4.13. Assume that $f \in C^k(\mathcal{M})$, $\|f\|_{C^k} \leq 1$ (for some $k \geq 0$) and

$$\sup_{\gamma \in \mathcal{G}} |If(\gamma)| < \varepsilon,$$

for some $\varepsilon > 0$ small enough. Is it then possible to decompose $f = Xu+h$ with $\|h\|_{C^k} \leq \varepsilon^\tau$?

Proof of Theorem 4.12. The following lemma states that we can find a sufficiently dense and yet separated orbit in the manifold $\mathcal{M}$. The separation holds transversally to the flow direction, and is defined as follows. We introduce

$$W_\varepsilon(x) := \bigcup_{y \in W^u_\varepsilon(x)} W^s_\varepsilon(x).$$

We then say that a set S is ε*-transversally separated* if, for any $x \in S$, we have $S \cap W_\varepsilon(x) = \{x\}$.

Lemma 4.14. *There exist* $\beta_s, \beta_d > 0$ *such that the following holds. Let* $\varepsilon > 0$ *be small enough. There exists a periodic orbit* $\mathcal{O}(x_0) := (\varphi_t x_0)_{0 \leq t \leq T}$ *with* $T \leq \varepsilon^{-1/2}$ *such that this orbit is* ε^{β_s}*-transversally separated and* $(\varphi_t x_0)_{0 \leq t \leq T-1}$ *is* ε^{β_d}*-dense. If* $\kappa > 0$ *is some fixed constant, then one can also require that there exists a piece of* $\mathcal{O}(x_0)$ *of length* $\leq C(\kappa)$ *which is* κ*-dense in the manifold.*

This Lemma is the cornerstone of the argument. Since it is technical, we do not intend to prove it here and refer to [GL19a, Lemma 3.4] for a proof. It uses the specification Theorem 4.2.

We can always assume that ε is small enough (i.e. $\varepsilon \leq \varepsilon_0$) to apply Lemma 4.14, with

$\kappa = \varepsilon_0$. On the orbit $\mathcal{O}(x_0)$ given by this lemma, we define $\tilde{u}$ by

$$\tilde{u}(\varphi_t x_0) = \int_0^t f(\varphi_s x_0)\mathrm{d}s.$$

Since it may not be continuous at x_0, we will rather denote by $\mathcal{O}(x_0)$ the set $(\varphi_t x_0)_{0\leq t\leq T-1}$.

Lemma 4.15. *There exist $\beta_1, C > 0$ independent of ε such that $\|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))} \leq C$.*

Proof. We first study the Hölder regularity of $\tilde{u}$, namely we want to control $|\tilde{u}(x) - \tilde{u}(y)|$ by $Cd(x,y)^{\beta_1}$ for some well-chosen exponent β_1, when $d(x,y) \leq \varepsilon_0$ (where ε_0 is the scale under which the shadowing Theorem 4.2 holds). If x and y are on the same local flow line, then the result is obvious since f is bounded by 1, so we are left to prove that $\tilde{u}$ is transversally C^{β_1}. Consider $x = \varphi_{t_0} x_0 \in \mathcal{O}(x_0)$ and $y = \varphi_{t_0+t} \in W_{\varepsilon_0}(x)$. By transversal separation of $\mathcal{O}(x_0)$, these points satisfy $d(x,y) \geq \varepsilon^{\beta_s}$. We can close the segment $[xy]$ i.e., we can find a periodic point p such that $d(p,x) \leq Cd(x,y)$ with period $t_p = t + \tau$, where $|\tau| \leq Cd(x,y)$ which shadows the segment. Then:

$$|\tilde{u}(y) - \tilde{u}(x)| \leq \underbrace{\left| \int_0^t f(\varphi_s x) ds - \int_0^{t_p} f(\varphi_s p) ds \right|}_{=\text{(I)}} + \underbrace{\left| \int_0^{t_p} f(\varphi_s p) ds \right|}_{=\text{(II)}}$$

The first term (I) is bounded by $Cd(x,y)^{\beta_1'}$ for some $\beta_1' > 0$ depending on the dynamics, whereas the second term (II) is bounded — by assumption — by εt_p, as in the proof of the usual Livšic Theorem 4.11. But $\varepsilon t_p \lesssim \varepsilon t \lesssim \varepsilon T \lesssim \varepsilon^{1/2} \lesssim d(x,y)^{1/2\beta_s}$. We thus obtain the sought result with $\beta_1 := \min(\beta_1', 1/2\beta_s)$.

We now prove that $\tilde{u}$ is bounded for the C^0-norm. We know that there exists a segment of the orbit $\mathcal{O}(x_0)$ — call it S — of length $\leq C$ which is ε_0-dense in $\mathcal{M}$. In particular, for any $x \in \mathcal{O}(x_0)$, there exists $x_S \in S$ with $d(x, x_S) \leq \varepsilon_0$, and therefore $|\tilde{u}(x) - \tilde{u}(x_S)| \leq Cd(x,x_S)^{\beta_1} \leq C\varepsilon_0^{\beta_1}$ thanks to the Hölder control of the previous paragraph. Using the same argument with x_0, we get as $\tilde{u}(x_0) = 0$

$$|\tilde{u}(x)| = |\tilde{u}(x) - \tilde{u}(x_0)| \leq |\tilde{u}(x) - \tilde{u}(x_S)| + |\tilde{u}(x_S) - \tilde{u}((x_0)_S)| + |\tilde{u}(x_0) - \tilde{u}((x_0)_S)|.$$

The first and last term are bounded by $C\varepsilon_0^{\beta_1}$, and the middle one is bounded by C as S has a bounded length and $\|f\|_{C^0} \leq 1$. □

We now cover the manifold $\mathcal{M}$ by a finite union of flowboxes $\mathcal{U}_i := \cup_{t\in(-\delta,\delta)}\varphi_t(\Sigma_i)$ (of some small $\delta > 0$), where $\Sigma_i := W_{\varepsilon_0}(x_i)$ and $x_i \in \mathcal{M}$. For each i, we extend the function $\tilde{u}$ (defined on $\mathcal{O}(x_0)$) to a Hölder function u_i on Σ_i, by the formula $u_i(x) = \sup_y \tilde{u}(y) - \|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))} d(x,y)^{\beta_1}$, where the supremum is taken over all $y \in \mathcal{O}(x_0)$. With this formula, it is classical that the extension is Hölder continuous, with $\|u_i\|_{C^{\beta_1}(\Sigma_i)} \leq$

$\|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))}$. We then push the function u_i by the flow in order to define it on $\mathcal{U}_i$ by setting for $x \in \Sigma_i, \varphi_t x \in \mathcal{U}_i$:

$$u_i(\varphi_t x) = u_i(x) + \int_0^t f(\varphi_s x) ds.$$

Note that the extension is still Hölder with the same regularity. We now set $u := \sum_i u_i \theta_i$ and $h := f - Xu = -\sum_i u_i X\theta_i$. The functions $X\theta_i$ are uniformly bounded in C^∞, independently of ε so the functions $u_i X\theta_i$ are in C^{β_1} with a Hölder norm independent of $\varepsilon > 0$ and thus $\|h\|_{C^{\beta_1}} \leq C$.

Lemma 4.16. $\|h\|_{C^{\beta_1/2}} \leq \varepsilon^{\beta_3/2}$

Proof. We claim that h vanishes on $\mathcal{O}(x_0)$: indeed, on $\mathcal{U}_i \cap \mathcal{O}(x_0)$ one has $u_i \equiv \tilde{u}$ and thus

$$h = -\tilde{u} \sum_i X\theta_i = -\tilde{u} X \sum_i \theta_i = -\tilde{u} X \mathbf{1} = 0.$$

Since $\mathcal{O}(x_0)$ is ε^{β_d}-dense and $\|h\|_{C^{\beta_1}} \leq C$, we get that $\|h\|_{C^0} \leq C\varepsilon^{\beta_1\beta_d} = C\varepsilon^{\beta_3}$, where $\beta_3 = \beta_1\beta_d$. By interpolation, we eventually obtain that $\|h\|_{C^{\beta_1/2}} \leq \varepsilon^{\beta_3/2}$. □

The previous lemma provides the desired estimate on the remainder h and completes the proof of Theorem 4.12. □

4.2.4 Livšic theory for cocycles

This paragraph is a particular case of the discussion of the following paragraph relative to connections. Let G be a Lie group. We consider a smooth cocycle $C : \mathcal{M} \times \mathbb{R} \to G$ over the flow $(\varphi_t)_{t\in\mathbb{R}}$ generated by X i.e. a map satisfying:

$$C(\varphi_s x, t)C(x, s) = C(x, s + t),$$

for all $x \in \mathcal{M}, s, t \in \mathbb{R}$. Its infinitesimal generator is defined to be

$$f(x) := \frac{\mathrm{d}}{\mathrm{d}t} C(x, t)|_{t=0} \in C^\infty(\mathcal{M}, \mathfrak{g}),$$

and C can be recovered from f as the unique solution the following ODE:

$$C(x, 0) = \mathbf{1}_G, \qquad \frac{\mathrm{d}}{\mathrm{d}t} C(x, t) = \mathrm{d}R_{C(x,t)}(f(\varphi_t(x))),$$

where e_G denotes the neutral element in G and R_g is the multiplication on the right by $g \in G$. As we shall see below in §7, a typical example of a cocycle is provided by parallel transport of sections of a vector bundle $\mathcal{E} \to \mathcal{M}$ along the flowlines of X, and with respect to a connection $\nabla^{\mathcal{E}}$. In the particular case where $\mathcal{E} = \mathbb{C}^r \times \mathcal{M}$ is trivial (of rank r) and the connection is unitary, the parallel transport is indeed a cocycle $C : \mathcal{M} \times \mathbb{R} \to G$, where $G = \mathrm{U}(r)$, the group of unitary matrices. We now introduce the *periodic orbit obstruction*:

Definition 4.17. We say that C satisfies the *periodic orbit obstruction* if $C(x, T) = e_G$ for any periodic point $x \in \mathcal{M}$ (where T denotes the period of x).

The previous X-ray transform of $f \in C^0(\mathcal{M})$ can be integrated in this framework by considering the cocycle:

$$C(x,t) := \exp\left(\int_0^t f(\varphi_{-s}(x))\, \mathrm{d}s\right).$$

Then $If = 0$ if and only if C satisfies the periodic orbit obstruction in the Lie group $(\mathbb{R}_+^*, \times)$. There is a generalization of Livsic's Theorem to this framework, due to [Liv72, NT98], which we will call the *Livšic cocycle Theorem* or *non-Abelian Livšic Theorem*:

Theorem 4.18 (Livšic '72, Nitica-Torok '98)**.** *Let G be a compact Lie group, let $C : \mathcal{M} \times \mathbb{R} \to G$ be a α-Hölder continuous cocycle which satisfies the periodic orbit obstruction. Then C is cohomologically trivial, i.e. there exists $u \in C^\alpha(\mathcal{M}, G)$ such that*

$$C(x,t) = u(\varphi_t x) u(x)^{-1},$$

for all $x \in \mathcal{M}, t \in \mathbb{R}$. Moreover, if C is smooth, then u is also smooth.

The same proof as that of Theorem 4.11 can be mimicked in order to deal with the case of Hölder regularity, see [MP, Chapter 6.2] for further details; proving that u is smooth can then be achieved thanks to Theorem 4.9 from [GL]. We leave the proof as an exercise for the reader.

Exercise 4.19. Complete the proof of Theorem 4.18.

Let us also mention here that non-Abelian Livšic theory is a very vast topic which is still being investigated. The literature is mostly concerned with the discrete-time case, namely hyperbolic (or Anosov) diffeomorphisms or even partially hyperbolic diffeomorphisms: most papers are concerned with the study of cocycles with values in a non-compact Lie group (and sometimes satisfying a "slow-growth" assumption) such as the group of diffeomorphisms $G = \mathrm{Diff}(N)$, where N is another closed manifold, see [dlLW10, Kal11, Sad13, Sad17, AKL18]. This is based on the so-called *Pesin theory*[10]. Some other articles (such as [NT95, NT98, NP99, Wal00, PW01]) are concerned with regularity issues on the map u, namely bootstrapping its regularity under some weak *a priori* assumption (such as measurability only).

[10]See https://encyclopediaofmath.org/wiki/Pesin_theory for a broad overview.

As in the case of the Abelian Livšic Theorem, one can also prove an approximate version of the Livšic cocycle Theorem following the arguments of the approximate Livšic theorem [GL19a]:

Theorem 4.20 (Cekic-L. '20). *Let G be a compact Lie group, let $C : \mathcal{M} \times \mathbb{R} \to G$ be a α-Hölder continuous cocycle. Assume that*

$$d_G(C(x,T), \mathbf{1}_G) \leq \varepsilon T,$$

for all periodic point $x \in \mathcal{M}$ (where T is the period of x), where $\varepsilon > 0$ is small enough. Then, there exists $u \in C^\beta(\mathcal{M}, G)$ (where $0 < \beta \leq \alpha$ only depends on the vector field X and on α) and a β-Hölder continuous cocycle $C' : \mathcal{M} \times \mathbb{R} \to G$ such that:

$$C(x,t) = u(\varphi_t x) C'(x,t) u(x)^{-1},$$

and C' is generated by $f' \in C^\beta(\mathcal{M}, \mathfrak{g})$ such that:

$$\|f'\|_{C^\beta(\mathcal{M},\mathfrak{g})} \leq \varepsilon^\tau.$$

Here $\tau > 0$ only depends on the flow.

As in the Abelian case, it is not clear yet if this Theorem holds in higher regularity, namely if C is smooth (or bounded in some C^k regularity), can one show that $\|f'\|_{C^k} \leq C\varepsilon^\tau$? It could also be interested to deal with the case of a general Lie group G.

4.3 Advanced Livšic theory

This paragraph intends to give more precise results than Theorem 4.18 in the specific case where the cocycle C is given by the parallel transport with respect to some arbitrary unitary connection along the flowlines of an Anosov flow. In the following, we will denote by $\mathcal{G}$ the set of all periodic orbits for the flow and by $\mathcal{G}^\sharp$ the set of all *primitive* orbits, namely orbits which cannot be written as a shorter orbit to some positive power greater or equal than 2.

4.3.1 Trace-equivalent cocycles

We consider $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$, two distinct Hermitian vector bundles equipped with the respective connections $\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2}$, then the connections are said to be *equivalent* (if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, we will rather say that the connections are *gauge-equivalent*) if there exists $p \in C^\infty(\mathcal{M}, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$ such that $\nabla^{\mathcal{E}_1} = p_* \nabla^{\mathcal{E}_2} = p \nabla^{\mathcal{E}_2}(p^{-1} \cdot)$. In this case, parallel transport along the flowlines of $(\varphi_t)_{t\in\mathbb{R}}$ satisfies the commutation relation:

$$C_1(x,t) = p(\varphi_t x)C_2(x,t)p(x)^{-1}. \tag{4.2}$$

We say that cocycles satisfying (4.2) are *cohomologous*. In particular, the holonomies of the connections along closed orbits are *conjugate*. One can also wonder if the converse holds true, namely: if the parallel transport maps are conjugate along closed orbits, are the cocycles C_1 and C_2 cohomologous as in (4.2)? To formalize this, let us introduce the following definition:

Definition 4.21. We say that the connections $\nabla^{\mathcal{E}_{1,2}}$ have *trace-equivalent holonomies* if for all *primitive* closed orbits $\gamma \in \mathcal{G}^\sharp$, there exists $x_\gamma \in \gamma$ such that:

$$\operatorname{Tr}\left(C_1(x_\gamma, T)\right) = \operatorname{Tr}\left(C_2(x_\gamma, T)\right), \tag{4.3}$$

where T is the period of γ.

It can be easily checked that this definition is independent of the choice of basepoint x_γ. Also note that, replacing primitive orbits in the previous definition by *all orbits* $\gamma \in \mathcal{G}$ (i.e. all positive powers of the primitive orbits), we obtain all the traces $\operatorname{Tr}\left(C(x_\gamma,T)^n\right)$ for $n \geq 0$, along all primitive orbits. It is then an exercise to check that this data is equivalent to the data of the conjugacy class of $C(x_\gamma, T)$.

Exercise 4.22. Let $A, B \in \mathrm{U}(r)$. Assume that $\operatorname{Tr}(A^n) = \operatorname{Tr}(B^n)$ for all $n \geq 0$. Show that there exists $p \in \mathrm{U}(r)$ such that $A = pBp^{-1}$.

We will prove the following, obtained in [CLa]:

Theorem 4.23. *Let $\mathcal{M}$ be a smooth manifold endowed with a smooth transitive Anosov flow. Let $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$ be two Hermitian vector bundles over $\mathcal{M}$ equipped with respective unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. If the connections have trace-equivalent holonomies in the sense of Definition 4.21, then there exists $p \in C^\infty(\mathcal{M}, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$ such that: for all $x \in \mathcal{M}, t \in \mathbb{R}$,*

$$C_1(x,t) = p(\varphi_t x)C_2(x,t)p(x)^{-1}, \tag{4.4}$$

i.e. the cocycles induced by parallel transport are cohomologous.

Theorem 4.23 improves known results on Livšic cocycle theory [Liv72, Par99, Sch99]. Surprisingly, the rather weak condition (4.3) implies in particular that the bundles are isomorphic:

Corollary 4.24. *Assume $\mathcal{M}$ is endowed with a smooth transitive Anosov flow. Let $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$ be two Hermitian vector bundles over $\mathcal{M}$ equipped with respective unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}}_2$. If the connections are trace-equivalent in the sense of Definition (4.21), then $\mathcal{E}_1 \simeq \mathcal{E}_2$, namely the vector bundles are isomorphic.*

The idea relies on a key notion introduced by Parry [Par99] which we call *Parry's free monoid* and denote by $\mathbf{G}$. Recall that a monoid is an algebraic structure with a neutral element $\mathbf{1}_{\mathbf{G}}$, an associative law but without inverses (which is the only difference with a *group*), see [Lan02] for the algebraic background. The monoid we will consider will be the formal free monoid whose elements are homoclinic orbits. Note that one can always enforce the group structure by formally adding inverses but this will not be of any interest; moreover, this would not have any dynamical meaning. One can then think of the "multiplication" of two homoclinic orbits as the concatenation of these (although this is only formal and cannot really happen at the dynamical level). We will prove that a unitary connection $\nabla^{\mathcal{E}}$ induces a (family of) unitary representation

$$\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_*}),$$

where x_* is some arbitrary periodic point. Recall that the *character* of the representation ρ is defined as $\chi_\rho := \mathrm{Tr}(\rho(\bullet))$. A lot of geometric information can be read off this representation: they are detailed in subsequent paragraphs, see §4.3.3 and §4.3.4. In particular, Theorem 4.23 will be obtained thanks to the following observation: if two unitary connections are trace-equivalent in the sense of Definition 4.21, then they have same character. A general statement in algebra (see [Lan02, Corollary 3.8]) then asserts that two unitary and finite-dimensional representations of a monoid with same character must be isomorphic; combining this with some ingredients from hyperbolic dynamical systems, we will eventually obtain Theorem 4.23.

4.3.2 Parry's free monoid

As we shall see, Parry's free monoid is the key notion to understand the holonomy of unitary connections. Recall from §4.1.2 that $x_* \in \mathcal{M}$ is a periodic point of period T_*. Let $\mathbf{G}$ be the free monoid generated by the set of homoclinic orbits $\mathcal{H}$ to x_*, namely the formal set of words

$$\mathbf{G} := \{\gamma_1^{m_1} ... \gamma_k^{m_k} \mid k \in \mathbb{N}, m_1, ..., m_k \in \mathbb{N}, \gamma_1, ..., \gamma_k \in \mathcal{H}\},$$

endowed with the obvious monoid structure. The primitive periodic orbit γ_* corresponding to x_* also belongs to the set of homoclinic orbits and the identity element corresponds to the empty word. We also introduce $\mathbf{G}^* := \mathbf{G} \setminus \left\{\gamma_*^k, k \geq 1\right\}$. We call $\mathbf{G}$ *Parry's free monoid* as the idea (although not written like this) was first introduced in his work [Par99]. The main result of this paragraph is the following:

Proposition 4.25. *Let $\nabla^{\mathcal{E}}$ be a unitary connection on the Hermitian vector bundle $\mathcal{E} \to \mathcal{M}$. Then $\nabla^{\mathcal{E}}$ induces a representation*

$$\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_*}).$$

This could also be stated as a Definition. Actually, the representation ρ is not unique; the representations rather come in family but we shall see below that the properties of the representation do not depend on choices.

Proof. Since $\mathbf{G}$ is a free monoid, it suffices to define ρ on the set of generators of $\mathbf{G}$, namely for all homoclinic orbits $\gamma \in \mathcal{H}$. For the neutral element $\mathbf{1_G}$, we set $\rho(\mathbf{1_G}) = \mathbb{1}_{\mathcal{E}_{x_*}}$. For the periodic orbit γ_* of x_*, we set $\rho(\gamma_*) := C(x_*, T_*)$.

Let $\gamma \in \mathcal{H}$ (and $\gamma \neq \gamma_*$) and consider a parametrization $\mathbb{R} \ni t \mapsto \gamma(t)$. Following the notations of §4.1.2, we let $x_n^{\pm} := \gamma(A_\pm \pm nT_*), x_n^+ = \varphi_{T_n}(x_n^-)$ for some $T_n < 0$, and the points $(x_n^{\pm})_{n\in\mathbb{N}}$ converge exponentially fast to x_0 as $n \to \infty$. As we shall see, there is a small technical issue coming from the fact that $C(x_*, T_*)$ is not trivial and this can be overcome by considering subsequences $k_n \to \infty$ such that $C(x_*, T_*)^{k_n} \to \mathbb{1}_{\mathcal{E}_{x_*}}$[11].

We define $\rho_n(\gamma) \in \mathrm{U}(\mathcal{E}_{x_*})$ as follows:

$$\rho_n(\gamma) := C_{x^+_{k_n} \to x_*} C(x^-_{k_n}, T_n) C_{x_* \to x^-_{k_n}}.$$

Lemma 4.26. *There exists $\rho(\gamma) \in \mathrm{U}(\mathcal{E}_{x_*})$ such that $\rho_n(\gamma) \to_{n\to\infty} \rho(\gamma)$.*

Proof. We have by construction:

$$\begin{aligned}
\rho_n(\gamma) &= C_{x^+_{k_n} \to x_0} C(x^-_{k_n}, T_n) C_{x_0 \to x^-_{k_n}} \\
&= C_{x^+_{k_n} \to x_*} C(x_0^+, k_n T_*) C(x_0^-, T_\gamma) C(x^-_{k_n}, k_n T_*) C_{x_* \to x^-_{k_n}} \\
&= C_{x^+_{k_n} \to x_*} C(x_0^+, k_n T_*) C_{x_* \to x_0^+} C_{x_0^+ \to x_*} C(x_0^-, T_\gamma) C_{x_* \to x_0^-} C_{x_0^- \to x_*} C(x^-_{k_n}, k_n T_*) C_{x_* \to x^-_{k_n}} \\
&= \left[C_{x^+_{k_n} \to x_*} C(x_0^+, k_n T_*) C_{x_* \to x_0^+} C(x_*, k_n T_*)^{-1} \right] \\
&\qquad \times C(x_*, k_n T_*) C_{x_0^+ \to x_*} C(x_0^-, T_\gamma) C_{x_* \to x_0^-} C(x_*, k_n T_*) \\
&\qquad \times \left[C(x_*, k_n T_*)^{-1} C_{x_0^- \to x_*} C(x^-_{k_n}, k_n T_*) C_{x_* \to x^-_{k_n}} \right],
\end{aligned}$$

where T_γ is independent of n. Observe that the term

$$C(x_*, k_n T_*) C_{x_0^+ \to x_*} C(x_0^-, T_\gamma) C_{x_* \to x_0^-} C(x_*, k_n T_*)$$

converges to $C_{x_0^+ \to x_*} C(x_0^-, T_\gamma) C_{x_* \to x_0^-}$ as n goes to $+\infty$, and the convergence of the terms between brackets follow from the spiral Lemma 4.8. ☐

This concludes the proof.

☐

[11] Note that for any compact group G, if $g \in G$, there exists a subsequence $k_n \in \mathbb{N}$ such that $g^{k_n} \to_{n\to\infty} \mathbf{1}_G$.

Recall that the character of a representation ρ is defined by $\chi_\rho := \mathrm{Tr}(\rho(\bullet))$. We now prove the following:

Proposition 4.27. *Let $\nabla^{\mathcal{E}_{1,2}}$ be two unitary connections on the Hermitian vector bundles $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$. Assume that the connections have trace-equivalent holonomies in the sense of Definition 4.21. Then, the induced representations $\rho_{1,2} : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{1,2_{x_*}})$ have the same character. In particular, this implies that they are isomorphic, i.e. there exists $p_* \in \mathrm{U}(\mathcal{E}_{2x_*}, \mathcal{E}_{1x_*})$ such that:*

$$\forall \gamma \in \mathbf{G}, \quad \rho_1(\gamma) = p_* \rho_2(\gamma) p_*^{-1}.$$

In Lemma 4.26, we had to consider a subsequence $(k_n)_{n\in\mathbb{N}}$ such that $C(x_*, T_*)^{k_n} \to \mathbb{1}$. Since we now have two representations, we assume that (up to taking another subsequence) $C_i(x_*, T_*)^{k_n} \to \mathbb{1}$.

Proof. Once we know that the representations have the same character, the conclusion is straightforward consequence of a general fact of representation theory, see [Lan02, Corollary 3.8]. The idea is to apply the shadowing Theorem 4.2 in order to concatenate homoclinic orbits but we will have to be careful so that we only use *primitive* periodic orbits. Let us first prove that the characters coincide on the submonoid $\mathbf{G}^*$. For the sake of simplicity, we take $\gamma = \gamma_1 \cdot \gamma_2 \in \mathbf{G}^*$, where $\gamma_{1,2} \in \mathcal{H}$ but the generalization to longer words is straightforward as we shall see. We have by Lemma 4.26:

$$\rho_1(\gamma) = \rho_1(\gamma_1)\rho_1(\gamma_2) = \rho_{1,n}(\gamma_1)\rho_{1,n}(\gamma_2) + o(1)$$

Let $x_n^\pm(i)$ be the points on the orbit γ_i that are exponentially close to x_*, given by §4.1.2. Consider the concatenation of the orbits $S := [x_{k_n}^-(2) x_{k_n}^+(2)] \ \cup [x_{k_n}^-(1) x_{k_n}^+(1)]$. Note that the starting points and endpoints of these segments are at distance at most $\mathcal{O}(e^{-\lambda k_n})$. Thus by the shadowing Theorem 4.2, there exists a genuine periodic orbit $\widetilde{\gamma}_n$ and a point $y_n \in \widetilde{\gamma}_n$ (of period T_n') which $\mathcal{O}(e^{-\lambda k_n})$-shadows the concatenation S (here, if we have a longer word of length k, it suffices to apply the shadowing Theorem 4.2 with k segments). It is not clear that the shadowing orbit of y_n is primitive: actually, we claim (and leave it as an exercise to the reader) that by a slight modification of the argument, namely by taking a concatenation $[x_{k_n}^-(2) x_{k_n}^+(2)] \ \cup [x_{k_{N(n)}}^-(1) x_{k_n}^+(1)]$ for some subsequence $N(n) \geq n$ large enough, one can ensure that the orbit is primitive. Hence by the first item of Lemma 4.6, we have:

$$\rho_{1,n}(\gamma_1)\rho_{1,n}(\gamma_2) = C_{1,y_n \to x_*} C_1(y_n, T_n') C_{1,y_n \to x_*}^{-1} + \mathcal{O}(e^{-\lambda k_n}).$$

By assumption, we have $\mathrm{Tr}(C_1(y_n,T'_n)) = \mathrm{Tr}(C_2(y_n,T'_n))$. This yields:

$$\begin{aligned}\mathrm{Tr}(\rho_1(\gamma)) &= \mathrm{Tr}(C_{1,y_n\to x_*}C_1(y_n,T'_n)C^{-1}_{1,y_n\to x_*}) + o(1)\\ &= \mathrm{Tr}(C_1(y_n,T'_n)) + o(1)\\ &= \mathrm{Tr}(C_2(y_n,T'_n)) + o(1) = \mathrm{Tr}(\rho_2(\gamma)) + o(1).\end{aligned}$$

This shows that the characters of the representations of the submonoid $\mathbf{G}^*$ are equal. Hence, by [**?**, Corollary 3.8], there exists p_* such that for all $\gamma \in \mathbf{G}^*$, $\rho_1(\gamma) = p_*\rho_2(\gamma)p_*^{-1}$. Observe then that for some arbitrary $\gamma \in \mathbf{G}^* \setminus \{\mathbf{1_G}\}$, we obtain:

$$\rho_1(\gamma_*\gamma) = \rho_1(\gamma_*)\rho_1(\gamma) = p_*\rho_2(\gamma_*\gamma)p_*^{-1} = p_*\rho_2(\gamma_*)p_*^{-1}p_*\rho_2(\gamma)p_*^{-1}.$$

Using that $\rho_1(\gamma) = p_*\rho_2(\gamma)p_*^{-1}$, we get that $\rho_1(\gamma_*) = p_*\rho_2(\gamma_*)p_*^{-1}$. This concludes the proof. □

Exercise 4.28. Show that the shadowing orbit of y_n can be chosen to be primitive.

In particular, this gives that $C_1(x_*,T_*) = p_*C_2(x_*,T_*)p_*^{-1}$, that is $P(x_*,T_*)p_* = p_*$, where P is the parallel transport with respect to the mixed connection. We can now complete the proof of Theorem 4.23.

Proof of Theorem 4.23. Let $\mathcal{W}$ be the set of all points belonging to homoclinic orbits in $\mathcal{H}$. By Lemma 4.4, $\mathcal{W}$ is dense in $\mathcal{M}$ and we are going to define the map p (which will conjugate the cocycles) on $\mathcal{W}$ and then show that p is Lipschitz-continuous on $\mathcal{W}$ so that it extends naturally to $\mathcal{M}$. The map p is defined as the parallel transport of p_* with respect to the mixed connection.

By assumptions, we have $C_i(x_*,T_*)^{k_n} \to \mathbb{1}_{\mathcal{E}_*}$, and thus $P(x_*,T_*)^{k_n} \to \mathbb{1}_{\mathrm{Hom}(\mathcal{E}_{2x_*},\mathcal{E}_{1x_*})}$ (where we use the notation $\mathbb{1}_{\mathrm{Hom}(\mathcal{E}_{2x_*},\mathcal{E}_{1x_*})}(q) = q$ for $q \in \mathrm{Hom}(\mathcal{E}_{2x_*},\mathcal{E}_{1x_*})$). Consider a point $x \in \gamma$, where $\gamma \in \mathcal{H}$ is a homoclinic orbit and also consider a parametrization of γ as in §4.1.2. For $n \in \mathbb{N}$, consider the point $x_n^- \in \gamma$ (which is exponentially close to x_*) and write $x = \varphi_{T_n^-}(x_n^-)$ for some $T_n^- > 0$. Define:

$$p_n^-(x) := P(x_{k_n}^-, T_{k_n}^-)P_{x_*\to x_{k_n}^-}p_* \in \mathrm{U}(\mathcal{E}_{2x},\mathcal{E}_{1x}).$$

Lemma 4.29. *Fix $\gamma \in \mathcal{H}$. Then for all $x \in \gamma$, there exists $p_-(x) \in \mathrm{U}(\mathcal{E}_{2x},\mathcal{E}_{1x})$ such that $p_n^-(x) \to_{n\to\infty} p_-(x)$. Moreover, ${}^{\mathrm{M}}\nabla_X^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}p_- = 0$ on γ.*

In particular, this shows that p_- is smooth in restriction to γ as ${}^{\mathrm{M}}\nabla_X^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}$ is elliptic on γ.

Proof. By construction, the differential equation is clearly satisfied if the limit exists. Moreover, we have for some time T_0 (independent of n, $T_{k_n}^- = T_0 + k_n T_*$):

$$\begin{aligned} p_n^-(x) &= P(x_{k_n}^-, T_{k_n}^-) P_{x_* \to x_{k_n}^-} p_* \\ &= P(x_0^-, T_0) P(x_{k_n}^-, k_n T_*) P_{x_* \to x_{k_n}^-} p_* \\ &= P(x_0^-, T_0) P_{x_* \to x_0} P(x_*, T_*)^{k_n} \left[P(x_*, T_*)^{-k_n} P_{x_0 \to x_*} P(x_{k_n}^-, k_n T_*) P_{x_* \to x_{k_n}^-} p_* \right] \end{aligned}$$

By assumption, the term outside the bracket converges as $n \to \infty$ and the term between brackets converges by the spiral Lemma 4.8. □

We now claim the following:

Lemma 4.30. *There exists a uniform constant $C > 0$ such that the following holds. Assume that x and z belong to two homoclinic orbits in $\mathcal{H}$ and $z \in W^u_{\mathrm{loc}}(x)$. Then:*

$$\| P_{x \to z} p_-(x) - p_-(z) \| \ \leq C d(x, z).$$

By the previous proofs, the point x is associated to points x_n^- on the homoclinic orbit and we will use the same notations for the point z associated to the points z_n^-.

Proof. There is here a slight subtlety coming from the fact that the parametrizations of the homoclinic orbits γ were chosen in a non-canonical way (via a choice of $A_\pm$). In particular, it is not true that the flowlines of $z_{k_n}^-$ and $x_{k_n}^-$ shadow each other; in other words, we might not have $T_{k_n}^-(z) = T_{k_n}^-(x) + \mathcal{O}(d(x,z))$ but we rather have $T_{k_n}^-(z) = T_{k_n}^-(x) + mT_* + \mathcal{O}(d(x,z))$ for some integer $m \in \mathbb{Z}$ depending on both x and z.

We have:

$$\begin{aligned} &\| P_{x \to z} p_-(x) - p_-(z) \| \\ &= \| P_{x \to z} p_n^-(x) - p_n^-(z) \| + o(1) \\ &= \| P_{x \to z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_* \to x_{k_n}^-} p_* - P(z_{k_n}^-, T_{k_n}^-(z)) P_{x_* \to z_{k_n}^-} p_* \| + o(1) \\ &\leq C \| P_{z_{k_n}^- \to x_*} P(z_{k_n}^-, T_{k_n}^-(z))^{-1} P_{x \to z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_* \to x_{k_n}^-} p_* - p_* \| + o(1) \\ &\leq C \| P_{z_{k_n}^- \to x_*} P(z_{k_n}, mT_*)^{-1} P_{x_* \to z_{k_n - m}^-} \\ &\qquad \times \left[P_{z_{k_n-m}^- \to x_*} P(z_{k_n-m}^-, T_{k_n}^-(z) - mT_*)^{-1} P_{x \to z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_* \to x_{k_n}^-} \right] p_* - p_* \| + o(1) \end{aligned}$$

Applying the first item of Lemma 4.6, we have that:

$$\| P_{z_{k_n-m}^- \to x_*} P(z_{k_n-m}^-, T_{k_n}^-(z) - mT_*)^{-1} P_{x \to z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_* \to x_{k_n}^-} - \mathbb{1}_{\mathrm{End}(\mathcal{E}_{x_*})} \| \ \leq C d(x, z),$$

where the constant $C > 0$ is uniform in n. Moreover, observe that

$$\lim_{n\to\infty} P_{z^-_{k_n}\to x_*} P(z_{k_n}, mT_*)^{-1} P_{x_*\to z^-_{k_n-m}} = P(x_*, mT_*)^{-1}.$$

Hence:

$$\|P_{x\to z} p_-(x) - p_-(z)\| \leq C \left(\|P(x_*, mT_*)^{-1} p_* - p_*\| + d(x,z) + o(1)\right)$$

Using that $P(x_*, T_*)p_* = p_*$, we get that the first term on the right-hand side vanishes. Taking the limit as $n \to +\infty$, we obtain the announced result. □

Note that we could have done the same construction "in the future" by considering instead:

$$p_+(x) = \lim_{n\to\infty} P(x, T^+_{k_n})^{-1} P_{x_*\to x^+_{k_n}} p_* \in \mathrm{U}(\mathcal{E}_{2x}, \mathcal{E}_{1x}),$$

where $x^+_n := \varphi_{T^+_n}(x)$ is exponentially closed to x_* as in §4.1.2. A similar statement as Lemma 4.30 holds with the unstable manifold being replaced by the stable one. We have:

Lemma 4.31. *For all $x \in \mathcal{W}$, $p_-(x) = p_+(x)$.*

Proof. This follows from Proposition 4.27. Indeed, we have:

$$\begin{aligned}
\|p_-(x) - p_+(x)\| &= \|P(x^-_{k_n}, T^-_{k_n}) P_{x_*\to x^-_{k_n}} p_* - P(x, T^+_{k_n})^{-1} P_{x_*\to x^+_{k_n}} p_*\| + o(1) \\
&\leq C\|P_{x^+_{k_n}\to x_*} P(x, T^+_{k_n}) P(x^-_{k_n}, T^-_{k_n}) P_{x_*\to x^-_{k_n}} p_* - p_*\| + o(1) \\
&\leq C\|P_{x^+_{k_n}\to x_*} P(x^-_{k_n}, T_{k_n}) P_{x_*\to x^-_{k_n}} p_* - p_*\| + o(1),
\end{aligned}$$

where $T_n := T^-_n + T^+_n$. Observe that:

$$\begin{aligned}
&P_{x^+_{k_n}\to x_*} P(x^-_{k_n}, T_{k_n}) P_{x_*\to x^-_{k_n}} p_* \\
&\qquad = C_{1,x^+_{k_n}\to x_*} C_1(x^-_{k_n}, T_{k_n}) C_{1,x_*\to x^-_{k_n}} p_* \left(C_{2,x^+_{k_n}\to x_*} C_2(x^-_{k_n}, T_{k_n}) C_{2,x_*\to x^-_{k_n}}\right)^{-1} \\
&\qquad = \rho_{1,n}(\gamma) p_* \rho_{2,n}(\gamma)^{-1} = \rho_1(\gamma) p_* \rho_2(\gamma)^{-1} + o(1) = p_* + o(1),
\end{aligned}$$

by Proposition 4.27. Hence $\|p_-(x) - p_+(x)\| = o(1)$, that is $p_-(x) = p_+(x)$. □

We can now prove the following lemma:

Lemma 4.32. *The map p_- is Lipschitz-continuous.*

Proof. Consider $x, y \in \mathcal{W}$ which are close enough. Let $z := [x, y] = W^{wu}_{\mathrm{loc}}(x) \cap W^s_{\mathrm{loc}}(y)$ and define τ such that $\varphi_\tau(z) \in W^u_{\mathrm{loc}}(x)$. Note that $|\tau| \leq Cd(x, y)$ for some uniform constant

$C > 0$; also observe that the point z is homoclinic to the periodic orbit x_*. We have:

$$\begin{aligned}
&\|p_-(x) - P_{y\to x}p_-(y)\| \\
&\leq \|p_-(x) - P_{z\to x}p_-(z)\| + \|p_-(z) - p_+(z)\| + \|P_{z\to x}p_+(z) - P_{y\to x}p_+(y)\| + \|p_+(y) - p_-(y)\| \\
&\leq \|p_-(x) - P_{z\to x}p_-(z)\| + \|P_{x\to y}P_{z\to x}p_+(z) - p_+(y)\| \\
&\leq \|p_-(x) - P_{\varphi_\tau(z)\to x}p_-(\varphi_\tau(z))\| + \|P_{\varphi_\tau(z)\to x}p_-(\varphi_\tau(z)) - P_{z\to x}p_-(z)\| + \|P_{x\to y}P_{z\to x}p_+(z) - p_+(y)\|
\end{aligned}$$

where the terms disappear between the second and third line by Lemma 4.31. By Lemma 4.30, the first term is controlled by:

$$\|p_-(x) - P_{\varphi_\tau(z)\to x}p_-(\varphi_\tau(z))\| \leq Cd(x, \varphi_\tau(z)) \leq Cd(x, y).$$

As to the second term, using the second item of Lemma 4.6, we have:

$$\|P_{\varphi_\tau(z)\to x}p_-(\varphi_\tau(z)) - P_{z\to x}p_-(z)\| = \|P_{x\to z}P_{\varphi_\tau(z)\to x}P(z,\tau)p_-(z) - p_-(z)\| \leq Cd(x, y).$$

Eventually, the last term $\|P_{x\to y}P_{z\to x}p_+(z) - p_+(y)\|$ is controlled similarly to the first term by applying Lemma 4.30 (but with the stable manifold instead of unstable). ☐

As $\mathcal{W}$ is dense, p_- extends to a Lipschitz-continuous map on $\mathcal{M}$ which satisfies the transport equation ${}^{\mathrm{M}}\nabla_X^{\mathrm{End}(\mathcal{E})}p_- = 0$ and by the regularity Theorem 4.9, this implies that p_- is smooth. This concludes the proof of the Theorem.

☐

4.3.3 Transparent connections

Let us introduce the terminology of *transparent* connections:

Definition 4.33. Let $\mathcal{E} \to \mathcal{M}$ be a smooth vector bundle. We say that a connection $\nabla^{\mathcal{E}}$ is *transparent* if $C(x, T) = \mathbb{1}$, for all periodic point $x \in \mathcal{M}$ (where T is the period of x).

A dull example of a transparent connection is given by the trivial bundle $\mathbb{C}^r \times \mathcal{M} \to \mathcal{M}$ equipped with the trivial flat connection d. We shall see in another section that there are non-trivial examples of transparent connections on the unit tangent bundle of Anosov surfaces. Nevertheless, it is still not known if there are examples of transparent connections in dimension ≥ 3. The following result shows that not all vector bundles can carry transparent connections:

Lemma 4.34. *Assume that $\nabla^{\mathcal{E}}$ is transparent. Then $\mathcal{E} \to \mathcal{M}$ is trivial and there exists a global orthonormal basis $f_1, ..., f_r \in C^\infty(\mathcal{M}, \mathcal{E})$ such that $\mathbf{X}f_i = 0$.*

The proof is a direct consequence of Theorem 4.23:

Proof. Indeed, let $\mathcal{E}_1 := \mathcal{E}$ equipped with $\nabla^{\mathcal{E}_1} = \nabla^{\mathcal{E}}$ and let $\mathcal{E}_2 = \varepsilon^r$ be the trivial bundle, where $\varepsilon^r = \mathbb{C}^r \times \mathcal{M}$ is equipped with the trivial flat connection $\nabla^{\mathcal{E}_2} = d$. Then,

$$\mathrm{Tr}(C_1(x_\gamma, T)) = r = \mathrm{Tr}(C_2(x_\gamma, T)).$$

By Corollary 4.24, we obtain that the bundles are isomorphic. Moreover, by Theorem 4.23, there exists $p \in C^\infty(\mathcal{M}, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$ such that $C_1(x,t) = p(\varphi_t x)p(x)^{-1}$. Taking $\mathbf{e}_1, ..., \mathbf{e}_r$ an orthonormal basis of ε^r, we set $f_i := p(\mathbf{e}_i)$. We then have:

$$C_1(x,t)f_i(x) = p(\varphi_t x)p(x)^{-1}p(\mathbf{e}_i) = p(\varphi_t x)\mathbf{e}_i = f_i(\varphi_t x),$$

that is $\mathbf{X} f_i = 0$. □

4.3.4 Opaque connections

We now introduce the "opposite" of transparent connections.

Definition 4.35 (Invariant subbundles)**.** We say that a smooth vector subbundle $\mathcal{F} \subset \mathcal{E}$ is *invariant* (with respect to the flow $(\varphi_t)_{t\in\mathbb{R}}$ and the connection $\nabla^{\mathcal{E}}$) if for any $x \in \mathcal{M}$, $f \in \mathcal{F}_x$, one has $C(x,t)f \in \mathcal{F}_{\varphi_t(x)}$ for all $t \in \mathbb{R}$.

Opaque connections are such that parallel transport along flowlines of $(\varphi_t)_{t\in\mathbb{R}}$ does not preserve any invariant subbundles (except the trivial ones $\mathcal{E}$ and $\{0\}$):

Definition 4.36 (Opaque connections)**.** Let $\mathcal{M}$ be a smooth manifold carrying an Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ and $\mathcal{E} \to \mathcal{M}$ be a smooth vector bundle. We say that a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ is *opaque* if any invariant subbundle $\mathcal{F} \subset \mathcal{E}$ is trivial i.e. $\mathcal{F} = \mathcal{E}$ or $\{0\}$.

The goal of this paragraph is to further describe these connections. It will be convenient to work with the connection $\nabla^{\mathrm{End}(\mathcal{E})}$ induced by $\nabla^{\mathcal{E}}$ on the vector bundle $\mathrm{End}(\mathcal{E}) \to \mathcal{M}$, as it was introduced in §2.3.2. Observe that always have the equality $\nabla^{\mathrm{End}(\mathcal{E})}\mathbb{1}_{\mathcal{E}} = 0$, and this can be checked in local coordinates for instance since

$$\nabla^{\mathrm{End}(\mathcal{E})}\mathbb{1}_{\mathcal{E}} = dI_r + [\Gamma, I_r] = 0,$$

where I_r denotes the identity matrix of rank r. The following observations are immediate:

Lemma 4.37. *The following hold for a subbundle $\mathcal{F} \subset \mathcal{E}$:*

1. *If $\mathcal{F}$ is invariant, then so is $\mathcal{F}^\perp$ (defined pointwise by taking the orthogonal subspace).*

2. *$\mathcal{F}$ is invariant if and only if for all $f \in C^\infty(\mathcal{M}, \mathcal{F})$, $\nabla^{\mathcal{E}}_X f \in C^\infty(\mathcal{M}, \mathcal{F})$.*

3. *$\mathcal{F}$ is invariant if and only if $\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_{\mathcal{F}} = 0$, where $\Pi_{\mathcal{F}}$ denotes the pointwise orthogonal projection onto $\mathcal{F}$.*

Proof. (1) Assume $\mathcal{F}$ is invariant and consider $x \in \mathcal{M}$ and $f_2 \in \mathcal{F}_x^\perp$. For $t \in \mathbb{R}$, consider $f_1' \in \mathcal{F}_{\varphi_t(x)}$; since $\mathcal{F}$ is invariant, it can be written as $f_1' = C(x,t)f_1$, for some $f_1 \in \mathcal{F}_x$. Then:

$$\langle f_1', C(x,t)f_2\rangle_{\mathcal{F}_{\varphi_t x}} = \langle C(x,t)f_1, C(x,t)f_2\rangle_{\mathcal{F}_{\varphi_t x}} = \langle f_1, f_2\rangle_{\mathcal{F}_x} = 0,$$

and thus $C(x,t)f_2 \in \mathcal{F}^\perp_{\varphi_t(x)}$, that is $\mathcal{F}^\perp$ is invariant.

(2) Assume that $\mathcal{F}$ is invariant. Consider $f_1 \in C^\infty(\mathcal{M}, \mathcal{F})$ and $x \in \mathcal{M}$. Consider $f_2 \in \mathcal{F}_x^\perp$ and extend f_2 by parallel transport along $(\varphi_t(x))_{t\in(-\varepsilon,\varepsilon)}$ for some $\varepsilon > 0$. By the first item, f_2 is a section of $\mathcal{F}^\perp$. Thus:

$$X \cdot \langle f_1, f_2\rangle_{\mathcal{E}}(x) = 0 = \langle \nabla_X^{\mathcal{E}} f_1, f_2\rangle_{\mathcal{E}_x} + \langle f_1, \underbrace{\nabla_X^{\mathcal{E}} f_2}_{=0}\rangle_{\mathcal{E}_x},$$

and thus in particular $\langle \nabla_X^{\mathcal{E}} f_1, f_2\rangle_{\mathcal{E}_x} = 0$. Conversely, assume $\mathcal{F}$ is a subbundle of $\mathcal{E}$ such that for all $f_1 \in C^\infty(\mathcal{M}, \mathcal{F})$, $\nabla_X f_1 \in C^\infty(\mathcal{M}, \mathcal{F})$. This is also true for $\mathcal{F}^\perp$: indeed, if $f_2 \in C^\infty(\mathcal{M}, \mathcal{F}^\perp)$, then $\langle f_1, \nabla_X f_2\rangle_{\mathcal{E}} = X \cdot \langle f_1, f_2\rangle_{\mathcal{E}} - \langle \nabla_X f_1, f_2\rangle_{\mathcal{E}} = 0$, i.e. $\nabla_X f_2 \in C^\infty(\mathcal{M}, \mathcal{F}^\perp)$. Now, consider $x_0 \in \mathcal{M}$, a local chart U_{x_0} around x_0, and a local orthonormal frame $(e_1, ..., e_r)$ of $\mathcal{E}|_{U_{x_0}} = U_{x_0} \times \mathbb{C}^r$such that $(e_1, ..., e_k)$ is a frame for $\mathcal{F}|_{U_{x_0}}$ and $(e_{k+1}, ..., e_r)$ a frame for $\mathcal{F}^\perp|_{U_{x_0}}$. On U_{x_0}, the connection can be written as $\nabla^{\mathcal{E}} = d + \Gamma$. We claim that for every $x \in U_{x_0}$, $\Gamma(X)(\mathcal{F}_x) \subset \mathcal{F}_x$. Indeed, consider $f = \sum_{i=1}^k f_i e_i \in \mathcal{F}_x$ and smooth functions $\widetilde{f}_1, ..., \widetilde{f}_k$ defined around x_0 such that $\mathrm{d}\widetilde{f}_i(x) = 0$ and $\widetilde{f}_i(x) = f_i$, and set $\widetilde{f} := \sum_{i=1}^k \widetilde{f}_i e_i$. Then $\nabla_X^{\mathcal{E}} \widetilde{f}(x) = \Gamma(X)\widetilde{f}(x) = \Gamma(X)f \in \mathcal{F}_x$ by assumption. Analogously, $\Gamma(X)(\mathcal{F}_x^\perp) \subset \mathcal{F}_x^\perp$ for every $x \in U_{x_0}$. We then obtain that for $f \in \mathcal{E}_x$ and $x \in U_{x_0}$, writing $f(t) := C(x,t)f = (f_1(t), f_2(t))$ with $(f_1(t), 0) \in \mathcal{F}_{\varphi_t(x)}, (0, f_2(t)) \in \mathcal{F}^\perp_{\varphi_t(x)}$, we have two separate differential equations for the parallel transport: $\dot{f}_1(t) = -\Gamma_{\mathcal{F}}(t)f_1(t), \dot{f}_2(t) = -\Gamma_{\mathcal{F}^\perp}(t)f_2(t)$. As a consequence, if $f_2(0) = 0$, then $f_2(t) = 0$ for all t. This proves the claim.

(3) Assume $\mathcal{F}$ is invariant. Then any $f \in C^\infty(M, \mathcal{E})$ can be decomposed as $f = f_1 + f_2$, where $f_1 = \Pi_{\mathcal{F}} f$, $f_2 = \Pi_{\mathcal{F}^\perp} f$ and by the second item:

$$(\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_{\mathcal{F}})f = \nabla_X^{\mathcal{E}}(\Pi_{\mathcal{F}} f) - \Pi_{\mathcal{F}}(\nabla_X^{\mathcal{E}} f) = \nabla_X^{\mathcal{E}} f_1 - \Pi_{\mathcal{F}}(\underbrace{\nabla_X^{\mathcal{E}} f_1}_{\in\mathcal{F}} + \underbrace{\nabla_X^{\mathcal{E}} f_2}_{\in\mathcal{F}^\perp}) = \nabla_X^{\mathcal{E}} f_1 - \nabla_X^{\mathcal{E}} f_1 = 0.$$

Conversely, if $\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_{\mathcal{F}} = 0$, then for any $f \in C^\infty(\mathcal{M}, \mathcal{F})$, one has

$$0 = (\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_{\mathcal{F}})f = \nabla_X^{\mathcal{E}} f - \Pi_{\mathcal{F}}(\nabla_X^{\mathcal{E}} f),$$

that is $\nabla^{\mathcal{E}}_X f \in C^\infty(\mathcal{M}, \mathcal{F})$ so $\mathcal{F}$ is invariant by the second item. $\square$

If $u \in C^\infty(\mathcal{M}, \mathrm{End}(\mathcal{E}))$, then $u = u_R + u_I$, where $u_R := \frac{u+u^*}{2}$ is hermitian and $u_I := \frac{u-u^*}{2}$ is skew-Hermitian. Since $\nabla^{\mathrm{End}(\mathcal{E})}_X(u^*) = (\nabla^{\mathrm{End}(\mathcal{E})}_X u)^*$, one obtains that $\nabla^{\mathrm{End}(\mathcal{E})}_X u = \nabla^{\mathrm{End}(\mathcal{E})}_X u_R + \nabla^{\mathrm{End}(\mathcal{E})}_X u_I$ is the decomposition into Hermitian and skew-Hermitian parts of $\nabla^{\mathrm{End}(\mathcal{E})}_X u$. Thus, $\nabla^{\mathrm{End}(\mathcal{E})}_X u = 0$, if and only if $u = u_1 + iu_2$, where $\nabla^{\mathrm{End}(\mathcal{E})}_X u_j = 0$ and $u_j^* = u_j$ for $j = 1, 2$. In other words,

$$\ker(\nabla^{\mathrm{End}(\mathcal{E})}_X)_{\mathbb{C}} = \Big(\ker(\nabla^{\mathrm{End}(\mathcal{E})}_X) \cap \ker(\bullet^* - \mathbb{1}_{\mathcal{E}})\Big)_{\mathbb{R}} \oplus i \times \Big(\ker(\nabla^{\mathrm{End}(\mathcal{E})}_X) \cap \ker(\bullet^* - \mathbb{1}_{\mathcal{E}})\Big)_{\mathbb{R}}, \tag{4.5}$$

where the subscript $\mathbb{R}$ or $\mathbb{C}$ indicates that it is seen as an $\mathbb{R}$- or $\mathbb{C}$-vector space. We have the following picture:

Lemma 4.38. *If $u \in \ker(\nabla^{\mathrm{End}(\mathcal{E})}_X)$, $u = u^*$, and u is smooth, then:*

- *At each point $x \in \mathcal{M}$, there exists a smooth orthogonal splitting $\mathcal{E}_x = \oplus_{i=1}^k \mathcal{E}_i(x)$ such that each $\mathcal{E}_i$ is invariant and $\mathcal{E}_i \to \mathcal{M}$ is a well-defined subbundle of $\mathcal{E} \to \mathcal{M}$,*
- *For all $x \in \mathcal{M}$, $u(x) = \sum_{i=1}^k \lambda_i \Pi_i(x)$, where $\Pi_i(x)$ is the orthogonal projection onto $\mathcal{E}_i$ (with kernel $\oplus_{j=1, j\neq i}^k \mathcal{E}_i$), λ_i are the* distinct *eigenvalues of u,*
- *Each projection satisfies $\nabla^{\mathrm{End}(\mathcal{E})}_X \Pi_i = 0$.*

Note that in the particular case where X is volume-preserving, (co)resonant states at 0 are smooth so this gives a description of *all* resonant states of $\nabla^{\mathrm{End}(\mathcal{E})}_X$.

Proof. Consider a dense orbit $\mathcal{O}(x_0)$, and a basis $(e_i)_{i=1}^r$ of $\mathcal{E}|_{\mathcal{O}(x_0)}$ that is invariant by parallel transport along the orbit. Then u can be written as $u = \sum_{i,j=1}^r \lambda_{ij} e_i^* \otimes e_j$ for some smooth functions $\lambda_{ij} \in C^\infty(\mathcal{O}(x_0))$ and:

$$\begin{aligned}\nabla^{\mathrm{End}(\mathcal{E})}_X u &= \sum_{i,j=1}^r X\lambda_{ij} e_i^* \otimes e_j + \sum_{i,j=1}^r \lambda_{ij} (\nabla_X e_i)^* \otimes e_j + \sum_{i,j=1}^r \lambda_{ij} e_i^* \otimes \nabla_X e_j \\ &= \sum_{i,j=1}^r X\lambda_{ij} e_i^* \otimes e_j = 0,\end{aligned}$$

thus λ_{ij} are constant along $\mathcal{O}(x_0)$. This implies that the distinct eigenvalues of u are constant along $\mathcal{O}(x_0)$ and thus constant on $\mathcal{M}$ (the eigenvalues counted with multiplicity are continuous on $\mathcal{M}$, thus uniformly continuous since $\mathcal{M}$ is compact; since they are constant on a dense set, they are constant everywhere). We denote the distinct ones by $\lambda_1, ..., \lambda_k$ and introduce for all $x \in \mathcal{M}$:

$$\Pi_i(x) := \frac{1}{2\pi i} \int_{\gamma_i} (u(x) - \lambda_i \mathbb{1}_{\mathcal{E}})^{-1} \mathrm{d}\lambda,$$

where γ_i is a small (counter clockwise oriented) circle around λ_i. One has: $u = \sum_i \lambda_i \Pi_i$. Observe that

$$\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_i = -\frac{1}{2\pi i}\int_{\gamma_i}(u(x)-\lambda_i \mathbb{1}_{\mathcal{E}})^{-1}\left(\nabla_X^{\mathrm{End}(\mathcal{E})}(u(x)-\lambda_i\mathbb{1}_{\mathcal{E}})\right)(u(x)-\lambda_i\mathbb{1}_{\mathcal{E}})^{-1}\mathrm{d}\lambda = 0.$$

□

We have the following characterization of opaque connections:

Lemma 4.39. *The connection $\nabla^{\mathcal{E}}$ is opaque if and only if* $\ker \nabla_X^{\mathrm{End}(\mathcal{E})}|_{C^\infty(\mathcal{M},\mathrm{End}(\mathcal{E}))} = \mathbb{C}\cdot\mathbb{1}_{\mathcal{E}}$.

In particular, if X is volume-preserving, $\nabla^{\mathcal{E}}$ is opaque if and only if the Pollicott-Ruelle (co)resonant states of $\nabla_X^{\mathrm{End}(\mathcal{E})}$ are reduced to $\mathbb{C}\cdot\mathbb{1}_{\mathcal{E}}$.

Proof. " $\Longrightarrow$ " Assume that the connection is opaque and $\ker(\nabla_X^{\mathrm{End}(\mathcal{E})}|_{\mathcal{H}^s_\pm}) \neq \mathbb{C}\cdot\mathbb{1}_{\mathcal{E}}$, then one can consider $0 \neq u \in \ker(\nabla_X^{\mathrm{End}(\mathcal{E})}|_{\mathcal{H}^s_\pm})$ which is orthogonal to $\mathbb{C}\cdot\mathbb{1}_{\mathcal{E}}$ (i.e. its trace vanishes everywhere on $\mathcal{M}$).

Taking its self-adjoint or i times the skew-adjoint part, by the previous discussion we may additionally assume $u^* = u$ and $u \neq 0$. By Lemma 4.38, it can be decomposed as $u = \sum_{i=1}^k \lambda_i \Pi_i$, where each Π_i is the orthogonal projection corresponding to an invariant subbundle $\mathcal{E}_i \to \mathcal{M}$, i.e. $\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_i = 0$. Observe that since $\mathrm{Tr}(u) = 0$, this decomposition cannot be the trivial one i.e. $\mathcal{E} = \mathcal{E} \oplus^\perp \{0\}$ (in which case u would be a multiple of $\mathbb{1}_{\mathcal{E}}$). Thus, $\mathcal{E}_1$ is an invariant subbundle which is neither $\{0\}$ nor $\mathcal{E}$ which contradicts the fact that the connection is opaque.

" $\Longleftarrow$" Conversely, if the connection is not opaque, then it admits an invariant subbundle $\mathcal{F}$ and $\mathcal{E} = \mathcal{F} \oplus^\perp \mathcal{F}^\perp$ is an invariant decomposition. The orthogonal projection $\Pi_{\mathcal{F}}$ satisfies $\nabla_X^{\mathrm{End}(\mathcal{E})}\Pi_{\mathcal{F}} = 0$ by Lemma 4.37, thus $\ker(\nabla_X^{\mathrm{End}(\mathcal{E})}|_{\mathcal{H}^s_\pm}) \neq \mathbb{C}\cdot\mathbb{1}_{\mathcal{E}}$. □

There is actually a connection between opacity and the representation-theoretic description introduced before:

Lemma 4.40. *The connection $\nabla^{\mathcal{E}}$ is opaque if and only if the representation $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_*})$ is irreducible.*

The proof is postponed to the next paragraph, where a more general result will be stated. As a conclusion, let us mention the following: we let $\mathbb{CP}^{r-1}$ be the complex projective space and $P(\mathcal{E}) \to \mathcal{M}$ be the $\mathbb{CP}^{r-1}$-bundle of the projective space of $\mathcal{E}$ over $\mathcal{M}$. The flow φ_t induces a natural flow $\Phi_t : P(\mathcal{E}) \to P(\mathcal{E})$. More precisely, if $[v] \in P(\mathcal{E}_x)$ is the complex line generated by the non-zero vector v, then $\Phi_t([v]) := [C(x,t)v]$. We have the following result, which is left as an exercise for the reader:

Exercise 4.41. Show that if $(\Phi_t)_{t\in\mathbb{R}}$ is transitive on $P(\mathcal{E})$, then $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_*})$ is irreducible.

4.3.5 General algebraic description

We refer to [Lan02] for a background exposition on representation theory. The representation $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_\star})$ gives rise to an orthogonal splitting

$$\mathcal{E}_{x_\star} = \oplus_{i=1}^{K} \mathcal{E}_i^{\oplus n_i},$$

where $\mathcal{E}_i \subset \mathcal{E}_{x_\star}$ and $n_i \geq 1$; each factor $\mathcal{E}_i$ is $\mathbf{G}$-invariant and the induced representation on each factor is irreducible; for $k, k' \in \{1, ..., n_i\}$, the induced representations on $\mathcal{E}_i^{(k)}$ and $\mathcal{E}_i^{(k')}$ are isomorphic; furthermore, for $i \neq j$, the induced representations on $\mathcal{E}_i$ and $\mathcal{E}_j$ are not isomorphic. Let $\mathbb{C}[\mathbf{G}]$ be the formal algebra generated by $\mathbf{G}$ over $\mathbb{C}$ and let $\mathbf{R} := \rho\left(\mathbb{C}[\mathbf{G}]\right)$. By Burnside's Theorem (see [Lan02, Corollary 3.3] for instance), one has that:

$$\mathbf{R} = \oplus_{i=1}^{K} \Delta_{n_i} \operatorname{End}(\mathcal{E}_i),$$

where $\Delta_{n_i} u = u \oplus ... \oplus u$ for $u \in \operatorname{End}(\mathcal{E}_i)$, the sum being repeated n_i-times. We introduce the *commutant* $\mathbf{R}'$ of $\mathbf{R}$, defined as:

$$\mathbf{R}' := \{u \in \operatorname{End}(\mathcal{E}_{x_\star}) \mid \forall v \in \mathbf{R}, uv = vu\}.$$

We then have:

Theorem 4.42. *There exists a natural isomorphism:*

$$\Phi : \mathbf{R}' \to \ker \nabla_X^{\operatorname{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \operatorname{End}(\mathcal{E}))}.$$

In particular these spaces have same dimension, that is

$$\dim\left(\ker \nabla_X^{\operatorname{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \operatorname{End}(\mathcal{E}))}\right) = \dim(\mathbf{R}') = \sum_{i=1}^{K} n_i^2.$$

The last assertion is left as an exercise to the reader:

Exercise 4.43. Check that the dimension of the commutant is indeed $\sum_{i=1}^{K} n_i^2$.

Proof. The linear map $\Phi : \mathbf{R}' \to \ker \nabla_X^{\operatorname{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \operatorname{End}(\mathcal{E}))}$ is defined in the following way. Consider an element $u_* \in \mathbf{R}'$ and define, as in Lemma 4.29, for x on a homoclinic orbit, $u_-(x)$ as the parallel transport of u_* from x_* to x along the orbit (with respect to the endomorphism connection $\nabla^{\operatorname{End}(\mathcal{E})}$). Similarly, one can define $u_+(x)$ by parallel transport from the future. The fact that $u_* \in \mathbf{R}'$ is then used in the following observation (see Lemma 4.31):

$$\|u_-(x) - u_+(x)\| = \|\rho(\gamma) u_* \rho(\gamma)^{-1} - u_*\| = 0.$$

Formally, observe that $\rho(\gamma)u_*\rho(\gamma)^{-1}$ corresponds to the parallel transport of u_* with respect to $\nabla^{\mathrm{End}(\mathcal{E})}$ from x_* to x_* along the homoclinic orbit γ. Hence, following the proof of Lemma 4.32, we get that u_- is Lipschitz-continuous and satisfies $\nabla_X^{\mathrm{End}(\mathcal{E})}u_- = 0$. By the regularity Theorem 4.9, it is smooth and we set $u_- := \Phi(u_*) \in \ker \nabla_X^{\mathrm{End}(\mathcal{E})}|_{C^\infty(\mathcal{M},\mathrm{End}(\mathcal{E}))}$.

Also observe that this construction is done by using parallel transport with respect to the unitary connection $\nabla^{\mathrm{End}(\mathcal{E})}$. As a consequence, if $u_*, u_*' \in \mathbf{R}$ are orthogonal (i.e. $\mathrm{Tr}(u_*^* u_*') = 0$), then $\Phi(u_*)$ and $\Phi(u_*')$ are also pointwise orthogonal. This proves that Φ is injective.

It now remains to show the surjectivity of Φ. Let $u \in \ker \nabla_X^{\mathrm{End}(\mathcal{E})}|_{C^\infty(\mathcal{M},\mathrm{End}(\mathcal{E}))}$. By the discussion of §4.3.4, we can write $u = u_R + iu_I$, where $u_R^* = u_R, u_I^* = u_I$ and $\nabla_X^{\mathrm{End}(\mathcal{E})}u_R = \nabla_X^{\mathrm{End}(\mathcal{E})}u_I = 0$. By Lemma 4.38, we can then further decompose $u_R = \sum_{i=1}^p \lambda_i \pi_{\mathcal{F}_i}$ (and same for u_I), where $\lambda_i \in \mathbb{R}$, $p \in \mathbb{N}$ and $\mathcal{F}_i \subset \mathcal{E}$ is a maximally invariant subbundle of $\mathcal{E}$ (i.e. it does not contain any non-trivial subbundle that is invariant under parallel transport along the flowlines of $(\varphi_t)_{t\in\mathbb{R}}$ with respect to $\nabla^{\mathcal{E}}$), and $\pi_{\mathcal{F}_i}$ is the orthogonal projection onto $\mathcal{F}_i$. Setting $(\pi_{\mathcal{F}_i})_* := \pi_{\mathcal{F}_i}(x_*)$, invariance of $\mathcal{F}_i$ by parallel transport implies that $\rho(\gamma)(\pi_{\mathcal{F}_i})_* = (\pi_{\mathcal{F}_i})_*\rho(\gamma)$, for all $\gamma \in \mathbf{G}$, that is $(\pi_{\mathcal{F}_i})_* \in \mathbf{R}'$. Moreover, we have $\Phi((\pi_{\mathcal{F}_i})_*) = \pi_{\mathcal{F}_i}$. This proves that both u_R and u_I are in $\mathrm{ran}(\Phi)$. This concludes the proof. □

We now investigate the existence of invariant sections.

Lemma 4.44. *Assume that there exists $u_* \in \mathcal{E}_{x_*}$ such that $\rho(\gamma)u_* = u_*$ for all $\gamma \in \mathbf{G}$. Then, there exists (a unique) $u \in C^\infty(\mathcal{M},\mathcal{E})$ such that $u(x_*) = u_*$ and $\nabla_X^{\mathcal{E}}u = 0$.*

Observe that the converse is obviously true: if there exists an invariant section u, then $u_* := u(x_*)$ is invariant by the $\mathbf{G}$-action.

Proof. Uniqueness is immediate since $\nabla_X^{\mathcal{E}}u = 0$ implies that $X|u|^2 = \langle \nabla_X^{\mathcal{E}}u, u\rangle = \langle u, \nabla_X^{\mathcal{E}}u\rangle = 0$, that is $|u|$ is constant. Now, given u_* which is $\mathbf{G}$-invariant, we can define $u_-(x)$ for x on a homoclinic orbit γ by parallel transport of u_* from x_* to x along γ with respect to $\nabla^{\mathcal{E}}$, similarly to Lemma 4.29 and to the proof of Theorem 4.42. We can also define $u_+(x)$ in the same fashion (by parallel transport in the other direction). Then one gets that $\|u_-(x) - u_+(x)\| = \|u_* - \rho(\gamma)u_*\|$ and the same arguments as before show that u_- extends to a smooth function in the kernel of $\nabla_X^{\mathcal{E}}$. □

Such an approach turns out to be useful when trying to understand a weak version of Livšic theory. For instance: assume that $\mathcal{E} \to \mathcal{M}$ is a vector bundle equipped with the unitary connection $\nabla^{\mathcal{E}}$ and that for each periodic orbit γ, there exists a section $u_\gamma \in C^\infty(\gamma, \mathcal{E}|_\gamma)$ such that $\nabla_X^{\mathcal{E}}u_\gamma = 0$. Does it imply he existence of a global smooth section $u \in C^\infty(\mathcal{M},\mathcal{E})$ in the kernel of $\nabla_X^{\mathcal{E}}$? It turns out that the answer is yes when $\mathrm{rank}(\mathcal{E}) \leq 2$ and no for $\mathrm{rank}(\mathcal{E}) \geq 3$:

Lemma 4.45. *Assume that* $\mathrm{rank}(\mathcal{E}) \leq 2$ *and that for all periodic orbits* $\gamma \in \mathcal{G}$*, there exists* $u_\gamma \in C^\infty(\gamma, \mathcal{E}|_\gamma)$ *such that* $\nabla^{\mathcal{E}}_X u_\gamma = 0$*. Then, there exists* $u \in C^\infty(\mathcal{M}, \mathcal{E})$ *such that* $\nabla^{\mathcal{E}}_X u = 0$.

Proof. Thanks to the tools developed so far, we can provide a purely representation-theoretic proof which completely avoids the need to understand dynamics and the distribution of periodic orbits. Indeed, we claim first of all that the following holds:

Lemma 4.46. *Assume that for all periodic orbits* $\gamma \in \mathcal{G}$*, there exists* $u_\gamma \in C^\infty(\gamma, \mathcal{E}|_\gamma)$ *such that* $\nabla^{\mathcal{E}}_X u_\gamma = 0$*. Then for all* $g \in \mathbf{G}$*, there exists* $u_g \in \mathcal{E}_{x_*}$ *such that* $\rho(g)u_g = u_g$.

Proof. Recall that by the construction of §4.3.2 and Proposition 4.27, each element $\rho(g) \in \mathrm{U}(\mathcal{E}_{x_*})$ can be approximated by the holonomy $C_{y_n \to x_*} C(y_n, T'_n) C_{x_* \to y_n}$ along a sequence of periodic orbits of points y_n converging to x_*. Now, each $C(y_n, T'_n)$ has 1 as eigenvalue by assumption and taking the limit as $n \to \infty$, we deduce that 1 is an eigenvalue of $\rho(g)$. □

As a consequence, we can write for all $g \in \mathbf{G}$:

$$\rho(g) = \alpha_g \begin{pmatrix} 1 & 0 \\ 0 & s(g) \end{pmatrix} \alpha_g^{-1},$$

for some $\alpha_g \in \mathrm{U}(\mathcal{E}_{x_*})$ and $s(g)$ is an $(r-1)\times(r-1)$ matrix. For $\mathrm{rank}(\mathcal{E}) = 1$, the Lemma is then a straightforward consequence of Lemma 4.44 since the conjugacy α_g does not appear. For $\mathrm{rank}(\mathcal{E}) = 2$, one has the remarkable property that $s(g)$ is *still* a representation of $\mathbf{G}$ since $\det \rho(g) = s(g) \in \mathrm{U}(1)$. As a consequence, $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_*})$ has the same character as $\rho' : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_*})$ defined by:

$$\rho'(g) := \begin{pmatrix} 1 & 0 \\ 0 & s(g) \end{pmatrix}.$$

By [Lan02, Corollary 3.8], we then conclude that these representations are isomorphic, that is there exists $p_* \in \mathrm{U}(\mathcal{E}_{x_*})$ such that $\rho(g) = p_* \rho'(g) p_*^{-1}$. If $u'_* \in \mathrm{U}(\mathcal{E}_{x_*})$ denotes the vector globally fixed by ρ', then $u_* := p_* u'_\gamma$ is globally fixed by ρ. We then conclude by Lemma 4.44. □

We leave as an exercise for the reader the fact that Lemma 4.45 does not hold when $\mathrm{rank}(\mathcal{E}) \geq 3$.

Exercise 4.47. Show the existence of a manifold $\mathcal{M}$ with Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$, a Hermitian vector bundle $\mathcal{E} \to \mathcal{M}$ equipped with a unitary connection $\nabla^{\mathcal{E}}$ such that the following holds: for all periodic orbits $\gamma \in \mathcal{G}$, there exists $u_\gamma \in C^\infty(\gamma, \mathcal{E})$ such that $\nabla^{\mathcal{E}}_X u_\gamma = 0$ but there is no global invariant smooth section $u \in C^\infty(\mathcal{M}, \mathcal{E})$ such that $\nabla^{\mathcal{E}}_X u = 0$. *Hint: Use the fact that matrices in* $\mathrm{SO}(3)$ *always preserve an axis...*

Part II

Geometric inverse problems

5 Geodesic X-ray transform

5.1 Definition and first properties

This paragraph is an application of the Abelian Livšic theory of §4.2 to the geodesic case, i.e. when X is the geodesic vector field on the unit tangent. We assume that (M, g) is an Anosov Riemannian manifold and set $\mathcal{M} := SM$ and X is the geodesic vector field. In this case, we know by Lemma 2.5 that there exists a unique closed geodesic by free homotopy class $c \in \mathcal{C}$ (where $\mathcal{C}$ denotes the set of free homotopy classes) and we can therefore identify the set $\mathcal{G}$ of periodic orbits of the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ with $\mathcal{C}$. The Abelian X-ray transform of Definition 4.10 can therefore be seen as a map

$$I : C^0(SM) \to \ell^\infty(\mathcal{C}), \quad If(c) := \frac{1}{L_g(c)} \int_0^{L_g(c)} f(\varphi_t(x, v))\mathrm{d}t,$$

where (x, v) is an arbitrary point on the unique closed geodesic $\gamma_g(c) \in c$ in the free homotopy class $c \in \mathcal{C}$. We will be particularly interested in the case where the functions are pullback via the map π_m^* of symmetric tensors on the base M, as introduced in §2.4. We therefore consider:

$$I_m := I \circ \pi_m^*.$$

Of course, using the relation $X\pi_m^* = \pi_{m+1}^* D$ of Lemma 2.11, it is clear that potential tensors are always in the kernel of I_m, namely:

$$\left\{ Dp \mid p \in C^\infty(M, \otimes_S^{m-1} T^*M) \right\} \subset \ker(I_m). \tag{5.1}$$

Definition 5.1. We say that I_m is solenoidal injective (or s-injective in short) if the inclusion (5.1) is an equality.

Observe that by the Livšic Theorem 4.11, if $I_m f = 0$, then $\pi_m^* f = Xu$, for some smooth function $u \in C^\infty(SM)$. An equation of the form $Xu = F$ (where $F \in C^\infty(SM)$) is called a *cohomological equation*. By the discussion on symmetric tensors of §2.4.1, we know that $\pi_m^* f \in C^\infty(SM)$ has degree at most m (see Lemma 2.10) and more precisely, $\pi_m^* f = f_m + f_{m-2} + \ldots$ where $f_{m-2i} \in C^\infty(M, \Omega_{m-2i})$. By the mapping properties of X (see Lemma 2.15) it immediately implies that u has only odd Fourier components (resp. even) if m is even (resp. if m is odd). The question is then wether u has degree $m - 1$ or not. If it is the case, then this proves that f is a potential tensor as u can be written in the form

$u = \pi_{m-1}^* \tilde{u}$ and thus $Xu = \pi_{m-1}^* \tilde{u} = \pi_m^* D\tilde{u} = \pi_m^* f$, that is $f = D\tilde{u}$.

We will explain the solenoidal injectivity of the X-ray transform in the following cases (see [CS98, DS03]):

Theorem 5.2 (Croke-Sharafutdinov '98, Dairbekov-Sharafutdinov '03)**.** *Assume (M, g) is Anosov. Then I_0 and I_1 are solenoidal-injective. If we further assume that (M, g) has non-positive curvature, then I_m is solenoidal-injective for every $m \in \mathbb{N}$.*

In the two-dimensional case, the curvature assumption can be relaxed and I_m is known to be injective or any $m \in \mathbb{N}$ as long as (M, g) is Anosov (see [PSU14, Gui17a]). It is conjectured that this should also hold in higher dimension:

Question 5.3. Is I_m solenoidal-injective when (M, g) is Anosov (in any dimension)?

In the general Anosov case, the best known result is the following:

Theorem 5.4 (Cekic-L. '21)**.** *For any $m \in \mathbb{N}$, there is an open dense set of Anosov metrics for which I_m is solenoidal-injective. In particular, the set of Anosov metrics such that all X-ray transforms I_* are solenoidal-injective is residual.*

We will not attempt to prove this last theorem. The strategy of proof of Theorem 5.2 relies on an energy identity called the *Pestov identity* and is done in two steps. First of all, one proves that given a cohomological equation $Xu = f$, where $f = f_0 + ... f_m$ (and $f_i \in C^\infty(M, \Omega_i)$) has finite degree $m \in \mathbb{N}$, then u also does have finite degree.

Lemma 5.5. *Assume $f, u \in C^\infty(SM)$ and $Xu = f$ with $\deg(f) < \infty$. Then $\deg(u) < \infty$.*

The proof is explained in the next paragraph. As a consequence, we can write $u = u_0 + ... + u_N$ with $u_i \in C^\infty(M, \Omega_i)$. We now assume by contradiction that $N \geq m$. Projecting the equality $Xu = f$ onto the spherical harmonics of degree $N + 1$ and using the mapping properties of X (see Lemma 2.15), we obtain that $X_+ u_N = 0$, that is u_N is a CKT of degree N, as they were introduced in Definition 2.17. As a consequence, if one can prove that there are no CKTs of degree $m \geq 1$, this implies that $u_N = 0$ which is a contradiction, hence $N \leq m - 1$. The second step is to prove:

Lemma 5.6. *If (M, g) has ergodic geodesic flow and non-positive curvature, there are no CKTs of degree $m \geq 1$.*

The proof will rely on the Pestov identity. An alternative approach to the previous Lemma using Weitzenböck formulas can also be found in [?]. It is remarkable that this strategy still works when when twisting with an arbitrary vector bundle $\mathcal{E}$ (modulo some extra work). This is explained in §5.3. Both Lemmas 5.5 and 5.6 rely on the so-called *Pestov energy identity.*

5.2 Pestov identity, cohomological equations

We start with the case of the trivial line bundle $\mathbb{C} \times M \to M$.

Lemma 5.7 (Pestov identity). *Let $u \in H^2(SM)$. Then*

$$\|\nabla_{\mathbb{V}} X u\|^2_{L^2(SM,\mathcal{N})} = \|X \nabla_{\mathbb{V}} u\|^2_{L^2(SM,\mathcal{N})} - \langle R\nabla_{\mathbb{V}} u, \nabla_{\mathbb{V}} u\rangle_{L^2(SM,\mathcal{N})} + (n-1)\|Xu\|^2_{L^2(SM)}.$$

Proof. For $u \in C^\infty(SM)$, using the commutator formulas (2.1):

$$\begin{aligned}
\|\nabla_{\mathbb{V}} X u\|^2 - \|\nabla_X \nabla_{\mathbb{V}} u\|^2 &= \langle \nabla_{\mathbb{V}} X u, \nabla_{\mathbb{V}} X u\rangle - \langle \nabla_X \nabla_{\mathbb{V}} u, \nabla_X \nabla_{\mathbb{V}} u\rangle \\
&= \langle (X \operatorname{div}_{\mathbb{V}} \nabla_{\mathbb{V}} X - \operatorname{div}_{\mathbb{V}} X^2 \nabla_{\mathbb{V}}) u, u\rangle \\
&= \langle (-\operatorname{div}_{\mathbb{H}} \nabla_{\mathbb{V}} X + \operatorname{div}_{\mathbb{V}} X \nabla_{\mathbb{H}}) u, u\rangle \\
&= \langle (-\operatorname{div}_{\mathbb{H}} \nabla_{\mathbb{V}} X + \operatorname{div}_{\mathbb{V}} \nabla_{\mathbb{H}} X + \operatorname{div}_{\mathbb{V}} R \nabla_{\mathbb{V}}) u, u\rangle \\
&= -(n-1)\langle X^2 u, u\rangle + \langle \operatorname{div}_{\mathbb{V}} R \nabla_{\mathbb{V}} u, u\rangle \\
&= (n-1)\|Xu\|^2 - \langle R\nabla_{\mathbb{V}} u, \nabla_{\mathbb{V}} u\rangle
\end{aligned}$$

□

An important point is the following:

Lemma 5.8. *Assume (M, g) is Anosov. Then, there exists $C > 0$ such that for all $Z \in C^\infty(SM, \mathcal{N})$:*

$$\|XZ\|^2_{L^2(SM,\mathcal{N})} - \langle RZ, Z\rangle_{L^2(SM,\mathcal{N})} \geq C\|Z\|^2_{L^2(SM,\mathcal{N})}.$$

Proof. First of all, observe that for $Z \in C^\infty(SM, \mathcal{N})$, pointwise in SM:

$$X\langle Z, UZ\rangle = 2\langle UZ, XZ\rangle + \langle Z, (XU)Z\rangle$$

Consider $U \in C^\alpha(SM, \operatorname{End}(\mathcal{N}))$, one of the two solutions of the Ricatti equation (2.3). Then for $(x, v) \in SM$:

$$\begin{aligned}
|XZ - UZ|^2(x, v) &= |XZ(x, v)|^2 + |UZ(x, v)|^2 - 2\langle XZ(x, v), UZ(x, v)\rangle \\
&= |XZ(x, v)|^2 + \langle U^2 Z(x, v), Z(x, v)\rangle - 2\langle XZ(x, v), UZ(x, v)\rangle \\
&= |XZ(x, v)|^2 - \langle RZ(x, v), Z(x, v)\rangle \\
&- \langle (XU)Z(x, v), Z(x, v)\rangle - 2\langle XZ(x, v), UZ(x, v)\rangle \\
&= |XZ(x, v)|^2 - \langle RZ(x, v), Z(x, v)\rangle - X\langle Z(x, v), UZ(x, v)\rangle.
\end{aligned}$$

Integrating over SM, we obtain:

$$\|XZ\|^2_{L^2} - \langle RZ, Z\rangle_{L^2} = \|(X - U)Z\|^2_{L^2}.$$

We now specify $U = U_+$ and consider the operator $-X + U_+ : C^\infty(SM, \mathcal{N}) \to C^\infty(SM, \mathcal{N})$. By (3.2), the resolvent of this operator is initially defined on $\{\Re(z) \gg 0\}$ by:

$$(-X + U_+ - z)^{-1} = -\int_0^\infty e^{-tz} e^{t(-X+U_+)} \mathrm{d}t, \tag{5.2}$$

where $e^{t(-X+U_+)} = R_{U_+}(t)$ is the propagator introduced in Lemma 2.3, namely

$$\dot{R}_{U_+}(t) = (-X + U_+) R_{U_+}(t), \qquad R_{U_+}(0) = \mathbb{1}.$$

Using the bound of Lemma 2.3, we then obtain:

$$\|(-(X - U_+) - z)^{-1}\|_{L^2(SM,\mathcal{N}) \to L^2(SM,\mathcal{N})} \leq C \int_0^{+\infty} e^{-\Re(z)t} e^{-\lambda t} \mathrm{d}t = \frac{C}{\Re(z) + \lambda}.$$

This shows that the resolvent (5.2) is holomorphic in $\{\Re(z) > -\lambda\}$. In particular, it is well-defined at 0 and thus:

$$\|Z\|_{L^2(SM,\mathcal{N})} \leq C/\lambda \times \|(X - U_+)Z\|_{L^2(SM,\mathcal{N})}.$$

□

Combined with the previous Lemma, a direct consequence of the Pestov identity is the following injectivity results for the geodesic X-ray transform:

Lemma 5.9. *Assume (M, g) is Anosov. Then I_0 is injective on $C^\infty(M)$ and I_1 is solenoidal injective on $C^\infty(M, T^*M)$.*

Proof. Assume $f \in C^\infty(M)$ satisfies $I_0 f = 0$. Then, by the smooth Livšic Theorem 4.11, there exists $u \in C^\infty(SM)$ such that $\pi_0^* f = Xu$. Applying the Pestov identity of Lemma 5.7, we obtain:

$$\|\nabla_{\mathbb{V}} Xu\|^2_{L^2(SM,\mathcal{N})} = 0 = \underbrace{\|X\nabla_{\mathbb{V}} u\|^2_{L^2(SM,\mathcal{N})} - \langle R\nabla_{\mathbb{V}} u, \nabla_{\mathbb{V}} u\rangle_{L^2(SM,\mathcal{N})}}_{\geq 0} + (n-1)\|Xu\|^2_{L^2(SM)},$$

and thus by Lemma 5.8, we obtain $Xu = 0$, hence $f \equiv 0$.

Let us now assume $f \in C^\infty(M, T^*M)$ satisfies $I_1 f = 0$. Then $\pi_1^* f = Xu$ for $u \in C^\infty(SM)$. An easy computation shows that:

$$\|\nabla_{\mathbb{V}} Xu\|^2_{L^2(SM,\mathcal{N})} = \langle (-\Delta_{\mathbb{V}})\pi_1^* f, \pi_1^* f\rangle_{L^2(SM)} = (n-1)\|\pi_1^* f\|^2_{L^2(SM)}.$$

Hence, by the Pestov identity: $\|X\nabla_{\mathbb{V}} u\|^2_{L^2(SM,\mathcal{N})} - \langle R\nabla_{\mathbb{V}} u, \nabla_{\mathbb{V}} u\rangle_{L^2(SM,\mathcal{N})} = 0$, and Lemma 5.8 implies that $\nabla_{\mathbb{V}} u = 0$, that is u is of degree 0. □

The previous Pestov identity of Lemma 5.7 specified to a function $u \in C^\infty(M, \Omega_m)$ yields the following, see [PSU15]:

Lemma 5.10 (Localized Pestov identity). *Let* $u \in C^\infty(M, \Omega_m)$. *Then:*

$$(2m+n-3)\|X_-u\|^2 + \|\nabla_{\mathbb{H}} u\|^2 - \langle R\nabla_{\mathbb{V}} u, \nabla_{\mathbb{V}} u\rangle_{L^2} = (2m+n-1)\|X_+u\|^2$$

Lemma 5.10 is remarkable in the sense that one can prove that summing all the contributions for $m \in \mathbb{N}$, one retrieves the Pestov identity, i.e. Lemma 5.7. We say that the Pestov identity can be *localized in frequency*. The proof is similar to that of Lemma 5.7 using the commutator identities (we refer to [PSU15, Proposition 3.4]). The crucial observation (which is a straightforward consequence of the previous Lemma 5.10) is that if the sectional curvatures are non-positive:

$$\|X_-u\|^2 \leq c(m,n)\|X_+u\|^2_{L^2}, \tag{5.3}$$

for u of degree m, where

$$c(m,n) = \frac{2m+n-1}{2m+n-3}.$$

We can now prove Lemma 5.6.

Proof of Lemma 5.6. If $u \in C^\infty(M, \Omega_m)$, $X_+u = 0$ and the sectional curvatures are non-positive, (5.3) implies that $X_-u = 0$. Thus $Xu = (X_- + X_+)u = 0$. By ergodicity, u is constant, thus $u = 0$ if $m \geq 1$. □

We now go on with the proof of Lemma 5.5:

Proof of Lemma 5.5. We assume that $Xu = f$ and f has finite degree. We want to show that u has finite degree too and we argue by contradiction. We decompose $u = u_0 + u_1 + ...$ As f has finite degree, the cohomological equation $Xu = f$ gives $X_+u_{k-1} + X_-u_{k+1} = 0$ for all $k \geq k_0$ for k_0 is chosen large enough (greater than $\deg(f)$). As a consequence, using (5.3):

$$\begin{aligned}
\|X_+u_{k-1}\|^2_{L^2} &= \|X_-u_{k+1}\|^2_{L^2} \\
&\leq c(k+1,n)\|X_+u_{k+1}\|^2_{L^2} \\
&= c(k+1,n)\|X_-u_{k+3}\|^2_{L^2} \\
&\leq c(k+1,n)c(k+3,n)\|X_+u_{k+3}\|^2_{L^2} \leq ... \leq \prod_{j=0}^{N} c(k+1+2j)\|X_+u_{k+1+2N}\|^2_{L^2}
\end{aligned}$$

Since u is smooth, we know by Lemma 2.19 that $\|X_+u_N\|^2_{L^2} \leq \dfrac{C_\alpha}{N^\alpha}$ for any $\alpha > 0$. It is then sufficient to prove that the product $\prod_{j=1}^{N} c(k+1+2j)$ diverges polynomially fast, i.e.

$\prod_{j=1}^{N} c(k+1+2j) \leq N^{\beta}$, for some exponent $\beta > 0$ (left to the reader). As a consequence, we deduce that

$$\prod_{j=0}^{N} c(k+1+2j)\|X_+ u_{k+1+2N}\|_{L^2}^2 \to_{N\to 0} 0.$$

This implies that $X_+ u_{k-1} = 0$, hence $u_{k-1} = 0$ since there are no CKTs by Lemma 5.6. Hence u has finite degree. □

We also observe that an alternative approach using *Carleman estimates in frequency* was developed in [PS18]. It also relies on the fact that the Pestov identity allows to localize in frequency, see Lemma 5.10.

5.3 Twisted cohomological equations

It is remarkable that the previous strategy still works in the case where one introduces a twist by a vector bundle $\mathcal{E} \to M$, as in §2.3.4. This was discovered in [GPSU16].

Lemma 5.11 (Twisted Pestov identity)**.** *Let $u \in H^2(SM, \pi^*\mathcal{E})$. Then*

$$\|\nabla_{\mathbb{V}}^{\mathcal{E}} \mathbf{X} u\|_{L^2}^2 = \|\mathbf{X}\nabla_{\mathbb{V}}^{\mathcal{E}} u\|_{L^2}^2 - \langle R\nabla_{\mathbb{V}}^{\mathcal{E}} u, \nabla_{\mathbb{V}}^{\mathcal{E}} u\rangle_{L^2} - \langle F^{\mathcal{E}} u, \nabla_{\mathbb{V}}^{\mathcal{E}} u\rangle_{L^2} + (n-1)\|\mathbf{X}u\|_{L^2}^2.$$

We also have a localized twisted Pestov identity when specified to $u \in C^\infty(M, \Omega_m \otimes \mathcal{E})$:

Lemma 5.12 (Localized twisted Pestov identity)**.** *Let $u \in H^2(M, \Omega_m \otimes \mathcal{E})$. Then:*

$$(2m+n-3)\|\mathbf{X}_- u\|^2 + \|\nabla_{\mathbb{H}}^{\mathcal{E}} u\|^2 - \langle R\nabla_{\mathbb{V}}^{\mathcal{E}} u, \nabla_{\mathbb{V}}^{\mathcal{E}} u\rangle_{L^2} - \langle F^{\mathcal{E}} u, \nabla_{\mathbb{V}}^{\mathcal{E}} u\rangle_{L^2} = (2m+n-1)\|\mathbf{X}_+ u\|^2$$

The proofs of these lemmas can be found in [GPSU16, Section 3]. Some extra-work is then required but, using this identity, one can still prove a similar result to Lemma 5.5, asserting that solutions to twisted cohomological equations $\mathbf{X}u = f$ have finite degree when f has finite degree:

Lemma 5.13. *Assume $f, u \in C^\infty(SM, \pi^*\mathcal{E})$ and $\mathbf{X}u = f$ with $\deg(f) < \infty$. Then $\deg(u) < \infty$.*

We refer to [GPSU16, Theorem 4.1] for a proof. As before, if f is of degree m and $\mathbf{X}u = f$, the (finite) degree of u is determined by the (non)existence of twisted CKTs, i.e. elements in $\ker \mathbf{X}_+$. By [CL20], a connection $\nabla^{\mathcal{E}}$ has generically no CKTs, which implies in particular that for such a generic connection, the cohomological equations are solvable, that is if $\mathbf{X}u = f$ with f is of degree m, then u is of degree $m-1$. However, there are exceptional connections which always carry CKTs and it is not always easy to compute them. Nevertheless, still using the twisted Pestov identity, one can show the following:

Lemma 5.14. *Assume that (M,g) has negative sectional curvature bounded from above by $-\kappa < 0$. Let $\mathcal{E} \to M$ be a smooth vector bundle with a unitary connection $\nabla^{\mathcal{E}}$. Then if $m \geq 1$ satisfies*

$$\lambda_m := m(m+n-2) \geq \frac{4\|F^{\mathcal{E}}\|^2_{L^\infty}}{\kappa^2},$$

one has $\ker \mathbf{X}_+|_{C^\infty(M,\Omega_m\otimes\pi^*\mathcal{E})} = \{0\}$. In other words, there is always a finite number of CKTs.

We refer to [GPSU16, Theorem 4.5] for a proof. Passing from the negatively-curved case to the Anosov case is still a big step to accomplish. In particular, one can wonder if analogous results to Lemma 5.13 and 5.14 can be proved without any assumption on the curvature:

Question 5.15. If (M,g) is Anosov and $\mathbf{X}u = f$, with $\deg(f) < \infty$, does it imply that $\deg(u) < \infty$?

Question 5.16. If (M,g) is Anosov, is there always a finite number of CKTs?

Eventually, it is not clear at all whether Lemma 5.14 is optimal and we ask the following:

Question 5.17. Is Lemma 5.14 sharp? Can one find a better condition to ensure absence of CKTs?

We will also need the following:

Lemma 5.18. *Assume (M,g) is Anosov and $\nabla^{\mathcal{E}}$ is a flat and unitary connection on the vector bundle $\mathcal{E} \to M$. If $\mathbf{X}u = f$ with $f = f_0 + f_1 \in C^\infty(M,(\Omega_0 \oplus \Omega_1)\otimes\mathcal{E})$ and $u \in C^\infty(SM,\pi^*\mathcal{E})$, then u is of degree 0.*

Proof. An important point is that the following inequality holds for Anosov manifolds:

$$\|\mathbf{X}\nabla^{\mathcal{E}}_{\mathbb{V}}u\|^2_{L^2} - \langle R\nabla^{\mathcal{E}}_{\mathbb{V}}u, \nabla^{\mathcal{E}}_{\mathbb{V}}u\rangle_{L^2} \geq C\|\nabla^{\mathcal{E}}_{\mathbb{V}}u\|^2_{L^2},$$

where $C > 0$ is independent of u, similarly to Lemma 5.8. We thus obtain with the twisted Pestov identity of Lemma 5.11:

$$\|\nabla^{\mathcal{E}}_{\mathbb{V}}\mathbf{X}u\|^2_{L^2} \geq C\|\nabla^{\mathcal{E}}_{\mathbb{V}}u\|^2_{L^2} + (n-1)\|\mathbf{X}u\|^2_{L^2}. \tag{5.4}$$

By assumption, $\mathbf{X}u = f_0+f_1 \in C^\infty(M,(\Omega_0\oplus\Omega_1)\otimes\mathcal{E})$. Observe that this equation can be split into odd/even parts, namely: $\mathbf{X}u_{\text{even}} = f_1, \mathbf{X}u_{\text{odd}} = f_0$, and $u_{\text{even,odd}} \in C^\infty(SM,\pi^*\mathcal{E})$ have respective even/odd Fourier components. Applying (5.4) with u_{odd}, we obtain $f_0 = 0$, $\mathbf{X}u_{\text{odd}} = 0$ and $\nabla^{\mathcal{E}}_{\mathbb{V}}u_{\text{odd}} = 0$, that is u_{odd} is of degree 0 but 0 is even so $u_{\text{odd}} = 0$. As far as u_{even} is concerned, observe that $\nabla^{\mathcal{E}}_{\mathbb{V}}\mathbf{X}u_{\text{even}} = \nabla^{\mathcal{E}}_{\mathbb{V}}f_1$ and:

$$\|\nabla^{\mathcal{E}}_{\mathbb{V}}f_1\|^2_{L^2} = \langle -\Delta^{\mathcal{E}}_{\mathbb{V}}f_1, f_1\rangle_{L^2} = (n-1)\|f_1\|^2_{L^2}.$$

Hence, applying the twisted Pestov identity with u_{even}, we obtain:

$$0 = \|\mathbf{X}\nabla^{\mathcal{E}}_{\mathbb{V}} u_{\mathrm{even}}\|^2_{L^2} - \langle R\nabla^{\mathcal{E}}_{\mathbb{V}} u_{\mathrm{even}}, \nabla^{\mathcal{E}}_{\mathbb{V}} u_{\mathrm{even}}\rangle_{L^2} \geq C\|\nabla^{\mathcal{E}}_{\mathbb{V}} u_{\mathrm{even}}\|^2_{L^2},$$

that is u_{even} is of degree 0. □

5.4 The normal operator

We now discuss in greater details the properties of the geodesic Abelian X-ray transform introduced in §5.1 via the introduction of the *normal operator*, also called *generalized X-ray transform*. Although most of the results presented in this paragraph could be easily extended to the twisted case involving a vector bundle $\mathcal{E} \to M$, we stick to the trivial line bundle. This paragraph relies heavily on microlocal analysis. We refer the reader to Appendix A for further details.

5.4.1 Definition, first properties

Let

$$\Pi_m := \pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})\pi_m^*, \tag{5.5}$$

be the *normal operator*, where we recall that $\Pi = R_0^+ + R_0^-$ is defined thanks to the holomorphic parts of the resolvents at $z = 0$. It was introduced by Guillarmou [Gui17a]. We will see that it enjoys very good analytical properties.

Recall from §2.4.2 that given $(x, \xi) \in T^*M$, the space $\otimes^m_S T^*_x M$ decomposes as the direct sum

$$\begin{aligned}\otimes^m_S T^*_x M &= \operatorname{ran}\left(i\sigma_D(x,\xi)|_{\otimes^{m-1}_S T^*_x M}\right) \oplus \ker\left(i\sigma_{D^*}(x,\xi)|_{\otimes^m_S T^*_x M}\right)\\ &= \operatorname{ran}\left(j_\xi|_{\otimes^{m-1}_S T^*_x M}\right) \oplus \ker\left(\imath_{\xi^\sharp}|_{\otimes^m_S T^*_x M}\right)\end{aligned}$$

The projection on the right space parallel to the left space is denoted by $\pi_{\ker i_\xi}$ and $\mathrm{Op}(\pi_{\ker i_\xi}) = \pi_{\ker D^*} + \mathcal{O}(\Psi^{-1})$ by Lemma 2.14. The following theorem will be crucial in the sequel.

Theorem 5.19. *Π_m is a pseudodifferential operator of order -1 with principal symbol*

$$\sigma_m := \sigma_{\Pi_m} : (x,\xi) \mapsto \frac{2\pi}{C_{n,m}}|\xi|^{-1}\pi_{\ker i_\xi}\pi_{m*}\pi_m^*\pi_{\ker i_\xi},$$

with:

$$C_{n,m} = \int_0^\pi \sin^{n-2+2m}(\varphi)d\varphi.$$

We now need some wavefront set computations. For that, we are going to rely on §A.3. We introduce

$$\mathbb{V}^*(\mathbb{V}) = 0, \mathbb{H}^*(\mathbb{H} + \mathbb{R} \cdot X) = 0.$$

Recall that $\pi : SM \to M$ denotes the projection. We have the following

Lemma 5.20. *One has:*

$$\mathrm{WF}'(\pi_m^*) \subset \left\{ \left(((x,v), (\underbrace{\mathrm{d}\pi^\top \xi}_{\in \mathbb{V}^*}, \underbrace{0}_{\in \mathbb{H}^*})), (x, \xi) \right) \mid (x,\xi) \in T^*M \setminus \{0\} \right\}$$

In particular, if $u \in C^{-\infty}(M, \otimes_S^m T^*M)$ *then,* $\mathrm{WF}(\pi_m^* u) \subset \mathbb{V}^*$.

Proof. The case $m = 0$ is rather immediate and follows from Lemma A.16, since $\mathrm{d}\pi(\mathbb{V}) = 0$. We have for $z = (x, v) \in SM$:

$$\mathrm{WF}(\pi_0^* u) \subset \left\{ (z, \mathrm{d}\pi(z)^T \eta), (\pi(z), \eta) \in \mathrm{WF}(u) \right\} \subset \mathbb{V}^*$$

As to the case $m \geq 1$, it actually boils down to the case $m = 0$. Indeed, consider a point $x_0 \in M$ and a local smooth orthonormal basis $(e_1(x), ..., e_{N(m)}(x))$ of $\otimes_S^m T^*M$ in a neighborhood V_{x_0} of x_0, where $N(m) = \begin{pmatrix} n-1+m \\ m \end{pmatrix}$ denotes the rank of $\otimes_S^m T^*M$. Consider a smooth cutoff function χ such that $\chi \equiv 1$ in a neighborhood $W_{x_0} \subset V_{x_0}$ of x_0 and $\mathrm{supp}(\chi) \subset V_{x_0}$. Any smooth section ψ of $\otimes_S^m T^*M$ can be decomposed in V_{x_0} as:

$$\psi(x) = \sum_{j=1}^{N(m)} \langle \psi(x), e_j(x) \rangle_g e_j(x)$$

Thus:

$$\pi_m^*(\chi\psi) = \sum_{j=1}^{N(m)} \pi_0^* \left(\langle \psi(x), \chi e_j(x) \rangle_g \right) \pi_m^* e_j = \sum_{j=1}^{N(m)} \pi_0^* \left(A_j \psi \right) \pi_m^* e_j,$$

where the $A_j : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, \mathbb{R})$ are pseudodifferential operators of order 0 with support in $\mathrm{supp}(\chi)$. This expression still holds for a distribution u. Note that $\pi_m^* e_j$ is a smooth function on SM, thus the wavefront is given by the $\pi_0^*(A_j \psi)$ and by our previous remark for $m = 0$:

$$\mathrm{WF}(\pi_m^*(\chi u)) \subset \mathbb{V}^*$$

□

In other words, π_m^* localizes the wavefront set in $\mathbb{V}^*$. Moreover, since π_{m*} consists in integrating in the fibers S_xM, one has by Lemma A.14

$$\mathrm{WF}(\pi_{m*}u) \subset \left\{ (x,\xi) \mid \exists v \in S_xM, ((x,v), \underbrace{\mathrm{d}\pi^\top \xi}_{\in \mathbb{V}^*}, \underbrace{0}_{\in \mathbb{H}^*}) \in \mathrm{WF}(f) \right\}, \tag{5.6}$$

so that π_{m*} only selects the wavefront set in $\mathbb{V}^*$ and kills the wavefront set in the other directions.

For $\varepsilon > 0$, we consider a smooth cutoff function χ such that $\chi \equiv 1$ on $[0, \varepsilon/2]$, and $\chi \equiv 0$ on $[\varepsilon, +\infty)$. For $\Re(z) > 0$, we write

$$\begin{aligned} R_+(z) &= -\int_0^{+\infty} \chi(t) e^{-tz-tX} \mathrm{d}t - \int_0^{+\infty} (1-\chi(t)) e^{-tz-tX} \mathrm{d}t \\ &= -\int_0^{+\infty} \chi(t) e^{-tz-tX} \mathrm{d}t - R_+(z) \int_0^{+\infty} \chi'(t) e^{-tz-tX} \mathrm{d}t. \end{aligned}$$

Taking the limit as $z \to 0$, we obtain:

$$R_0^+ = \int_0^{+\infty} \chi(t) e^{-tX} \mathrm{d}t - R_0^+ \int_0^{+\infty} \chi'(t) e^{-tX} \mathrm{d}t - \int_0^{+\infty} \chi(t) \mathrm{d}t \ \Pi_0^+.$$

Note that the last operator is obviously smoothing. Hence:

$$\pi_{m*} R_0^+ \pi_m^* = \pi_{m*} \int_0^{+\infty} \chi(t) e^{-tX} \mathrm{d}t \pi_m^* - \pi_{m*} R_0^+ \int_0^{+\infty} \chi'(t) e^{-tX} \mathrm{d}t \pi_m^* + \text{ smoothing}$$

Lemma 5.21. *The operator $\pi_{m*} R_0^+ \int_0^{+\infty} \chi'(t) e^{-tX} \mathrm{d}t \pi_m^*$ is smoothing.*

In order to simplify the proof, we will admit the following caracterization of the wavefront set of R_0^+ obtained in [DZ16, Proposition 3.3]:

Lemma 5.22 (Dyatlov-Zworski)**.**

$$\begin{aligned} \mathrm{WF}'(R_0^+) \subset &\{(x,\xi;x,\xi) \mid (x,\xi) \in T^*(SM)\} \cup \{(\Phi_t(x,\xi);x,\xi) \mid (x,\xi) \in T^*(SM), \langle \xi, X(x) \rangle = 0\} \\ &\cup E_u^* \times E_s^* \end{aligned}$$

Exercise 5.23. Compare the previous Lemma with Example A.18.

We can now prove Lemma 5.21:

Proof. Let $u \in C^{-\infty}(M, \otimes_S^m T^*M)$ be a distribution with values in $\otimes_S^m T^*M$. By Lemma 5.20, one has that $\mathrm{WF}(\pi_m^* u) \subset \mathbb{V}^*$. Observe that χ' has support in $[\varepsilon/2, \varepsilon]$ and thus by

Example A.18, we know that:

$$\mathrm{WF}\left(\int_{\varepsilon}^{\varepsilon} \chi'(t)e^{-tX}dt\pi_m^* u\right) \subset \{\Phi_t(x,\xi) \mid t \in [\varepsilon/2, \varepsilon], (x,\xi) \in \mathbb{V}^*, \langle \xi, X\rangle = 0\}.$$

Then, using Lemma 5.22, we get:

$$\mathrm{WF}\left(R_0^+ \int_{\varepsilon/2}^{\varepsilon} \chi'(t)e^{-tX}dt\pi_m^* u\right) \subset \{\Phi_t(x,\xi) \mid t \geq \varepsilon, (x,\xi) \in \mathbb{V}^*, \langle \xi, X\rangle = 0\}.$$

By Lemma A.14, we see that:

$$\begin{aligned}\mathrm{WF}\, (\pi_{m*}\ R_0^+ & \int_{\varepsilon/2}^{\varepsilon} \chi'(t)e^{-tX}dt\pi_m^*\Big) \\ & \subset \left\{\mathrm{d}\pi^{-\top}\Phi_t(x,\xi) \mid \Phi_t(x,\xi) \in \mathbb{V}^*, (x,\xi) \in \mathbb{V}^*, t \geq \varepsilon, \langle \xi, X\rangle = 0\right\}.\end{aligned}$$

Now, by the absence of conjugate points (see Definition 2.4), one has that if $(x,\xi) \in \mathbb{V}^*, \langle \xi, X\rangle = 0$, then $\Phi_t(x,\xi) \notin \mathbb{V}^*$ for all $t \neq 0$ (note that we apply the definition for the cotangent bundle rather than the tangent bundle here), hence the wavefront set is empty. □

The same argument works for R_0^+ and we have thus proved that

$$\Pi_m = \pi_{m*} \int_{-\varepsilon}^{+\varepsilon} \chi(t)e^{-tX}\mathrm{d}t\pi_m^* + \text{ smoothing},$$

where χ is a cutoff function chosen to be equal to 1 at 0 and 0 outside $(-\varepsilon, \varepsilon)$. We can now prove Theorem 5.19. We will only deal with the case of Π_0 since it is rather similar for higher order tensors but complications arise due to the fact that the rank of $\otimes_S^m T^*M \to M$ is strictly bigger than 1. However, the computation for the principal symbol will be carried out in full generality.

Proof of Theorem 5.19. By the previous discussion, we have to prove that $\pi_{0*}\int_{-\varepsilon}^{\varepsilon} e^{tX}dt\pi_0^*$ is a pseudodifferential operator of order 0, where we can choose $\varepsilon > 0$ small enough, less than the injectivity radius of (M, g). Note that π_{0*} is simply the integration in the fibers S_xM. We fix a local chart (U, φ) and compute everything in this chart. If χ is a cutoff function with support in $\varphi(U)$ such that $e^{tX}(\mathrm{supp}(\chi)) \subset \varphi(U)$ for all $t \in (-\varepsilon, \varepsilon)$, then for $f \in C_c^\infty(\varphi(U))$:

$$\begin{aligned}\left(\chi\pi_{0*}\int_{-\varepsilon}^{\varepsilon} e^{tX}\mathrm{d}t\pi_0^*\chi\right) f(x) &= \int_{S_xM} \chi(x) \int_{-\varepsilon}^{\varepsilon} \pi_0^*\chi f(\varphi_t(x,v))\mathrm{d}t\mathrm{d}v \\ &= 2\int_{S_xM} \chi(x) \int_0^{\varepsilon} \pi_0^*\chi f(\varphi_t(x,v))\mathrm{d}t\mathrm{d}v\end{aligned}$$

For fixed x, since $\varepsilon > 0$ is smaller than the injectivity radius of (M, g), the map $(t, v) \mapsto \pi_0(\varphi_t(x, v)) = \exp_x(tv)$ is a diffeomorphism from $[0, \varepsilon) \times S_xM$ onto $B(x, \varepsilon)$. By making a change of variable in the previous integral, we obtain

$$\chi\pi_{0*}\int_{-\varepsilon}^{\varepsilon} e^{tX}\mathrm{d}t\pi_0^*\chi f(x) = \int_{\varphi(U)} K(x, y)f(y)\mathrm{d}y,$$

with $K(x, y) = 2\chi(x)\chi(y)|\det \mathrm{d}(\exp_x^{-1})_y|\sqrt{\det g(y)}/d^n(x, y)$. We compute the left symbol

$$p(x, \xi) = \int_{\mathbb{R}^{n+1}} e^{-iz\cdot\xi}K(x, x - z)\mathrm{d}z,$$

and we want to prove that $p \in S^{-1}(\mathbb{R}_x^{n+1} \times \mathbb{R}_\xi^{n+1})$. We write $F(x, z) = K(x, x - z)$. By [Tay11, Proposition 2.7], this amounts to proving that

$$\forall\alpha, \beta,\ \exists C_{\alpha\beta} > 0, \forall x \in \varphi(U), \forall z \neq 0, \qquad |\partial_x^\beta\partial_z^\alpha F(x, z)| \leq C_{\alpha\beta}|z|^{-n-|\alpha|} \tag{5.7}$$

The singularity of F is induced by $(x, z) \mapsto d^{-n}(x, x - z)$ (remark that $F(x, z) \sim_{|z|\to 0} 2\chi(x)^2\sqrt{\det g(x)}|z|^{-n}$) so this boils down to proving (5.7) for this function. But by the usual argument relying on Leibniz formula for the derivative of a product, this amounts to proving

$$\forall\alpha, \beta,\ \exists C_{\alpha\beta} > 0, \forall x \in \varphi(U), \forall z \neq 0, \qquad |\partial_x^\beta\partial_z^\alpha d^n(x, z)| \leq C_{\alpha\beta}|z|^{n-|\alpha|}.$$

But this is a rather immediate consequence of the fact that in local coordinates, there exist smooth functions $(G^{ij})_{1\leq i,j\leq n+1}$ defined in the patch $\varphi(U)$ such that $d^2(x, x - z) = \sum_{i,j} G^{ij}(x, x - z)z_iz_j$. Combining everything, we obtain that $p \in S^{-1}(\mathbb{R}_x^{n+1} \times \mathbb{R}_\xi^{n+1})$ so Π_0 is a pseudodifferential operator of order -1. The same arguments allow to show that Π_m is also a ΨDO of order -1 for any $m \geq 0$.

We now compute the principal symbol of Π_m. Let us consider a smooth section $f_1 \in C^\infty(M, \otimes_S^m T^*M)$ defined in a neighborhood of $x \in M$ and $f_2 \in \otimes_S^m T_x^*M$, then:

$$\begin{aligned}\langle\sigma_m(x_0, \xi)f_1, f_2\rangle_{x_0} &= \lim_{h\to 0} h^{-1}e^{-iS(x_0)/h}\langle\Pi_m(e^{iS(x)/h}f_1), f_2\rangle_{x_0} \\ &= \lim_{h\to 0} h^{-1}e^{-iS(x_0)/h}\langle\Pi\pi_m^*(e^{iS(x)/h}f_1), \pi_m^*f_2\rangle_{L^2(S_{x_0}M)},\end{aligned}$$

where $\xi = \mathrm{d}S(x) \neq 0$. Here, it is assumed that Hess_xS is non-degenerate. According to the previous paragraph, we can only consider the integral in time between $(-\varepsilon, \varepsilon)$, where

$\varepsilon > 0$ is chosen small enough (less than the injectivity radius at the point x), namely:

$$\begin{aligned}
&\langle \sigma_m(x,\xi) f_1, f_2\rangle_{x_0} \\
&= \lim_{h\to 0} h^{-1} \int_{\mathbb{S}^n} \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t))-S(x))} \pi_m^* f_1(\gamma(t),\dot\gamma(t)) \pi_m^* f_2(x_0,v)\chi(t) \mathrm{d}t\mathrm{d}v \\
&= \lim_{h\to 0} h^{-1} \int_{\mathbb{S}^{n-1}} \Big(\int_0^\pi \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t))-S(x))} \pi_m^* f_1(\gamma(t),\dot\gamma(t)) \pi_m^* f_2(x_0,v) \\
&\qquad\qquad \sin^{n-1}(\varphi)\chi(t)\mathrm{d}t\mathrm{d}\varphi\Big)\, \mathrm{d}u
\end{aligned}$$

where χ is a cutoff function with support in $(-\varepsilon,\varepsilon)$, γ is the geodesic such that $\gamma(0) = x, \dot\gamma(0) = v$ and we have decomposed $v = \cos(\varphi)w + \sin(\varphi)u$ with $w = \xi^\sharp/|\xi| = \mathrm{d}S(x)^\sharp/|\mathrm{d}S(x)|$, $u \in \mathbb{S}^{n-2}$. We apply the stationnary phase lemma [Zwo12, Theorem 3.13] uniformly in the $u \in \mathbb{S}^{n-2}$ variable. For fixed u, the phase is $\Phi : (t,\varphi) \mapsto S(\gamma(t)) - S(x)$ so $\partial_t\Phi(t,\varphi) = \mathrm{d}S(\dot\gamma(t))$. More generally if $\tilde\Phi : (t,v) \mapsto S(\gamma(t)) - S(x)$, then

$$\partial_v \tilde\Phi(t,v)\cdot V = d\pi(d\varphi_t(x,v)\cdot V), \qquad \forall V \in \mathbb{V}.$$

Since (M,g) has no conjugate points, $\mathrm{d}\pi(\mathrm{d}\varphi_t(x,v))\cdot V \neq 0$ as long as $t \neq 0$ and $V \in \mathbb{V}\setminus\{0\}$. And $\mathrm{d}S(\dot\gamma(0)) = \mathrm{d}S(\cos(\varphi)w + \sin(\varphi)u) = \cos(\varphi)|\mathrm{d}S(x)| = 0$ if and only if $\varphi = \pi/2$. So the only critical point of Φ is $(t = 0, \varphi = \pi/2)$. Let us also remark that

$$\mathrm{Hess}_{(0,\pi/2)}\Phi = \begin{pmatrix} \mathrm{Hess}_x S(u,u) & -|\mathrm{d}S(x)| \\ -|\mathrm{d}S(x)| & 0 \end{pmatrix}$$

is non-degenerate with determinant $-|\xi|^2$, so the stationary phase lemma can be applied and we get:

$$\begin{aligned}
&\int_0^\pi \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t))-S(x_0))} \pi_m^* f_1(\gamma(t),\dot\gamma(t)) \pi_m^* f_2(x_0,v) \sin^{n-2}(\varphi)\mathrm{d}t\mathrm{d}\varphi \\
&\qquad \sim_{h\to 0} 2\pi h|\xi|^{-1} \pi_m^* f_1(x_0,u)\pi_m^* f_2(x_0,u).
\end{aligned}$$

Eventually, we obtain:

$$\langle \sigma_m(x,\xi) f_1, f_2\rangle_{x_0} = \frac{2\pi}{|\xi|} \int_{\{\langle \xi, v\rangle = 0\}} \pi_m^* f_1(v)\pi_m^* f_2(v)\mathrm{d}S_\xi(v),$$

where $\mathrm{d}S_\xi$ is the canonical measure induced on the $(n-2)$-dimensional sphere

$$\mathbb{S}_\xi M := \mathbb{S}_x M \cap \{\langle \xi, v\rangle = 0\}.$$

The result then follows from the following computation. We write $E = T_x M$.

We can write $f_1 = j_\xi f_p + f_s$ where $f_p \in \otimes_S^{m-1} E^*$, $f_s \in \ker\left(\imath_\xi|_{\otimes_S^m T_x^* M}\right)$, where $j_\xi f_p = \mathcal{S}(\xi \otimes f_p)$. Note that $\pi_m^*(j_\xi f_p)(v) = \langle \xi, v\rangle \pi_{m-1}^* f_p(v)$ and this vanishes on $\{\langle \xi, v\rangle = 0\}$ (and the same holds for f_2). In other words, $\pi_m^* f_1 = \pi_m^* \pi_{\ker \imath_\xi} f_1$ on $\{\langle \xi, v\rangle = 0\}$. We are thus left to check that for $f_1, f_2 \in \ker \imath_\xi$,

$$C_{n,m} \int_{\langle \xi, v\rangle = 0} \pi_m^* f_1(v) \pi_m^* f_2(v) \mathrm{d}S_\xi(v) = \int_{\mathbb{S}_E} \pi_m^* f_1(v) \pi_m^* f_2(v) \mathrm{d}S(v),$$

for some constant $C_{n,m} > 0$. We will use the coordinates $v' = (v, \varphi) \in \mathbb{S}_{E,\xi} \times [0, \pi]$ on $\mathbb{S}_E$ which allow to decompose $v' = \sin(\varphi) v + \cos(\varphi) \xi^\sharp / |\xi|$. Then the measure on $\mathbb{S}_E$ disintegrates as $\mathrm{d}S = \sin^{n-2}(\varphi) \mathrm{d}\varphi \mathrm{d}S_\xi(v)$. Also remark that $\pi_m^* f(v + \cos(\varphi)\xi^\sharp/|\xi|) = \pi_m^* f(v)$. Then, if $C_{n,m} := \int_0^\pi \sin^{n-2+2m}(\varphi) d\varphi$, we obtain:

$$\begin{aligned}
&\int_{\langle \xi, v\rangle = 0} \pi_m^* f_1(v) \pi_m^* f_2(v) \mathrm{d}S_\xi(v) \\
&\qquad = C_{n,m}^{-1} \int_0^\pi \sin^{n-2+2m}(\varphi) \mathrm{d}\varphi \int_{\langle \xi, v\rangle = 0} \pi_m^* f_1(v) \pi_m^* f_2(v) \mathrm{d}S_\xi(v) \\
&\qquad = C_{n,m}^{-1} \int_0^\pi \int_{\langle \xi, v\rangle = 0} \pi_m^* f_1(\sin(\varphi) v + \cos(\varphi)\xi^\sharp/|\xi|) \\
&\qquad\qquad \times \pi_m^* f_2(\sin(\varphi) v + \cos(\varphi)\xi^\sharp/|\xi|) \sin^{n-2}(\varphi) \mathrm{d}\varphi \mathrm{d}S_\xi(v) \\
&\qquad = C_{n,m}^{-1} \int_{\mathbb{S}_E} \pi_m^* f_1(v') \pi_m^* f_2(v') \mathrm{d}S(v')
\end{aligned}$$

□

5.4.2 Main properties of the normal operator

The crucial property of the normal operator Π_m is that it is elliptic on solenoidal tensors.

Lemma 5.24. *The operator Π_m is elliptic on solenoidal tensors, that is there exists pseudodifferential operators Q and R of respective order 1 and $-\infty$ such that:*

$$Q \Pi_m = \pi_{\ker D^*} + R$$

Proof. We define

$$\tilde{q}(x, \xi) = \begin{cases} 0, & \text{on } \operatorname{ran}(j_\xi) \\ \dfrac{C_{n,m}}{2\pi} |\xi| (\pi_{\ker \imath_{\xi^\sharp}} \pi_{m*} \pi_m^* \pi_{\ker \imath_{\xi^\sharp}})^{-1}, & \text{on } \ker(\imath_{\xi^\sharp}) \end{cases}$$

and $q(x, \xi) = (1 - \chi(x, \xi)) \tilde{q}(x, \xi)$ for some cutoff function $\chi \in C^\infty_{\mathrm{comp}}(T^*M)$ around the zero section. By construction, $\mathrm{Op}(q) \Pi_m = \pi_{\ker D^*} - R'$ with $R' \in \Psi^{-1}$. Let $r' = \sigma_{R'}$ and define

$a \sim \sum_{k=0}^{\infty} r'^k$. Then $\mathrm{Op}(a)$ is a microlocal inverse for $\mathbb{1} - R'$ that is $\mathrm{Op}(a)(\mathbb{1} - R') \in \Psi^{-\infty}$. Since $R'D = 0$, we obtain that $R' = R'\pi_{\ker D^*}$ and thus

$$\underbrace{\mathrm{Op}(a)\,\mathrm{Op}(q)}_{=Q}\,\Pi_m = \mathrm{Op}(a)(\mathbb{1} - R')\pi_{\ker D^*} = \pi_{\ker D^*} + R,$$

where R is a smoothing operator. □

We now study **injectivity** of the normal operator. From now on, we will add a subscript sol to denote the fact that we consider solenoidal tensors, i.e. elements in $\ker D^*$. The next lemma shows that the s-injectivity of the X-ray transform is equivalent to that of the normal operator Π_m:

Lemma 5.25. *I_m is solenoidal injective if and only if Π_m is injective on the space $H^s_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, for all $s \in \mathbb{R}$.*

Proof. There is a trivial implication: s-injectivity of Π_m implies that of I_m. Indeed, assume $f \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ is such that $I_m f = 0$, then $\pi_m^* f = Xu$ for some $u \in C^\infty(SM)$ by the smooth Livšic Theorem 4.11. But then $\Pi_m f = \pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})\pi_m^* f = \pi_{m*}\Pi X u = 0$ by Lemma 3.11. Thus $f = 0$.

Let us now prove the converse. We fix $s \in \mathbb{R}$. We assume that $\Pi_m f = 0$ for some $f \in H^s_{\mathrm{sol}}(M, S^m(T^*M))$. By ellipticity of the operator, we get that $f \in C^\infty_{\mathrm{sol}}(M, S^m(T^*M))$. And:

$$\begin{aligned}\langle \Pi_m f, f\rangle_{L^2} &= \langle \Pi \pi_m^* f, \pi_m^* f\rangle_{L^2} + \left(\int_{SM} \pi_m^* f \, d\mu\right)^2 \\ &= \langle (-\Delta + 1)^{-s}\Pi \pi_m^* f, \pi_m^* f\rangle_{H^s} + \left(\int_{SM} \pi_m^* f \, d\mu\right)^2 = 0.\end{aligned}$$

By Lemma 3.11, since $\langle \Pi\pi_m^* f, \pi_m^* f\rangle \geq 0$, we obtain that $\int_{SM} \pi_m^* f d\mu = 0$. Moreover, $(-\Delta + 1)^{-s}\Pi$ is bounded and positive on H^s so there exists a square root $R : H^s \to H^s$, that is a bounded positive operator satisfying $(-\Delta+1)^{-s}\Pi = R^* R$, where R^* is the adjoint on H^s. Then:

$$\langle (-\Delta + 1)^{-s}\Pi \pi_m^* f, \pi_m^* f\rangle_{H^s} = 0 = \|R\pi_m^* f\|^2_{H^s}$$

This yields $(-\Delta+1)^{-s}\Pi\pi_m^* f = 0$ so $\Pi\pi_m^* f = 0$. By Lemma 3.11, there exists $u \in C^\infty(SM)$ such that $\pi_m^* f = Xu$ so $f \in \ker I_m \cap \ker D^*$. By s-injectivity of the X-ray transform, we get $f \equiv 0$. □

In particular, the previous lemma directly implies the following, which was already known since [DS03]:

Proposition 5.26. *Let (M,g) be a smooth Anosov Riemannian manifold. Then, the kernel of I_m on $C^\infty_{\mathrm{sol}}(M,\otimes^m_S T^*M)$ is finite dimensional.*

Proof. By Lemma 5.25, s-injectivity of I_m is equivalent to that of Π_m, which is elliptic on solenoidal tensors. Hence its kernel is finite-dimensional, see Proposition A.5. □

Another direct consequence of Lemma 5.25 and Theorem 5.24 is the following:

Theorem 5.27. *If I_m is solenoidal injective, then there exists a pseudodifferential operator Q' of order 1 such that: $Q'\Pi_m = \pi_{\ker D^*}$.*

Proof. The operator Π_m is elliptic of order -1 on $\ker D^*$, thus Fredholm as an operator $H^s_{\mathrm{sol}}(M,\otimes^m_S T^*M) \to H^{s+1}_{\mathrm{sol}}(M,\otimes^m_S T^*M)$ for all $s \in \mathbb{R}$. It is selfadjoint on the Hilbert space $H^{-1/2}_{\mathrm{sol}}(M,\otimes^m_S T^*M)$, thus Fredholm of index 0 (the index being independent of the Sobolev space considered, see [Shu01, Theorem 8.1]), and injective, thus invertible on $H^s_{\mathrm{sol}}(M,\otimes^m_S T^*M)$. We multiply the equality $Q\Pi_m = \pi_{\ker D^*} + R$ on the right by $Q' := \pi_{\ker D^*}\Pi_m^{-1}\pi_{\ker D^*}$:

$$Q\Pi_m Q' = Q\underbrace{\Pi_m \pi_{\ker D^*}}_{=\Pi_m}\Pi_m^{-1}\pi_{\ker D^*} = Q\pi_{\ker D^*} = Q' + RQ'$$

As a consequence, $Q' = Q\pi_{\ker D^*} +$smoothing so it is a pseudodifferential operator of order 1. And $Q'\Pi_m = \pi_{\ker D^*}$. □

This yields the following stability estimate:

Lemma 5.28. *If I_m is solenoidal injective, then for all $s \in \mathbb{R}$, there exists a constant $C := C(s) > 0$ such that:*

$$\forall f \in H^s_{\mathrm{sol}}(M,\otimes^m_S T^*M), \qquad \|f\|_{H^s} \leq C\|\Pi_m f\|_{H^{s+1}}$$

We also obtain a coercivity lemma on the operator Π_m.

Lemma 5.29. *If I_m is solenoidal injective, then there exists a constant $C > 0$ such that:*

$$\forall f \in H^{-1/2}(M,\otimes^m_S T^*M), \qquad \langle \Pi_m f, f\rangle \geq C\|\pi_{\ker D^*} f\|^2_{H^{-1/2}}.$$

Proof. The operator $\pi_{m*}\pi_m^* : \otimes^m_S T^*_x M \to \otimes^m_S T^*_x M$ is positive definite and thus admits a square root $S(x) : \otimes^m_S T^*_x M \to \otimes^m_S T^*_x M$, self-adjoint and such that $S^m(x) = \pi_{m*}\pi_m^*$. We introduce the symbol $b \in C^\infty(T^*M)$ of order $-1/2$ defined by $b : (x,\xi) \mapsto \chi(x,\xi)|\xi|^{-1/2}S(x)$, where $\chi \in C^\infty(T^*M)$ vanishes near the 0 section in T^*M and equal to 1 for $|\xi| > 1$ and define $B := \mathrm{Op}(b) \in \Psi^{-1/2}(M,\otimes^m_S T^*M)$, where Op is a quantization on M. Using that the principal symbol of $\pi_{\ker D^*}$ is $\imath_{\xi^\sharp}$, the inner product with $\xi^\sharp$, we observe that $\Pi_m = \pi_{\ker D^*} B^* B \pi_{\ker D^*} + R$, where $R \in \Psi^{-2}(M, \otimes_S^m T^*M)$. Thus, given $f \in H^{-1/2}(M, \otimes_S^m T^*M)$:

$$\langle \Pi_m f, f \rangle_{L^2} = \|B\pi_{\ker D^*} f\|_{L^2}^2 + \langle Rf, f\rangle_{L^2} \tag{5.8}$$

By ellipticity of B, there exists a pseudodifferential operator Q of order $1/2$ such that $QB\pi_{\ker D^*} = \pi_{\ker D^*} + R'$, where $R' \in \Psi^{-\infty}(M, \otimes_S^m T^*M)$ is smoothing. Thus there is $C > 0$ such that for each $f \in C^\infty(M, \otimes_S^m T^*M)$

$$\|\pi_{\ker D^*} f\|_{H^{-1/2}}^2 \leq \|QB\pi_{\ker D^*} f\|_{H^{-1/2}}^2 + \|R'f\|_{H^{-1/2}}^2 \leq C\|B\pi_{\ker D^*} f\|_{L^2}^2 + \|R'f\|_{H^{-1/2}}^2.$$

Since Lemma 5.29 is trivial on potential tensors, we can already assume that f is solenoidal, that is $\pi_{\ker D^*} f = f$. Reporting in (5.8), we obtain that

$$\begin{aligned} \|f\|_{H^{-1/2}}^2 &\leq C\langle \Pi_m f, f\rangle_{L^2} - C\langle Rf, f\rangle_{L^2} + \|R'f\|_{H^{-1/2}}^2 \\ &\leq C\langle \Pi_m f, f\rangle_{L^2} + C\|Rf\|_{H^{1/2}}\|f\|_{H^{-1/2}} + \|R'f\|_{H^{-1/2}}^2. \end{aligned} \tag{5.9}$$

Now, assume by contradiction that the statement in Lemma 5.29 does not hold, that is we can find a sequence of tensors $f_n \in C^\infty(M, \otimes_S^m T^*M)$ such that $\|f_n\|_{H^{-1/2}} = 1$ with $D^* f_n = 0$ and

$$\|\sqrt{\Pi_m} f_n\|_{L^2}^2 = \langle \Pi_m f_n, f_n\rangle_{L^2} \leq \|f_n\|_{H^{-1/2}}^2/n = 1/n \to 0.$$

Up to extraction, and since R is of order -2, we can assume that $Rf_n \to v_1$ in $H^{1/2}$ for some v_1, and $R'f_n \to v_2$ in $H^{-1/2}$. Then, using (5.9), we obtain that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1/2}$ which thus converges to an element $v_3 \in H^{-1/2}$ such that $\|v_3\|_{H^{-1/2}} = 1$ and $D^* v_3 = 0$. By continuity, $\Pi_m f_n \to \Pi_2 v_3$ in $H^{1/2}$ and thus $\langle \Pi_2 v_3, v_3\rangle = 0$. Since v_3 is solenoidal, we get $\sqrt{\Pi_m} v_3 = 0$, thus $\Pi_2 v_3 = 0$. Note that I_m is s-injective by assumption, thus Π_m is also injective by Lemma 5.25. This implies that $v_3 \equiv 0$, thus contradicting $\|v_3\|_{H^{-1/2}} = 1$. □

We now study **surjectivity**. The normal operator Π_m is formally self-adjoint, elliptic on solenoidal tensors and is thus Fredholm of index 0. As a consequence, Π_m is injective on solenoidal tensors if and only if it is surjective. We can even be more precise on this statement:

Lemma 5.30. *I_m is solenoidal injective if and only if*

$$\pi_{m*} : C^{-\infty}_{\mathrm{inv}}(SM) \to C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M)$$

is surjective.

Here, $C^{-\infty}_{\mathrm{inv}}(SM) = \cup_{s\leq 0}, H^{-s}_{\mathrm{inv}}(SM)$ denotes the distributions which are invariant by the geodesic flow. We note that this lemma was first stated in the literature in the case of simple manifolds [PZ16].

Proof. Assume that $\pi_{m*} : C^{-\infty}_{\mathrm{inv}}(SM) \to C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ is surjective. Let $f \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ be such that $I_m f = 0$. Then $\pi_m^* f = Xu$ for some $u \in C^\infty(SM)$ by the smooth Livšic Theorem 4.11 and $f = \pi_{m*} h$ for some $h \in C^{-\infty}_{\mathrm{inv}}(SM)$ by assumption. Then:

$$0 = \langle Xh, u\rangle = -\langle h, Xu\rangle = -\langle h, \pi_m^* f\rangle = -\langle \pi_{m*} h, f\rangle = -\|f\|^2$$

Thus $f \equiv 0$.

We now prove the converse. If I_m is s-injective, then Π_m is s-injective and thus surjective on solenoidal tensors. Thus, given $f \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, there exists $u \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ such that $f = \Pi_m u = \pi_{m*}\Pi\pi_m^* u$, that is $f = \pi_{m*} h$ for $h = \Pi\pi_m^* u \in \cap_{s>0} H^{-s}(SM)$. □

Eventually, we will need this last lemma which we leave as an exercise for the reader:

Lemma 5.31. $\Pi\pi_m^* : H^{-s}(M, \otimes^m_S T^*M) \to H^{-s}(SM)$ *is bounded, for any* $s > 0$. *By duality,* $\pi_{m*}\Pi : H^s(SM) \to H^s(M, \otimes^m_S T^*M)$ *is bounded too, for any* $s > 0$.

5.4.3 Stability estimates for the X-ray transform

An useful consequence of the previous tools is that we can derive stability estimates for the X-ray transform:

Lemma 5.32. *There exists* $s_0 \in (0, 1)$ *and* $C, \tau > 0$ *such that for all* $f \in C^1(M, \otimes^m_S T^*M)$*:*

$$\|\pi_{\ker D^*} f\|_{H^{s_0}} \leq C\|I_m f\|^\tau_{\ell^\infty(\mathcal{C})}\|f\|^{1-\tau}_{C^1}$$

We did not try to optimize the constants in the previous Lemma; in particular, C^1 regularity could be lowered to some C^β for $0 < \beta < 1$.

Proof. Without loss of generality, we can always assume that f is solenoidal. By the approximate Livšic Theorem 4.12, we can write $\pi_m^* f = Xu + h$, where $\|h\|_{C^\alpha} \leq C\|I_m f\|^\tau_{\ell^\infty(\mathcal{C})}\|f\|^{1-\tau}_{C^1}$, for some $\alpha, C > 0$. Applying the operator $\pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})$, we then obtain, for $s < \alpha$:

$$\begin{aligned}
\|f\|_{H^{s-1}} &\lesssim \|\Pi_m f\|_{H^s}, && \text{by Lemma 5.28,}\\
&= \|\pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})(Xu + h)\|_{H^s} &&\\
&= \|\pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})h\|_{H^s}, && \text{by Lemma 3.11,}\\
&\lesssim \|h\|_{H^s}, && \text{by Lemma 5.31,}\\
&\lesssim \|h\|_{C^\alpha} \leq C\|I_m f\|^\tau_{\ell^\infty(\mathcal{C})}\|f\|^{1-\tau}_{C^1} &&
\end{aligned}$$

□

6 The marked length spectrum

The section is devoted to one of the most famous geometric inverse problems on closed manifolds: the *Burns-Katok conjecture* [BK85], also known as the *marked length spectrum rigidity conjecture*.

6.1 The Burns-Katok conjecture

We consider an Anosov Riemannian manifold (M, g). We recall that $\mathcal{C}$ denotes the set of free homotopy classes on M. This set is in one-to-one correspondance with the conjugacy classes of the fundamental group $\pi_1(M)$ and if (M, g) is Anosov, there exists a unique closed geodesic $\gamma_g(c) \in c$ in each free homotopy class $c \in \mathcal{C}$.

Definition 6.1. The marked length spectrum of the Anosov manifold (M, g) is the map

$$L_g : \mathcal{C} \to \mathbb{R}_+, \quad L_g(c) := \ell_g(\gamma_g(c)),$$

where $\ell_g(\gamma)$ denotes the Riemannian length of the curve γ computed with respect to the metric g.

Let $\mathrm{Met}_{\mathrm{An}}$ be the space of (smooth) Anosov metrics on M and let $\mathrm{Diff}^0(M)$ be the group of smooth diffeomorphisms that are isotopic to the identity. It is clear that the map

$$\mathrm{Met}_{\mathrm{An}} \ni g \mapsto L_g$$

is invariant by the action (by pullback) of $\mathrm{Diff}^0(M)$, namely $L_g = L_{\phi^* g}$ whenever $\phi \in \mathrm{Diff}^0(M)$. An element $[g] \in \mathrm{Met}_{\mathrm{An}}/\mathrm{Diff}^0(M)$ is called *an isometry class*. We are interested in the following conjecture, known as the *Burns-Katok conjecture* [BK85] or the *marked length spectrum rigidity conjecture*[12]:

Conjecture 6.2. *The map*

$$\mathrm{Met}_{\mathrm{An}}/\mathrm{Diff}^0(M) \ni [g] \mapsto L_{[g]}$$

is injective.

This conjecture was proved on negatively-curved surfaces [Cro90, Ota90] and in some other partial cases [Kat88, BCG95, Ham99] but remains open in full generality. Otal's proof in the two-dimensional case [Ota90] is remarkable insofar it combines in a clever and beautiful way elements of the theory of two-dimensional negatively-curved Riemannian spaces. Unfortunately, it is out of reach of the present survey and we encourage the curious

[12] Originally, it was only formulated for negatively-curved manifolds.

reader to have a look at Wilkinson's lecture notes [Wil14] where this is explained in great details. In this section, we will mainly explain how the previous theory of X-ray transform brings new and interesting (although partial) answers to the conjecture. In particular, we will prove the following local version:

Theorem 6.3 (Guillarmou-L. '18, Guillarmou-Knieper,L. '19)**.** *Let (M, g_0) be a smooth Anosov manifold and further assume it is non-positively-curved if* $\dim(M) \geq 3$*. Then, there exists $N, \varepsilon > 0$ such that the following holds. Let g be a metric such that $\|g - g_0\|_{C^N} < \varepsilon$ and $L_g = L_{g_0}$. Then $[g] = [g_0]$, i.e. the metrics are isometric.*

This result was initially obtained by Guillarmou and the author in [GL19d] and then refined in a joint work with Knieper [GKL19]. We also point out that similar results were then obtained on manifolds with hyperbolic cusps by Bonthonneau and the author in the sequence of papers [GL19b, GL19c]. Before proving Theorem 6.3, we study an easier version of the problem to which we will refer to as *linear* or *infinitesimal*.

6.2 Linear problem

The linear version of the Burns-Katok conjecture consists in looking at a family $(g_s)_{s\in(-1,1)}$ such that $L_{g_s} = L_{g_0}$. If the conjecture is true, then one should be able to find an isotopy $(\phi_s)_{s\in(-1,1)}$ such that $\phi_s^* g_s = g_0$. We call this problem the *infinitesimal rigidity of the marked length spectrum* and we say that a metric g_0 is infinitesimally rigid if this holds. The important remark is the following:

Lemma 6.4. *We have:*

$$\frac{\mathrm{d}}{\mathrm{d}s} L_{g_s} = 1/2 \times I_2^{g_s}(\dot{g}_s),$$

where $\dot{g}_s = \frac{\mathrm{d}}{\mathrm{d}s} g_s$.

The proof is left as an exercise to the reader; it uses the fact that geodesics are critical points of the length functional among a free homotopy class of curves. As a consequence, if the metrics $(g_s)_{s\in(-1,1)}$ share the same marked length spectrum, then we obtain:

$$I_2^{g_s}(\dot{g}_s) = 0.$$

If all the metrics $(g_s)_{s\in(-1,1)}$ are known to be solenoidal-injective, this implies that $\dot{g}_s = D_{g_s} p_s$, for some $p_s \in C^\infty(M, T^*M)$, where D_{g_s} is the symmetric derivative induced by the metric g_s. By duality, p_s can be identified with a vector field $-X_s \in C^\infty(M, TM)$ and $\dot{g}_s = -\mathcal{L}_{X_s} g_s$. As a consequence, if $(\phi_s)_{s\in(-1,1)}$ denotes the isotopy generated by the vector fields $(X_s)_{s\in(-1,1)}$, then we obtain that $\phi_s^* g_s = g_0$. Note that solenoidal injectivity is an open property with respect to the metric hence, s-injectivity of g_0 implies that of all g

in a C^k-neighborhood of g_0 (for k large enough). This can be proved by using the fact that s-injectivity of I^g_m is equivalent to that of Π^g_m (by Lemma 5.25) and that the operator $C^\infty(M, \otimes^2_S T^*M) \ni g \mapsto \Pi^g_m \in \Psi^{-1}$ is continuous (see [GKL19]). In other words, we obtain the following:

Lemma 6.5. *If (M, g_0) is an Anosov Riemannian manifold such that $I^{g_0}_2$ is solenoidal injective, then it is infinitesimally rigid in the sense that any smooth family of metrics $(g_s)_{s\in(-1,1)}$ such that $L_{g_s} = L_{g_0}$ satisfies $\phi^*_s g_s = g_0$ for some isotopy $(\phi_s)_{s\in(-1,1)}$.*

In particular, as mentioned earlier, $I^{g_0}_2$ is known to be solenoidal injective when (M, g) is Anosov, under the additional assumption that the sectional curvature is nonpositive if $\dim(M) \geq 3$.

6.3 Local geometry of the space of metrics

From now on, $SM := SM_{g_0}$ and the metric g_0 is fixed on M and assumed to be Anosov. We are interested in the local geometry of the space of (smooth) Anosov metrics $\mathrm{Met}_{\mathrm{An}}$ in a neighborhood of g_0.

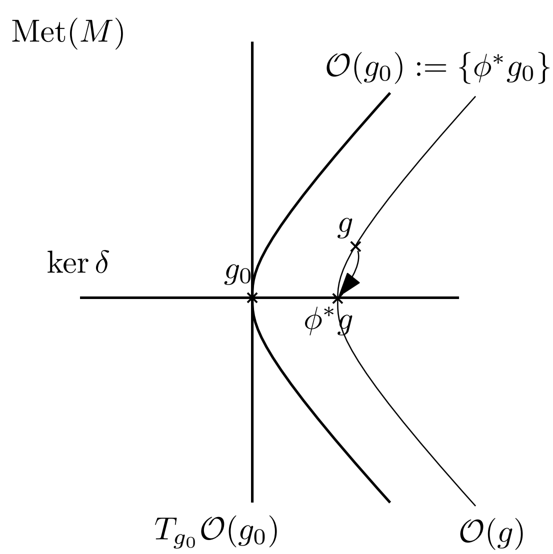


Figure 8: A local picture of the geometry of the space of all metrics.

Passing through g_0 are two important subspaces of $\mathrm{Met}_{\mathrm{An}}$ (see Figure 8): one is $\mathcal{O}(g_0) := \left\{\phi^* g_0 \mid \phi \in \mathrm{Diff}^0(M)\right\}$, the orbit of g_0 under the action of the group of smooth diffeomorphisms that are isotopic to the identity. The other one is $g_0 + \ker D^*_{g_0}$, the space of solenoidal tensors (with respect to g_0) which is obviously affine. It can be easily checked that

$$T_{g_0}\mathcal{O}(g_0) = \{\mathcal{L}_V g_0 \mid V \in C^\infty(M, TM)\} = \{D_{g_0} p \mid p \in C^\infty(M, T^*M)\}.$$

Moreover, as $T_{g_0}\mathrm{Met}_{\mathrm{An}} \simeq C^\infty(M, \otimes_S^2 T^*M)$, we see, using the decomposition of symmetric tensors of Theorem 2.13 into potential/solenoidal parts, that:

$$\begin{aligned} T_{g_0}\mathrm{Met}_{\mathrm{An}} &\simeq C^\infty(M, \otimes_S^2 T^*M) \\ &= D_{g_0} C^\infty(M, T^*M) \oplus \ker D_{g_0}^*|_{C^\infty(M, \otimes_S^2 T^*M)} \\ &= T_{g_0}\mathcal{O}(g_0) \oplus T_{g_0}\left(g_0 + \ker D_{g_0}^*|_{C^\infty(M,\otimes_S^2 T^*M)}\right), \end{aligned}$$

that is the two (Fréchet) submanifolds $\mathcal{O}(g_0)$ and $g_0 + \ker D_{g_0}^*|_{C^\infty(M,\otimes_S^2 T^*M)}$ of $\mathrm{Met}_{\mathrm{An}}$ are transverse at g_0. This is represented in Figure 8.

Moreover, it can be proved that the various orbits $\mathcal{O}(g)$ for g in a neighborhood of g_0 are all transverse to $g_0 + \ker D_{g_0}^*|_{C^\infty(M,\otimes_S^2 T^*M)}$. This can be seen in the content of the following Lemma who goes back to [Ebi68] (see also [GL19d]). We provide the proof in *finite* regularity as it is easier and relies on the implicit function Theorem for Banach spaces (below $C^{N,\alpha}$ denotes the space of C^N functions such that the N-th derivatives are α-Hölder continuous). Nevertheless, it still holds in the smooth category (i.e. taking $N = \infty$) by applying the Nash-Moser Theorem.

Lemma 6.6. *Assume g_0 is smooth. Let $N \geq 2$, $\alpha \in (0,1)$. Then, there exists $\varepsilon > 0$ such that the following holds. For any metric g such that $\|g - g_0\|_{C^{N,\alpha}} < \varepsilon$, there exists a (unique) diffeomorphism isotopic to the identity ϕ, of regularity $C^{N+1,\alpha}$, such that $D^*(\phi^* g) = 0$. The metric $\phi^* g$ (of regularity $C^{N,\alpha}$) is called the* solenoidal reduction *of g.*

Proof. Consider the map $C^{k+1,\alpha}(M, TM) \ni V \mapsto e_V := x \mapsto \exp_x(V(x)) \in \mathrm{Diff}^{k+1,\alpha}(M)$ (the exponential map is that induced by g_0); it is a well-defined smooth diffeomorphism for $V \in \mathcal{U}_0$ a small $C^{k+1,\alpha}$-neighborhood of the zero section onto a neighborhood of the identity in $\mathrm{Diff}^{k+1,\alpha}(M)$. We define

$$F_1 : \mathcal{U}_0 \times C^{k,\alpha}(M, \otimes_S^2 T^*M) \to C^{k-1,\alpha}(M, \otimes_S^2 T^*M), \qquad F_1(V, f) = D_{g_0}^*(e_V^*(g_0 + f))$$

and we want to solve locally the equation $F_1(V(f), f) = 0$. Note that $e_V^*(g + f) \in C^{k,\alpha}(M, \otimes_S^2 T^*M)$ if $V \in C^{k+1,\alpha}(M, TM)$. However, there is a subtle problem here coming from the fact that F_1 *is not smooth* in a neighborhood of $(0,0)$ but only differentiable. This would not prevent us from applying the inverse function theorem, but the regularity of the map $g \mapsto \phi$ would only be C^1. Indeed, if we take $f \neq 0$, then $g := g_0 + f \in C^{k,\alpha}(M, \otimes_S^2 T^*M)$ and in local coordinates

$$(e_V^* g)_{kl}(x) = g_{ij}(e_V(x)) \frac{\partial e_V^i}{\partial x_k}(x) \frac{\partial e_V^j}{\partial x_l} \tag{6.1}$$

As a consequence, by the chain rule, differentiating with respect to V makes a term $Z \mapsto \mathrm{d}_{e_V(x)} g_{ij}(\mathrm{d}_V e(Z)) \in C^{k-1,\alpha}(M, \otimes_S^2 T^*M)$ appear and differentiating twice, we would obtain a term in $C^{k-2,\alpha}(M, \otimes_S^2 T^*M)$ (so we would leave the Banach space $C^{k-1,\alpha}(M, \otimes_S^2 T^*M)$). However, remark that

$$e_{V*} \circ D_{g_0}^* \circ e_V^* = D_{e_{V*}g_0}^* \tag{6.2}$$

Thus, solving $D_{g_0}^* e_V^*(f + g_0) = 0$ is equivalent to solving $D_{e_{V*}g_0}^*(f + g_0) = 0$. Therefore, we rather consider

$$F_2 : \mathcal{U}_0 \times C^{k,\alpha}(M, \otimes_S^2 T^*M) \to C^{k-1,\alpha}(M, \otimes_S^2 T^*M), \qquad F_2(V, f) = D_{e_{V*}g_0}^*(f + g_0)$$

and we want to solve $F_2(V(f), f) = 0$ in a neighborhood of $(0, 0)$. The map F_2 is *smooth*. Indeed, it is immediately smooth in f, since it is linear and by (6.1), since g is smooth, it is smooth in V.

Since $\mathrm{d}_V e(0) = \mathbb{1}$ (because the differential of the exponential map $\exp_x$ at 0 is the identity), we see from (6.2) that $\mathrm{d}_V F_2(0, 0) = \mathrm{d}_V F_1(0, 0)$. As a consequence, by the implicit function theorem, solving $F_2(V(f), f) = 0$ in a neighborhood of $(0, 0)$ amounts to proving that $d_V F_1(0, 0)$ is an isomorphism. The differential of F_1 at $(0, 0)$ is given by

$$\mathrm{d}_V F_1(0, 0) \cdot Z = D_{g_0}^*(\mathcal{L}_Z g_0) = 2 \times D_{g_0}^* D_{g_0}(Z^\sharp),$$

for $Z \in C^{k+1,\alpha}(M, TM)$, where $\sharp : TM \to T^*M$ is the musical isomorphism induced by the metric g (and this maps $C^{k+1,\alpha}(M, TM) \to C^{k-1,\alpha}(M, \otimes_S^2 T^*M)$ which is coherent). But $D_{g_0}^* D_{g_0}$ is a differential operator of order 2 which is elliptic and injective — since D_{g_0} is, by Lemma 2.12. Moreover, it is formally selfadjoint and its Fredholm index is thus equal to 0 by Proposition A.5 so it is also surjective, hence invertible. As a consequence $D_{g_0}^* D_{g_0}$: $C^{k+1,\alpha}(M, T^*M) \to C^{k-1,\alpha}(M, T^*M)$ is an isomorphism. By the implicit function theorem for Banach spaces, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ and a smooth map $f \mapsto V(f)$ (from $C^{k,\alpha}(M, \otimes_S^2 T^*M) \to C^{k+1,\alpha}(M, \otimes_S^2 T^*M)$) such that $F_2(V(f), f) = 0$ for all $f \in \mathcal{U}$ (and thus $F_1(V(f), f) = 0$). Moreover, $V(f)$ is the unique solution to $F_{1,2}(Z, f) = 0$ in this neighborhood. □

Note that one has to use $C^{k,\alpha}$ as the spaces C^k for $k \in \mathbb{N}$ are not well-suited for microlocal analysis (or one has to resort to Hölder-Zygmund spaces C_*^k). The previous discussion also shows that the moduli space $\mathbb{M} := \mathrm{Met}_{\mathrm{An}}/\mathrm{Diff}^0(M)$ (i.e. the space of orbits $[g] = \mathcal{O}(g)$) can be locally identified with $g_0 + \ker D_{g_0}^*$. In other words, this last space is a *local chart* for $\mathbb{M}$ near $[g_0]$.

6.4 Local rigidity via the geodesic stretch

We fix the metric g_0 and consider a metric g in a C^2-neighborhood of g_0. By Anosov structural stability (see [GKL19, Appendix B] for instance), there exists an orbit-conjugacy of the geodesic flows i.e. a map

$$\psi_g : SM_{g_0} \to SM_g$$

such that

$$\mathrm{d}\psi_g(X_{g_0}(z)) = a_g(z) X_g(\psi_g(z)), \quad \forall z \in SM_{g_0}, \tag{6.3}$$

where a_g is a function on SM_{g_0} called the *infinitesimal stretch*. The map ψ_g is not unique and a_g is only defined up to a coboundary, namely a term of the form $X_{g_0} u$. Recall that two functions are said to be *cohomologous* if they differ by a coboundary. The infinitesimal stretch is linked to the marked length spectrum by the following equality: for all $c \in \mathcal{C}$,

$$L_g(c) = \int_{\gamma_{g_0}(c)} a_g(\varphi_t(z)) \mathrm{d}t,$$

where z is an arbitrary point on $\gamma_{g_0}(c)$. Observe that the previous integral is indeed invariant by adding a coboundary to a_g. The following lemma is well-known (see the discussion in [GKL19, Section 2.5] for instance):

Lemma 6.7. *The following statements are equivalent:*

1. $L_g = L_{g_0}$,
2. *The geodesic flows are conjugate i.e. there exists* $\widetilde{\psi}_g : SM_{g_0} \to SM_g$ *such that* $\widetilde{\psi}_g \circ \varphi_t^{g_0} = \varphi_t^g \circ \widetilde{\psi}_g$, *for all* $t \in \mathbb{R}$,
3. a_g *is cohomologous to the constant function* $\mathbf{1}$.

Proof. (1) $\Leftrightarrow$ (3) If $L_g = L_{g_0}$ then

$$L_g(c) = \int_0^{L_{g_0}(c)} a_g(\varphi_t(z)) \mathrm{d}t = L_{g_0}(c) = \int_0^{L_{g_0}(c)} \mathbf{1}(\varphi_t(z)) \mathrm{d}t.$$

As a consequence, by the Livšic Theorem 4.11, $a_g - \mathbf{1} = Xu$ for some Hölder-continuous $u \in C^\alpha(SM_{g_0})$. The converse is also immediate.

(2) $\Longrightarrow$ (1) is straightforward. Let us show that (3) $\Longrightarrow$ (2). First of all, the flows are always conjugate up to a time reparametrization, namely:

$$\varphi^g_{\kappa_g(z,t)}(\psi_g(z)) = \psi_g(\varphi_t^{g_0}(z)), \tag{6.4}$$

for all $t \in \mathbb{R}, z \in SM_{g_0}$, where

$$\kappa_g(z,t) = \int_0^t a_g(\varphi_s^{g_0}(z)) \mathrm{d}s.$$

As a consequence, if $a_g = \mathbf{1} + Xu$, then

$$\kappa_g(z,t) = t + u(\varphi_t^{g_0}(z)) - u(z),$$

and using (6.4):

$$\varphi_t^g\left(\varphi_{-u(z)}^g \circ \psi_g(z)\right) = \varphi^g_{-u(\varphi_t^{g_0}(z))} \circ \psi_g(\varphi_t^{g_0}(z)),$$

that is the flows are conjugate by $z \mapsto \varphi^g_{-u(z)} \circ \psi_g(z) =: \tilde{\psi}_g(z)$. □

Although these considerations seem to be only local (i.e. g close to g_0) insofar as they rely on the Anosov structural stability, one can prove that they are actually global and g does not need to be taken close to g_0. This is very specific to geodesic flows, see [GKL19, Appendix B] for instance.

As a consequence, it is natural to consider the cohomology class of the infinitesimal stretch minus one $[a_g - \mathbf{1}]$ as a faithful measure of the distance between the marked length spectra of the metrics g_0 and g. Let us give a more precise meaning to that. Given $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, we introduce the space of coboundaries D^α of regularity α, namely:

$$D^\alpha(SM_{g_0}) := \left\{X_{g_0}u \mid u \in C^\alpha(SM_{g_0}), X_{g_0}u \in C^\alpha(SM_{g_0})\right\}.$$

This is a closed subspace of $C^\alpha(SM_{g_0})$ and we can therefore consider the quotient space $C^\alpha(SM_{g_0})/D^\alpha(SM_{g_0})$ endowed with the natural norm

$$\|[f]\|_{C^\alpha/D^\alpha} := \inf_{X_{g_0}u \in D^\alpha} \|f + X_{g_0}u\|_{C^\alpha},$$

where $[f]$ denotes an element in C^α/D^α.

For g close to g_0, we can look at the map

$$C^k(M, \otimes_S^2 T^*M) \ni g \mapsto a_g \in C^\nu(SM),$$

where $\nu > 0$ is some fixed exponent and this map is known to be C^{k-2} by [Con92, Proposition 1.1]. In particular, for $k = 2$, it admits a Taylor expansion:

$$a_g - \mathbf{1} = 0 + \mathrm{d}a|_{g=g_0}(g - g_0) + \mathcal{O}_{C^\nu}(\|g - g_0\|^2), \tag{6.5}$$

and we have (see [GKL19, Lemma 3.3])

Lemma 6.8. $\mathrm{d}a|_{g=g_0}(g-g_0)$ *is cohomologous to* $1/2 \times \pi_2^*(g - g_0)$.

Proof. We consider $c \in \mathcal{C}$ and use both expressions for $L_g(c)$:

$$L_g(c) = \int_0^{L_{g_0}(c)} a_g(\varphi_t^{g_0}(z))\mathrm{d}t = \int_{\gamma_g(c)} g^{1/2}\mathrm{d}\gamma_g(c).$$

Taking the derivative with respect to g at g_0, we obtain:

$$\int_0^{L_{g_0}(c)} \mathrm{d}a_{g=0}(h)(\varphi_t^{g_0}(z))\mathrm{d}t = 1/2 \times \int_{\gamma_{g_0}(c)} h\mathrm{d}\gamma_g(c) + \partial_g \left(\int_{\gamma_g(c)} g_0^{1/2} \mathrm{d}\gamma_g(c) \right)(h),$$

and the last term vanishes as geodesics are critical points of the length functional. This completes the proof. □

We will prove the following which implies in turn Theorem 6.3.

Theorem 6.9. *Let* (M, g_0) *be an Anosov manifold such that* $I_2^{g_0}$ *is injective. There exists* $\nu, \varepsilon, N, \alpha > 0$ *such that the following holds. For any metric* g *such that* $\|g - g_0\|_{C^{N,\alpha}} < \varepsilon$, *there exists a* $C^{N+1,\alpha}$*-diffeomorphism* ϕ, *isotopic to the identity, such that*

$$\|\phi^* g - g_0\|_{H^{\nu/2-1}} \leq C \|[a_g - \mathbf{1}]\|_{C^\nu/D^\nu}.$$

Proof of Theorem 6.9. First of all, we define $g' = \phi^* g$ as the solenoidal reduction of g (with respect to g_0), i.e. $D^*(g' - g_0) = 0$, by Lemma 6.6. The following map is C^2 (see [Con92]) and we can Taylor-expand it:

$$C^2(M, \otimes_S^2 T^* M) \ni g \mapsto a_g \in C^\nu(SM),$$

and we obtain at $g = g_0$, using (6.5) and Lemma 6.8:

$$a_{g'} - 1 = 0 + 1/2 \times \pi_2^*(g' - g_0) + Xw + r,$$

where $r = \mathcal{O}_{C^\nu}(\|g' - g_0\|_{C^2}^2)$. Hence, for an arbitrary $f \in C^\nu$ such that $Xf \in C^\nu$, we obtain:

$$a_{g'} - 1 + Xf = 1/2 \times \pi_2^*(g' - g_0) + X(w + f) + r$$

Observe that a_g and $a_{g'}$ are cohomologous since g and g' have same marked length spectrum, i.e. $a'_g = a_g + Xf'$, hence:

$$a_g - 1 + Xf = 1/2 \times \pi_2^*(g' - g_0) + X(w + f - f') + r$$

Applying the operator $\pi_{2*}(\Pi + \mathbf{1} \otimes \mathbf{1})$, we obtain:

$$\pi_{2*}(\Pi + \mathbf{1} \otimes \mathbf{1})(a_g - 1 + Xf) = 1/2 \times \Pi_2(g' - g_0) + \pi_{2*}(\Pi + \mathbf{1} \otimes \mathbf{1})r.$$

Using Lemma 5.31, we obtain:

$$\begin{aligned}\|g' - g_0\|_{H^{\nu/2-1}} &\lesssim \|\Pi_2(g' - g_0)\|_{H^{\nu/2}} \\ &\lesssim \|\pi_{2*}(\Pi + \mathbf{1} \otimes \mathbf{1})(a_g - 1 + Xf)\|_{H^{\nu/2}} + \|\pi_{2*}(\Pi + \mathbf{1} \otimes \mathbf{1})r\|_{H^{\nu/2}} \\ &\lesssim \|a_g - 1 + Xf\|_{H^{\nu/2}} + \|r\|_{H^{\nu/2}} \\ &\lesssim \|a_g - 1 + Xf\|_{C^\nu} + \|r\|_{C^\nu} \\ &\lesssim \|a_g - 1 + Xf\|_{C^\nu} + \|g' - g_0\|^2_{C^2} \\ &\lesssim \|a_g - 1 + Xf\|_{C^\nu} + \|g' - g_0\|_{H^{\nu/2-1}}\|g' - g_0\|_{C^{N,\alpha}},\end{aligned}$$

for some $N \geq 0$ large enough, $\alpha \in (0,1)$, where the last inequality is obtained by interpolation. Assuming $\|g' - g_0\|_{C^{N,\alpha}} < \varepsilon$ is small enough, the second term on the right-hand side can be swallowed on the left-hand side and we obtain:

$$\|\phi^* g - g_0\|_{H^{\nu/2-1}} \leq C\|a_g - 1 + Xf\|_{C^\nu}$$

Since f was arbitrary, we can take the infimum over all such coboundaries Xf and we obtain the desired result. □

6.5 Generalized Weil-Petersson metric on $\mathbb{M}$

Recall that $\mathbb{M} := \mathrm{Met}_{\mathrm{An}}/\mathrm{Diff}^0(M)$. The operator Π_2 also allows to define a metric on the moduli space $\mathbb{M}$:

Definition 6.10 (Generalized Weil-Petersson metric)**.** Let $[g] \in \mathbb{M}$ and $[h] \in T_{[g]}\mathbb{M}$. We introduce the symmetric bilinear form:

$$G_{[g]}([h],[h]) := \langle \Pi_2^g h, h\rangle_{L^2},$$

where g is an element of the class $[g]$ and $h \in \ker D_g^*$ represents $[h]$.

We leave it as an exercise to the reader to check that this is indeed well-defined, independently of the choice of element g in the class $[g]$.

Lemma 6.11. *G defines a smooth metric on $\mathbb{M}$ called the* generalized Weil-Petersson metric.

Note that this is a metric on an infinite-dimensional space.

Proof. Smoothness is not trivial and follows from that of the map $C^\infty(M, \otimes_S^2 T^*M) \mapsto \Pi_2^g \in \Psi^{-1}$ is smooth. It was proved in [GKL19] that this map is indeed continuous but, inspecting the proof, the same arguments also show smoothness. The fact that G is a metric is a mere consequence of Lemma 5.29 i.e. $G_{[g]}([h],[h]) = \langle \Pi_2^g h, h\rangle_{L^2} \geq C\|h\|^2_{H^{-1/2}}$. □

We now consider the specific case where M is an orientable surface of genus $g \geq 2$. We denote by $\mathcal{T}(M)$ the Teichmüller space of M, i.e. the space of hyperbolic metrics (with constant curvature -1) quotiented by $\mathrm{Diff}^0(M)$. This space is endowed with canonical metric called the *Weil-Petersson metric*, see [Tro92, Chapter 5]. This is a smooth manifold diffeomorphic to $\mathbb{R}^{6g-6}$ which can be seen as a natural submanifold of $\mathbb{M}$.

Lemma 6.12. *The restriction $G|_{\mathcal{T}(M)}$ is equal to (a multiple of) the Weil-Petersson metric.*

We refer to [GKL19] for a proof (based on [BCLS15]). The Weil-Petersson metric on $\mathcal{T}(M)$ has been well-studied and some important properties are known. For instance, it is known that this metric has negative sectional curvature, see [Ahl62] and [Tro92, Theorem 5.4.15]. In the same vein, one can wonder if this still holds true for te generalized Weil-Petersson metric G.

Question 6.13. Is the sectional curvature of G negative?

7 Holonomy inverse problem

In this section, we study the inverse problem of recovering a connection from the knowledge of the trace of its holonomy along closed geodesics. Let (M, g) be a smooth Anosov Riemannian manifold. Given a vector bundle $\mathcal{E} \to M$, we let $\mathcal{A}_{\mathcal{E}}$ be the set of smooth unitary connections on $\mathcal{E}$. The gauge-group $\mathcal{G}(\mathcal{E}) := C^\infty(M, \mathrm{U}(\mathcal{E}))$ acts on an element $\nabla^{\mathcal{E}} \in \mathcal{A}_{\mathcal{E}}$ by pullback, that is if $p \in C^\infty(M, \mathrm{U}(\mathcal{E}))$, then $p^*\nabla^{\mathcal{E}} := p\nabla^{\mathcal{E}}(p^{-1}\bullet)$. An orbit

$$\mathfrak{a} := \left\{p^*\nabla^{\mathcal{E}} \mid p \in C^\infty(M, \mathrm{U}(\mathcal{E}))\right\}$$

is called a *gauge-class* of connections and two connections in the same orbit are *gauge-equivalent*. We let $\mathbb{A}_{\mathcal{E}} := \mathcal{A}_{\mathcal{E}}/\mathcal{G}(\mathcal{E})$ be the space of all orbits.

Two smooth vector bundles $\mathcal{E}_1, \mathcal{E}_2 \to M$ are *isomorphic* if there exists a non-vanishing section $p \in C^\infty(M, \mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1))$ which is everywhere invertible as a linear map. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hermitian (namely, they are equipped with a Hermitian metric), one can check that p can actually be taken unitary, that is $p \in C^\infty(M, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$. An *equivalence class* of vector bundles over M is the set of all vector bundles identified up to isomorphisms. We let $\mathrm{Vect}_r(M)$ be the set of all vector bundles over M with rank $r \geq 0$ and $\mathrm{Vect}(M) := \sqcup_{r \geq 0} \mathrm{Vect}_r(M)$. As a warm-up, the reader can try the following exercise:

Exercise 7.1. Compute $\mathrm{Vect}_r(\mathbb{S}^1)$.

The moduli space of all connections is then defined as:

$$\mathbb{A} := \bigsqcup_{[\mathcal{E}] \in \mathrm{Vect}(M)} \mathbb{A}_{[\mathcal{E}]}.$$

There is a natural additive structure $\oplus$ (by taking the direct sum for the vector bundle and the connection parts) making $(\mathbb{A}, \oplus)$ a *semi-group*. There is also a natural multiplicative structure $(\mathbb{A}, \otimes)$ by taking tensor products.

7.1 The primitive trace map

Consider a vector bundle $\mathcal{E} \to M$ equipped with a unitary connection $\nabla^{\mathcal{E}}$. We denote by C the parallel transport map along geodesics, that is, if $(x, v) \in SM$, we consider a geodesic segment $\gamma : [0, L] \ni t \mapsto \pi(\varphi_t(x, v))$ with endpoints $x_- = x$ and $x_+ = \pi(\varphi_L(x, v))$. We then define:

$$C((x, v), t) : \mathcal{E}_{x_-} \to \mathcal{E}_{x_+},$$

as the parallel transport map along the geodesic γ with respect to the connection $\nabla^{\mathcal{E}}$. An equivalent point of view is to consider the lift $(\pi^*\mathcal{E}, \pi^*\nabla^{\mathcal{E}})$ on SM. Then C is the cocycle over the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ obtained by parallel transport along the flowlines with respect to $\pi^*\nabla^{\mathcal{E}}$ as in §4.3.

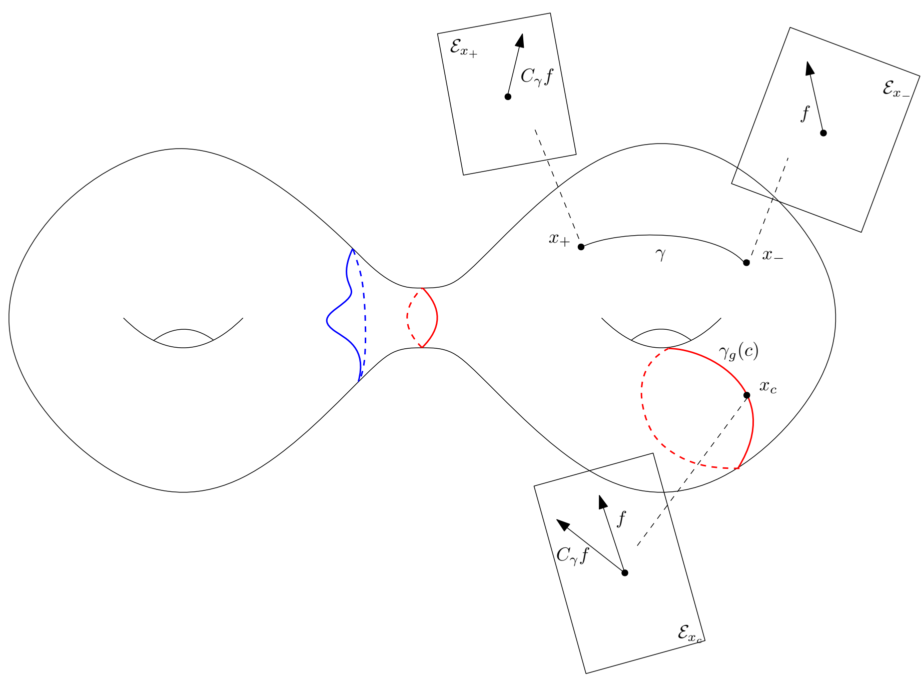


Figure 9: Parallel transport on a closed oriented surface.

In the same spirit as for the marked length spectrum problem, one can look at the holonomy induced by the connection on closed geodesics. However, it will now be important to make a distinction between primitive and non-primitive closed geodesics. We let $\mathcal{C}^\sharp$ be the set of *primitive free homotopy classes*. For each $c^\sharp \in \mathcal{C}^\sharp$, we choose an arbitrary point $x_{c^\sharp} \in \gamma_g(c^\sharp)$ on the unique closed geodesic $\gamma_g(c^\sharp) \in c^\sharp$; we denote by $v_{c^\sharp} \in S_{x_{c^\sharp}}M$ the unit vector generating $\gamma_g(c^\sharp)$. We then consider the trace of the holonomy along the curve, namely $\mathrm{Tr}\,(C((x_{c^\sharp}, v_{c^\sharp}), \ell_g(c)))$. It is straightforward to check that this map is invariant by the action of the gauge-group on connections (since two gauge-equivalent connections have conjugate holonomies). As a consequence, the trace descends as a map on the moduli space. We call the induced map the *primitive trace map*:

$$\mathcal{T}^\sharp : \mathbb{A} \longrightarrow \ell^\infty(\mathcal{C}^\sharp). \tag{7.1}$$

More precisely, given $\mathrm{x} = ([\mathcal{E}], \mathfrak{a}) \in \mathbb{A}$ (a point corresponds to a pair: class of vector bundles and gauge-class of connections) and any $\nabla^{\mathcal{E}} \in \mathfrak{a}$, $\mathcal{T}^\sharp(\mathfrak{x})(c^\sharp)$ is the trace of the holonomy of the connection along the unique closed geodesic in the class $c^\sharp \in \mathcal{C}^\sharp$. We ask the following general geometric inverse problem:

Question 7.2. Is the map $\mathcal{T}^\sharp$ injective?

In full generality, this is an open question and the goal of the following paragraphs is to provide some partial answers to it.

7.2 Global results

7.2.1 Line bundles

This paragraph is mainly inspired by [Pat09]. Recall that the topology of a line bundle is determined by its first Chern class that is

$$\mathrm{Vect}_1(M) \ni [\mathcal{L}] \mapsto c_1([\mathcal{L}]) \in H^2(M, \mathbb{Z})$$

is a bijection, where $[\mathcal{L}]$ stands for a class of isomorphic line bundles, see [Bry93, Theorems 2.2.14 and 2.2.15]. As a consequence, the moduli space of all connections on line bundles $\mathbb{A}_1$ can be decomposed as

$$\mathbb{A}_1 = \bigsqcup_{[\mathcal{L}] \in H^2(M,\mathbb{Z})} \mathbb{A}_{[\mathcal{L}]}.$$

It was observed by [Kos70] that $\mathbb{A}_1$ carries a natural Abelian group structure using the tensor product, see also [Bry93, Theorems 2.2.18] and below for further details.

Lemma 7.3. *There exists a natural group structure* $(\mathbb{A}_1, \otimes)$ *defined in the following way: for* $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{A}_1$,

$$\mathrm{x}_1 \otimes \mathrm{x}_2 := ([\mathcal{L}_1 \otimes \mathcal{L}_2], [\nabla^{\mathcal{L}_1} \otimes \nabla^{\mathcal{L}_2}]). \tag{7.2}$$

The inverse is obtained by duality, namely

$$\mathrm{x}^{-1} = ([\mathcal{L}^*], [\nabla^{\mathcal{L}^*}]).$$

The neutral element $\mathbf{1}_{\mathbb{A}_1}$ is given by the class of the trivial connection on the trivial line bundle.

Proof. Without taking into account the moduli space of gauge-equivalent connections, this is a standard result of algebraic topology, see [Bry93] for instance. The fact that the product is well-defined on the connection part is almost tautological; details are left as an exercise to the reader. $\square$

When restricted to line bundles, the primitive trace map $\mathcal{T}_1^\sharp : \mathbb{A}_1 \longrightarrow \ell^\infty(\mathcal{C}^\sharp)$ becomes a group homomorphism. More precisely:

Lemma 7.4. *The map $\mathcal{T}_1^\sharp : \mathbb{A}_1 \to \ell(\mathcal{C}^\sharp, \mathrm{U}(1))$ is a multiplicative group homomorphism.*

Proof. This is easily reduced to a local statement in coordinates. Now, in a patch $V \subset \mathbb{R}^n$ of coordinates, if $\nabla^{\mathcal{L}_{1,2}} = d + i\theta_{1,2}$ (for real-valued $\theta_{1,2} \in C^\infty(V, T^*V)$), then $\nabla^{\mathcal{L}_1} \otimes \nabla^{\mathcal{L}_2} = d + i\theta$, where $\theta = \theta_1 + \theta_2$. Using that $\mathrm{U}(1)$ is Abelian and that the holonomy with respect to a connection along a loop γ is given by exponentiating the integral of the connection 1-form on γ, we easily obtain the result. $\square$

The main result of this paragraph is the following, mainly due to Paternain [Pat09]:

Theorem 7.5 (Paternain '09)**.** *Let (M, g) be a smooth Anosov manifold of dimension ≥ 3. Then the restriction of the primitive trace map to line bundles*

$$\mathcal{T}_1^\sharp : \mathbb{A}_1 \longrightarrow \ell^\infty(\mathcal{C}^\sharp), \tag{7.3}$$

is globally injective.

Observe that on surfaces, the trivial line bundle $\mathbb{C} \times M \to M$ (with trivial connection) and the canonical line bundle $\kappa \to M$ (with the Levi-Civita connection) both have trivial holonomy but are not isomorphic, hence $\mathcal{T}_1^\sharp$ cannot be injective. We shall compute its kernel below in the surface case, see Lemma 7.9. Before proving Theorem 7.5, we shall need the following result showing that, in any dimension $n \geq 2$, on a *fixed* line bundle, the trace map is injective:

Proposition 7.6 (Paternain '09)**.** *Let (M, g) be an Anosov Riemannian manifold of dimension ≥ 2 and let $\mathcal{L} \to M$ be a Hermitian line bundle over M. Let $\nabla_1^{\mathcal{L}}$ and $\nabla_2^{\mathcal{L}}$ be two unitary connections such that $\mathcal{T}_1^\sharp(\nabla_1^{\mathcal{L}}) = \mathcal{T}_1^\sharp(\nabla_2^{\mathcal{L}})$. Then $\nabla_1^{\mathcal{L}}$ and $\nabla_2^{\mathcal{L}}$ are gauge-equivalent.*

As we will see, the proof relies on the solenoidal injectivity of the geodesic X-ray transform I_1 studied in §5.1.

Proof. As $\nabla^{\mathcal{L}}_{1,2}$ are unitary connections, we can write $\nabla^{\mathcal{L}}_2 = \nabla^{\mathcal{L}}_1 + i\theta$, where $\theta \in C^\infty(M, T^*M)$ is a real valued 1-form on M. Since the connections are transparent, we have

$$\int_{\gamma_g(c)} \theta \in 2\pi\mathbb{Z}, \tag{7.4}$$

for all $c \in \mathcal{C}$. We introduce the cocycle

$$C((x,v),t) := \exp\left(i \int_0^t \pi_1^*\theta(\varphi_s(x,v))\mathrm{d}s\right) \in \mathrm{U}(1), \tag{7.5}$$

which satisfies the periodic orbit obstruction (see Definition 4.17) by (7.4). As a consequence, by the smooth Livšic cocycle Theorem 4.18, there exists $u \in C^\infty(SM, \mathrm{U}(1))$ such that

$$C((x,v),t) = u(\varphi_t(x,v))u(x,v)^{-1}.$$

Differentiating with respect to t and evaluating at $t = 0$, we obtain:

$$i \times \pi_1^*\theta = (Xu)u^{-1}.$$

We now introduce the closed 1-form $\omega := \frac{du}{iu} \in C^\infty(SM, T^*(SM))$. If $\pi : SM \to M$ denotes the projection, then $\pi^* : H^1(M) \to H^1(SM)$ is an isomorphism by the Gysin sequence (one has to use here that M cannot be the two-torus). As a consequence, we can write $\omega = \pi^*\eta + df$, for some harmonic 1-form $\eta \in H^1(M)$ and $f \in C^\infty(SM)$. Applying the vector field X and the commutation relation of Lemma 2.11, we obtain:

$$\omega(X) = \pi_1^*\theta = \pi_1^*\eta + Xf,$$

that is $\pi_1^*(\theta - \eta) = Xf$ and thus $I_1(\theta - \eta) = 0$. By s-injectivity of the X-ray transform I_1 on Anosov manifolds [DS03], we obtain that $\theta - \eta = df'$ is exact. In particular, θ is closed. If we fix a basepoint $x_0 \in M$ and consider for $x \in M$ a geodesic γ joining x_0 to x, then we set:

$$G(x) := \exp\left(i \int_\gamma \theta \mathrm{d}\gamma\right),$$

and it can be checked that this definition is independent of γ (as θ is closed and $\int_{\gamma_g(c)} \theta \in 2\pi\mathbb{Z}$ for all closed geodesic). Such a $G \in C^\infty(M, \mathrm{U}(1))$ satisfies $\theta = dG/iG$ and the two connections are gauge-equivalent. □

The following lemma will also be needed:

Lemma 7.7. *Let (M, g) be a smooth closed Riemannian manifold of dimension ≥ 3 and let $\pi : SM \to M$ be the projection. Let $\mathcal{L}_1 \to M$ and $\mathcal{L}_2 \to M$ be two Hermitian lines bundles. If $\pi^*\mathcal{L}_1 \simeq \pi^*\mathcal{L}_2$ are isomorphic, then $\mathcal{L}_1 \simeq \mathcal{L}_2$ are isomorphic.*

Proof. The topology of line bundles is determined by their first Chern class. As a consequence, it suffices to show that $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$. By assumption, we have $c_1(\pi^*\mathcal{L}_1) = \pi^* c_1(\mathcal{L}_1) = c_1(\pi^*\mathcal{L}_2) = \pi^* c_1(\mathcal{L}_2)$ and thus it suffices to show that $\pi^* : H^2(M,\mathbb{Z}) \to H^2(SM,\mathbb{Z})$ is injective when $\dim(M) \geq 3$. But this is then a mere consequence of the Gysin short exact sequence [BT82, Proposition 14.33]:

$$... \longrightarrow \{0\} \longrightarrow H^2(M,\mathbb{Z}) \xrightarrow{\pi^*} H^2(SM,\mathbb{Z}) \longrightarrow ...$$

□

Observe that on surfaces, the Gysin exact sequence [BT82, Proposition 14.33] gives:

$$... \longrightarrow H^0(M,\mathbb{Z}) \xrightarrow{\wedge e} H^2(M,\mathbb{Z}) \xrightarrow{\pi^*} H^2(SM,\mathbb{Z}) \longrightarrow ...$$

where e is the Euler class. Note that $H^0(M,\mathbb{Z}) = H^2(M,\mathbb{Z}) = \mathbb{Z}$ and the Euler class corresponds to a multiplication by $2g-2$ (minus the Euler characteristic), that is the kernel of $\pi^* : H^2(M,\mathbb{Z}) \longrightarrow H^2(SM,\mathbb{Z})$ is given by $(2g-2)\mathbb{Z}$. Now, if κ denotes the canonical line bundle over M, then $c_1(\kappa) = 2g-2$ and thus $c_1(\kappa^{\otimes k}) = k(2g-2)$. In other words, we have proved:

Lemma 7.8. *Let M be an oriented surface. Then the kernel of $\pi^* : \mathrm{Vect}_1(M) \to \mathrm{Vect}_1(SM)$ is given by $\left\{[\kappa^{\otimes k}] \mid k \in \mathbb{Z}\right\}$.*

We can now prove Theorem 7.5:

Proof of Theorem 7.5. Assume that $\mathcal{T}_1^\sharp(\mathfrak{a}_1) = \mathcal{T}_1^\sharp(\mathfrak{a}_2)$, where $\mathfrak{a}_1 \in \mathbb{A}_{\mathcal{L}_1}$ and $\mathfrak{a}_2 \in \mathbb{A}_{\mathcal{L}_2}$ are two classes of connections defined on two (classes of) line bundles. By Theorem 4.23, we obtain that the pullback bundles $\pi^*\mathcal{L}_1$ and $\pi^*\mathcal{L}_2$ are isomorphic, hence $\mathcal{L}_1 \simeq \mathcal{L}_2$ are isomorphic by Lemma 7.7. Up to composing by a first bundle (unitary) isomorphism, we can therefore assume that $\mathcal{L}_1 = \mathcal{L}_2 =: \mathcal{L}$. Let $\nabla_1^{\mathcal{L}} \in \mathfrak{a}_1$ and $\nabla_2^{\mathcal{L}} \in \mathfrak{a}_2$ be two representatives of these classes. They satisfy $\mathcal{T}^\sharp(\nabla_1^{\mathcal{L}}) = \mathcal{T}^\sharp(\nabla_2^{\mathcal{L}})$. Applying Proposition 7.6, the primitive trace map $\mathcal{T}_{\mathcal{L}}^\sharp$ is known to be globally injective for connections on the same fixed bundle. Hence $\nabla_1^{\mathcal{L}}$ and $\nabla_2^{\mathcal{L}}$ are gauge-equivalent. □

In the two-dimensional case, we obtain:

Lemma 7.9. *If (M,g) is Anosov and $\dim(M) = 2$, then:*

$$\ker \mathcal{T}_1^\sharp = \left\{([\kappa^{\otimes n}], [\nabla^{\mathrm{LC}^{\otimes n}}]), n \in \mathbb{Z}\right\},$$

where $\kappa \to M$ denotes the canonical line bundle and ∇^{LC} is connection induced on κ by the Levi-Civita connection.

Proof. We write $\mathrm{x} = ([\mathcal{L}], \mathfrak{a})$. If $\mathcal{T}_1^\sharp(\mathrm{x}) = (1, 1, ...)$ (i.e. the connection is transparent), then by Theorem 4.23, one has that $\pi^*\mathcal{L} \to SM$ is trivial. By the Gysin sequence [BT82, Proposition 14.33], this implies that $c_1(\mathcal{L})$ is divisible by $2g-2$, where g is the genus of M (see [Pat09, Theorem 3.1]), hence $[\mathcal{L}] = [\kappa^{\otimes n}]$ for some $n \in \mathbb{Z}$. Moreover, the Levi-Civita connection on $\kappa^{\otimes n}$ has trivial holonomy and by the uniqueness result of Proposition 7.6, this implies that $\mathfrak{a} = [\nabla^{\mathrm{LC}}]$. □

7.2.2 Anosov 3-manifolds

We give a particular emphasis to the three-dimensional case. Actually, it is very likely that the primitive trace-map is injective on $\mathbb{A}$ as long as the dimension is greater or equal than 3; hence a first step would be to study the three-dimensional case. Observe that if $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{A}$ satisfy $\mathcal{T}^\sharp(\mathrm{x}_1) = \mathcal{T}^\sharp(\mathrm{x}_2)$, then one obtains as a direct consequence of Theorem 4.23 that $\pi^*[\mathcal{E}_1] \simeq \pi^*[\mathcal{E}_2]$ are isomorphic. As a consequence, when $\pi^* : \mathrm{Vect}(M) \to \mathrm{Vect}(SM)$ is injective, one obtains that $[\mathcal{E}_1] \simeq [\mathcal{E}_2]$. We shall see that this is indeed the case on 3-manifolds thus proving:

Proposition 7.10. *Let (M, g) be an Anosov Riemannian 3-manifold. Let $\mathrm{x}_1 = ([\mathcal{E}_1], \mathfrak{a}_1), \mathrm{x}_2 = ([\mathcal{E}_2], \mathfrak{a}_2) \in \mathbb{A}$ such that $\mathcal{T}^\sharp(\mathrm{x}_1) = \mathcal{T}^\sharp(\mathrm{x}_2)$. Then $[\mathcal{E}_1] = [\mathcal{E}_2]$.*

By the preliminary discussion, this boils down to:

Lemma 7.11. *Let (M, g) be a Riemannian 3-manifold and let $\pi : SM \to M$ be the projection. Then $\pi^* : \mathrm{Vect}(M) \to \mathrm{Vect}(SM)$ is injective.*

Proof. Observe that the following holds: if N is a smooth manifold of dimension n, and $\mathcal{E} \to N$ is a smooth complex vector bundle of rank r, then $\mathcal{E}$ admits a global non-vanishing section when $r > n/2$. As a consequence, on the 3-manifold M, we can write $\mathcal{E} = \varepsilon^{r-1} \oplus \det(\mathcal{E})$, where ε^{r-1} denotes the trivial bundle; the last bundle is a line bundle and it has to be isomorphic to $\det(\mathcal{E}) := \Lambda^r\mathcal{E}$ since $c_1(\mathcal{E}) = c_1(\det \mathcal{E})$ and c_1 is invariant by adding copies of the trivial bundle. Hence, taking $[\mathcal{E}_1], [\mathcal{E}_2] \in \mathrm{Vect}(M)$ such that $\pi^*[\mathcal{E}_1] = \pi^*[\mathcal{E}_2]$, we can write $[\mathcal{E}_1] = \varepsilon^{r-1} \oplus \det[\mathcal{E}_1], [\mathcal{E}_2] = \varepsilon^{r-1} \oplus \det[\mathcal{E}_2]$ and we obtain that:

$$\varepsilon^{r-1} \oplus \pi^* \det[\mathcal{E}_1] = \varepsilon^{r-1} \oplus \pi^* \det[\mathcal{E}_2],$$

where ε^{r-1} now denotes the trivial bundle over SM in the previous equality. Taking the first Chern class, we see that $c_1(\pi^* \det[\mathcal{E}_1]) = c_1(\pi^* \det[\mathcal{E}_2])$, that is $\pi^* \det \mathcal{E}_1 \simeq \pi^* \det \mathcal{E}_2$ are isomorphic. By Lemma 7.7, we get that $\det \mathcal{E}_1 \simeq \det \mathcal{E}_2$ are isomorphic. Hence $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic. □

7.2.3 Flat bundles

We discuss the particular case of flat vector bundles. It is well-known that the data of a vector bundle equipped with a unitary connection (modulo isomorphism) is equivalent to a unitary representation of the fundamental group (modulo inner automorphisms of the unitary group). More precisely, given $\rho \in (\pi_1(M), \mathrm{U}(r))$, one can associate a Hermitian bundle $\mathcal{E} \to M$ equipped with a flat unitary connection $\nabla^{\mathcal{E}}$ by the following process: let $\widetilde{M}$ be the universal cover of M; consider the trivial bundle $\mathbb{C}^r \times \widetilde{M}$ equipped with the flat connection d and define the relation $(x, v) \sim (x', v')$ if and only if $x' = \gamma(x), v' = \rho(\gamma)v$, for some $\gamma \in \pi_1(M)$; then $(\mathcal{E}, \nabla^{\mathcal{E}})$ is obtained by taking the quotient $\mathbb{C}^r \times \widetilde{M}/\sim$. Changing ρ by an isomorphic representation $\rho' = p \cdot \rho \cdot p^{-1}$ (for $p \in \mathrm{U}(r)$) changes the connection by a gauge-equivalent connection and this process gives a one-to-one correspondance between the moduli spaces.

For $r \geq 0$, we let

$$\mathcal{M}_r := \mathrm{Hom}(\pi_1(M), \mathrm{U}(r))/\sim,$$

be the moduli space of unitary representations of the fundamental group, where two representations are equivalent $\sim$ whenever they are isomorphic. The space $\mathcal{M}_r$ is called the *character variety*, see [**?**] for instance. For $r = 0$, it is reduced to a point; for $r = 1$, it is given by $\mathcal{M}_1 = \mathrm{U}(1)^{b_1(M)}$. Given $\mathrm{x} \in \mathcal{M}_r$, we let $\Psi(\mathrm{x}) = (\mathcal{E}_{\mathrm{x}}, \nabla^{\mathcal{E}_{\mathrm{x}}})$ be the data of a Hermitian vector bundle equipped with a unitary connection (up to gauge-equivalence) described by the above process. The primitive trace map $\mathcal{T}^\sharp$ can then be seen as a map:

$$\mathcal{T}^\sharp : \bigsqcup_{r \geq 0} \mathcal{M}_r \to \ell^\infty(\mathcal{C}^\sharp),$$

which is defined in the following way: given $\mathrm{x} \in \mathcal{M}_r$, one sets $\mathcal{T}^\sharp(\mathrm{x}) := \mathcal{T}^\sharp(\nabla^{\mathcal{E}_{\mathrm{x}}}) \in \ell^\infty(\mathcal{C}^\sharp)$, where the right-hand side is understood by (7.1). We then have the following:

Proposition 7.12. *Let (M, g) be an Anosov manifold of dimension ≥ 2. The primitive trace map*

$$\mathcal{T}^\sharp : \bigsqcup_{r \geq 0} \mathcal{M}_r \to \ell^\infty(\mathcal{C}^\sharp),$$

is globally injective.

Proof. We assume that $\mathcal{T}^\sharp(\mathrm{x}_1) = \mathcal{T}^\sharp(\mathrm{x}_2)$ or, equivalently, that $\mathcal{T}^\sharp(\nabla^{\mathcal{E}_{\mathrm{x}_1}}) = \mathcal{T}^\sharp(\nabla^{\mathcal{E}_{\mathrm{x}_2}})$. The exact Livšic cocycle Theorem 4.23 implies that the bundles $\pi^*\mathcal{E}_{\mathrm{x}_1}$ and $\pi^*\mathcal{E}_{\mathrm{x}_2}$ are isomorphic and yields the existence of a section $p \in C^\infty(SM, \mathrm{U}(\pi^*\mathcal{E}_{\mathrm{x}_2}, \pi^*\mathcal{E}_{\mathrm{x}_1}))$ such that: $C_{\mathrm{x}_1}(x, t) = p(\varphi_t x) C_{\mathrm{x}_2}(x, t) p(x)^{-1}$ for all $x \in \mathcal{M}, t \in \mathbb{R}$, which is equivalent to

$$\pi^{*\mathrm{M}} \nabla_X^{\mathrm{Hom}(\mathcal{E}_{\mathrm{x}_2}, \mathcal{E}_{\mathrm{x}_1})} p = 0,$$

where ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_{\mathrm{x}_2},\mathcal{E}_{\mathrm{x}_1})}$ is the mixed connection induced by $\nabla^{\mathcal{E}_{\mathrm{x}_1}}$ and $\nabla^{\mathcal{E}_{\mathrm{x}_1}}$ on $\mathrm{Hom}(\mathcal{E}_{\mathrm{x}_2},\mathcal{E}_{\mathrm{x}_1})$. Observe that by (2.9), the curvature of ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_{\mathrm{x}_2},\mathcal{E}_{\mathrm{x}_1})}$ vanishes as both curvatures $F_{\nabla^{\mathcal{E}_{\mathrm{x}_{1,2}}}}$ vanish. Applying Lemma 5.18 with $\mathbf{X} := \pi^{*\mathrm{M}}\nabla_X^{\mathrm{Hom}(\mathcal{E}_{\mathrm{x}_2},\mathcal{E}_{\mathrm{x}_1})}$ acting on the pullback bundle $\pi^*\mathrm{Hom}(\mathcal{E}_{\mathrm{x}_2},\mathcal{E}_{\mathrm{x}_1})$, we get that p is of degree 0, which is equivalent to the fact that the connections are gauge-equivalent. □

7.2.4 General results in higher rank and negative sectional curvature

Let (M,g) be a Riemannian manifold with *negative sectionla curvature*. We introduce the following condition:

Definition 7.13. We say that the pair of connections $(\nabla^{\mathcal{E}_1},\nabla^{\mathcal{E}_2})$ satisfies the *spectral condition* if the mixed connection ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}$ has no non-trivial twisted CKTs.

Observe that by (2.10), the previous condition is invariant by changing one of the two connections by $p_*\nabla^{\mathcal{E}_i}$, for some vector bundle isomorphism p, and thus this condition descends to the moduli space. We then define

$$\mathbf{S} \subset \mathbb{A} \times \mathbb{A},$$

the subspace of all pairs of equivalence classes of connections satisfying the spectral condition. Observe that the set $\mathbf{S}$ is non-empty as it contains at least all the pairs of flat unitary connections (modulo gauge-equivalence). We have the following statement:

Proposition 7.14. *Let (M,g) be a negatively-curved Riemannian manifold of dimension ≥ 2. Let $(\mathfrak{a},\mathfrak{a}') \in \mathbf{S}$ such that $\mathcal{T}^\sharp(\mathfrak{a}) = \mathcal{T}^\sharp(\mathfrak{a}')$. Then $\mathfrak{a} = \mathfrak{a}'$.*

In other words, two connections satisfying the spectral condition and whose images by the primitive trace map are equal, are actually gauge-equivalent.

Proof. Consider two representatives $\nabla^{\mathcal{E}_1} \in \mathfrak{a}$ and $\nabla^{\mathcal{E}_2} \in \mathfrak{a}'$. The exact Livšic cocycle Theorem 4.23 provides a section $p \in C^\infty(SM, \mathrm{U}(\pi^*\mathcal{E}_2, \pi^*\mathcal{E}_1))$ such that:

$$\pi^{*\mathrm{M}}\nabla_X^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)} p = 0.$$

By assumption, (M,g) has negative curvature and thus p has finite Fourier degree by Lemma 5.13. Moreover, since ${}^{\mathrm{M}}\nabla^{\mathrm{Hom}(\mathcal{E}_2,\mathcal{E}_1)}$ has no non-trivial twisted CKTs, p is of degree 0. This shows that the connections are gauge-equivalent. □

7.3 Holonomy of the Levi-Civita connection

We study more precisely in this paragraph the holonomy obtained along closed geodesics for the vector bundle $TM \to M$ equipped with the Levi-Civita connection.

7.3.1 Transparent manifolds

We now discuss a very particular case of transparent connections. We restrict ourselves to the study of the vector bundle $(TM, \nabla^{\mathrm{LC}})$ endowed with the Levi-Civita connection and introduce the following terminology:

Definition 7.15. We say that the manifold (M, g) is *transparent* if the tangent vector bundle $(TM, \nabla^{\mathrm{LC}})$ equipped with the Levi-Civita connection is transparent.

Of course, any oriented surface is transparent. In higher dimensions, flat tori are transparent for instance. One can legitimately conjecture that there are no transparent manifolds in negative curvature, and more generally, as long as the geodesic flow is Anosov, insofar as the chaotic properties of the flow should generate holonomy in the transverse direction to the flow.

Conjecture 7.16. *Assume (M, g) is an Anosov manifold of dimension ≥ 3. Then, (M, g) is not transparent.*

It is straightforward to check the following:

Lemma 7.17. *A transparent manifold is $2-$,$4-$ or $8-$dimensional.*

Proof. Indeed, if the manifold is transparent, then $\pi^*TM \to SM$ is trivial by Lemma 4.34 and trivialized by a global $(e_1, ..., e_n)$ such that $e_i \in C^\infty(SM, \pi^*TM)$, $(\pi^*\nabla)_X e_i = 0$ and the e_i are pointwise orthogonal (by Lemma 3.12). Observe that the tautological section $s(x, v) := v$ is always in the kernel of $\mathbf{X} := (\pi^*\nabla)_X$ and so we can always assume that $e_1 = s$, and $e_2(x, v), ..., e_n(x, v)$ are orthogonal to v. As a consequence, for fixed x, the vector fields $(e_2(x, \cdot), ..., e_n(x, \cdot))$ are tangent to the $(n-1)$-dimensional sphere S_xM and pointwise orthogonal. This implies that the $(n-1)$-dimensional sphere is parallelizable, hence $n - 1 = 1, 3$ or 7. □

For the moment, Conjecture 7.16 is an open question but a first step could be to study the case of negatively-curved manifolds. Also observe that non-transparency is an open condition (in the set of C^2 metrics). The only cases that are known are the following (see [CLb])

Theorem 7.18 (Cekic-L. '20)**.** *Let (M, g_0) be a hyperbolic metric on a 4- or 8-manifold. Then, there is an open C^2-neighborhood of g_0 such that there are no transparent manifolds in this neighborhood.*

The proof is rather simple although non elementary and relies on the following crucial fact (recall that $\mathbf{X} := (\pi^*\nabla)_X$):

Lemma 7.19. *If $\mathbf{X}$ has no CKTs of degree $m \geq 2$, then (M, g) is not transparent.*

Proof. The absence of CKTs of degree ≥ 2 implies that the e_i are of degree at most one. We now show that they are of degree exactly one, namely they have no zeroth Fourier mode in their spectral decomposition. We argue by contradiction, and consider $f \in C^\infty(SM, \pi^*TM)$ such that $\mathbf{X}f = 0$. Such a f has Fourier degree ≤ 1. Since $\mathbf{X}$ acts diagonally on odd/even Fourier modes, we can write $f = f_0 + f_1$ and $\mathbf{X}f_0 = \mathbf{X}f_1 = 0$. Now, f_0 can be identified with a section $f_0 \in C^\infty(M, TM)$ and the equation $\mathbf{X}f_0$ can be rewritten as $\nabla f_0 = 0$. Using the musical isomorphism $\sharp : TM \to T^*M$, f_0 can be identified with a one-form α such that $\nabla\alpha = 0$. Hence: $\pi_2^*\nabla\alpha = X\pi_1^*\alpha = 0$. By ergodicity of the geodesic flow, this implies that $\pi_1^*\alpha$ is constant but since $\pi_1^*\alpha(x,-v) = -\pi_1^*\alpha(x,v)$, this implies that $\alpha \equiv 0$. Hence $f = f_1$ is a pure mode of degree 1.

As a consequence, the section e_i for $i \geq 2$ are of pure degree 1. It is easy to see that this implies the existence of section $R_i \in C^\infty(M, \mathrm{End}(TM))$ such that $e_i(x,v) = R_i(x)v$, for all $(x,v) \in SM$. Moreover, using some ingredients from Clifford algebra theory, one can prove that $\mathbf{X}e_i = 0$ actually implies that the R_i are parallel, namely $\nabla^{\mathrm{End}(TM)}R_i = 0$. (This is an easy exercise in dimension 4. In dimension 8, further work is needed.) As a consequence, the triple (R_2, R_3, R_2R_3) endows (M,g) with the structure of a hyperkähler manifold and this implies that the manifold is Ricci-flat (see [CLb, Section 2.1] for instance). Hence it cannot be negatively-curved. □

In order to prove Theorem 7.18, we indeed prove that in the case of a hyperbolic manifold, there are no CKTs of degree $m \geq 2$ for the Levi-Civita connection. The proof simply relies on Lemma 5.14 as one can compute in an explicit fashion in this case the norm $\|F^{TM}\|_{L^\infty}$ for a hyperbolic manifold. Nevertheless, one can legitimately believe that this Lemma 5.14 is not sharp and that one could prove that there are no CKTs for this connection of degree $m \geq 2$.

7.3.2 Image of the representation

The previous discussion can be generalized in the following direction: in the case of the tangent bundle $TM \to M$ equipped with the Levi-Civita ∇^{LC}, the representation

$$\rho : \mathbf{G} \to \mathrm{O}(n),$$

introduced in §4.3.2 takes values in orthogonal matrices, where $\mathrm{O}(n) \simeq \pi^*TM(x_*, v_*)$ for some arbitrary periodic $(x_*, v_*) \in SM$. As we saw earlier, the pullback bundle π^*TM splits as the orthogonal sum:

$$\pi^*TM = \mathbb{R}s \oplus \mathcal{N}, \tag{7.6}$$

where $s(x,v) := v$ is the tautological section and $\mathcal{N}(x,v) := \{v\}^\perp$ is the normal bundle. Using (7.6) and the invariance of s by parallel transport, we get by Lemma 4.44 that the representation ρ splits as:

$$\rho(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & \rho_\perp(\gamma) \end{pmatrix},$$

for some subrepresentation $\rho_\perp(\gamma)$ taking values in the endomorphisms of $\mathcal{N}(x_*, v_*)$. Observe that $\mathcal{N}(x_*, v_*) \simeq \mathbb{R}^{n-1}$ and thus $\rho_\perp(\mathbf{G}) \subset \mathrm{O}(n-1)$. In the oriented case, this further restricts to $\mathrm{SO}(n-1)$. The transparent condition of Definition 4.33 is equivalent to the fact that $\rho_\perp$ is the trivial representation, namely $\rho_\perp(\gamma) = \mathbb{1}_{\mathcal{N}}$ for all $\gamma \in \mathbf{G}$.

For the sake of simplicity, we now assume that the manifolds are oriented. Following Berger [Ber53, Ber03] on the restricted holonomy group, one can then ask the following natural question: *what is the image of $\rho(\mathbf{G})$ inside $\mathrm{SO}(n-1)$?* There are obvious further restrictions: for instance, a Kähler manifold of negative sectional curvature (such as a compact quotient of the complex hyperbolic space) will have a representation:

$$\rho(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho'_\perp(\gamma) \end{pmatrix},$$

where the first blocks correspond to the fact that $\rho(\gamma)v_* = v_*, \rho(\gamma)(Jv_*) = Jv_*$ (J being the almost-complex structure), and $\rho'_\perp(\gamma)$ is an endomorphism of $(\mathbb{R}v_* + \mathbb{R}Jv_*)^\perp$ which sits inside the subgroup $\mathrm{U}((n-2)/2) = \mathrm{U}(n/2-1) \subset \mathrm{O}(n-2)$. From now on, we will write ρ instead of $\rho_\perp$ or $\rho'_\perp$, meaning that we forget about the trivial v_* component of ρ. We suggest the following conjecture:

Conjecture 7.20. *If (M, g) does not carry an almost complex structure, then $\rho(\mathbf{G})$ is a dense subset of $\mathrm{SO}(n-1)$.*

In particular, $\rho(\mathbf{G}) \subset \mathrm{SO}(n-1)$ should always be dense for odd-dimensional Anosov manifolds.

8 Open questions

We conclude this survey by summing up all the open questions:

On the Livšic theorem:

- Can one prove an approximate version of the Livšic theorem (both in the Abelian case or in the cocycle case) in high regularity?
- Can one prove a positive version of the Livšic theorem (in the Abelian case) in high regularity?

On the marked length spectrum:

- Can one prove the global rigidity of the marked length spectrum on Anosov surfaces? on negatively-curved manifolds? on Anosov manifolds?
- What can be said about the (infinite-dimensional) moduli space of isometry classes endowed with the general Weil-Petersson metric? Does it have negative sectional curvature for instance?

On Anosov manifolds:

- Are there no CKTs on the trivial line bundle of Anosov manifolds?
- Let $\mathcal{E} \to M$ be a vector bundle over the Anosov manifold (M, g) equipped with a unitary connection $\nabla^{\mathcal{E}}$ and let $\mathbf{X} := (\pi^*\nabla^{\mathcal{E}})_X$. Does any smooth element in $\ker(\mathbf{X})$ has finite Fourier degree?

In particular, a positive answer to these two questions would imply the injectivity of the X-ray transform I_m, for any $m \in \mathbb{N}$, which we also formulate as a question:

- In the 2D case, let $A \in C^\infty(M, \Omega_1)$ be linear (in v), let $f \in C^\infty(M)$ and assume that $u \in C^\infty(SM)$ solves $(X + A)u = \pi_0^* f$. Does u have finite degree? (Communicated by Gabriel Paternain).
- Is the X-ray transform I_m s-injective on Anosov manifolds?

This is only known for the moment in the cases $m = 0, 1$ [DS03]. In particular, for $m = 2$, this would prove that Anosov manifolds are locally rigid with respect to the marked length spectrum.

On twisted Conformal Killing Tensors:

- Given a fixed vector bundle $\mathcal{E} \to M$, equipped with a unitary connection $\nabla^{\mathcal{E}}$, it is true that generically with respect to the metric g, there are no CKTs for $\nabla^{\mathcal{E}}$?
- Let (M, g) be a negatively-curved manifold. Can one prove that ∇^{LC} has no CKTs of degree $m \geq 2$? What if (M, g) is only Anosov?

A positive answer to the last question would prove that there are no transparent manifolds, except surfaces.

On transparent connections and holonomy problems:

- Is the primitive trace map injective on Anosov manifolds of dimension $n \geq 3$? What about $n = 3$?
- Are there examples of non-trivial transparent connections on Anosov manifolds of dimension $n \geq 3$?
- What is the generic image of the representation $\rho : \mathbf{G} \to O(n-1)$?

Appendix

A Elements of microlocal analysis

A.1 Pseudodifferential operators in $\mathbb{R}^n$

We first recall the definition of pseudodifferential operators in the Euclidean space $\mathbb{R}^n$. We start with the usual classes of symbols.

Definition A.1. Let $m \in \mathbb{R}, \rho \in (1/2, 1]$. We define $S^m_\rho(\mathbb{R}^n)$ to be the set of smooth functions $p \in C^\infty(T^*\mathbb{R}^{n+1})$ such that for all $\alpha, \beta \in \mathbb{N}$:

$$\|p\|_{\alpha,\beta} := \sup_{|\alpha'| \leq \alpha, |\beta'| \leq \beta} \sup_{(x,\xi) \in T^*\mathbb{R}^n} \langle \xi \rangle^{-(m-\rho|\alpha'|+(1-\rho)|\beta'|)} |\partial_\xi^{\alpha'} \partial_x^{\beta'} p(x,\xi)| < \infty, \tag{A.1}$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$. For $\rho = 1$, we will simply write $S^m(\mathbb{R}^n)$.

This class is invariant by the action by pullback of properly supported diffeomorphisms. As a consequence, they are intrinsically defined on smooth closed manifolds. Namely, if M is a smooth closed manifold, then $p \in S^m(M)$ if and only if, in any local trivialization $\phi : U \to \phi(U) \subset \mathbb{R}^n$ (where $U \subset M$ is an open subset), $\chi \phi_* p \chi \in S^m(\mathbb{R}^n)$, where χ is any cutoff function supported in $\phi(U)$. These classes of symbols form a graded algebra of Fréchet spaces (for each $m \in \mathbb{R}$) with semi-norms given by (A.1).

Remark A.2. The order $m \in \mathbb{R}$ is fixed in the previous definition but it can actually be chosen to vary. This is used extensively is Section 3. Namely, if $m \in S^0(\mathbb{R}^n)$, then we define $S^m_\rho(\mathbb{R}^n)$ to be the set of smooth functions $p \in C^\infty(T^*\mathbb{R}^{n+1})$ such that for all indices α, β, there exists a constant $C_{\alpha\beta} > 0$ such that:

$$\forall (x,\xi) \in T^*\mathbb{R}^n, \qquad |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|}.$$

We refer to [FRS08] for further details. This class of symbols will appear in the proofs of the meromorphic extension of the generator of Anosov flows. It enjoys the usual features of more classical classes of symbols like the parametrix construction for instance, which are described below.

We say that P is a pseudodifferential operator of order $m \in \mathbb{R}$ on $\mathbb{R}^n$ if there exists $p \in S^m(\mathbb{R}^n)$ such that for any function $f \in C_c^\infty(\mathbb{R}^n)$:

$$Pf(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}} e^{i\xi\cdot(x-y)} p(x,\xi) f(y) dy d\xi \tag{A.2}$$

This integral does not converge absolutely and has to be understood as an *oscillatory integral*: for further details, we refer to [Abe12, Shu01]. In this case, we write $P = \mathrm{Op}(p)$

and we say that the operator P is the *quantization* of p. We denote by $\Psi^m(\mathbb{R}^n)$ the set of pseudodifferential operators of order m and we set $\Psi^{-\infty}(\mathbb{R}^n) := \cap_{m\in\mathbb{R}}\Psi^m(\mathbb{R}^n)$. These are operators with smooth Schwartz kernel (and fast decay at infinity off the diagonal $\{x = y\}$ in $\mathbb{R}^n \times \mathbb{R}^n$). Eventually, we denote by $\sigma_P : \Psi^m(\mathbb{R}^n) \to S^m(\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n)$ the principal symbol of P, defined by

$$\sigma_P(x,\xi) := \lim_{h\to 0} h^m e^{-iS/h} P(e^{iS/h})(x),$$

for $(x,\xi) \in T^*\mathbb{R}^n$, if $S : C^\infty(\mathbb{R}^n)$ is such that $\mathrm{d}S(x) = \xi$.

The space $\Psi^m(\mathbb{R}^n)$ is in one-to-one correspondance with $S^m(\mathbb{R}^n)$ (see [Mel03, Theorem 2.1]) via the quantization formula (A.2). This allows to transfer the Fréchet topology of $S^m(\mathbb{R}^n)$ to the space $\Psi^m(\mathbb{R}^n)$. As a consequence, $\Psi^m(\mathbb{R}^n)$ is a Fréchet space endowed with the topology given by the semi-norms of its full symbol (A.1).

A symbol $p \in S^m(\mathbb{R}^n)$ is said to be *globally elliptic* if there exists constants $C, R > 0$ such that:

$$\forall |\xi| \ \geq R, \forall x \in \mathbb{R}^n, \qquad |p(x,\xi)| \ \geq C\langle\xi\rangle^m.$$

It is said to be *locally elliptic* at (x_0, ξ_0) if there exists a conic neighborhood V of (x_0, ξ_0)[13] such that:

$$\forall (x,\xi) \in V, |\xi| \ \geq R, \qquad |p(x,\xi)| \ \geq C\langle\xi\rangle^m.$$

Given $P \in \Psi^m(\mathbb{R}^n)$, we say that it is locally elliptic at (x_0, ξ_0) if its principal symbol σ_P is. We denote by $\mathrm{ell}(P)$ the set of points $(x_0, \xi_0) \in T^*M$ at which P is locally elliptic. Note that this is by construction an open conic subset of $T^*M \setminus \{0\}$.

A.2 Pseudodifferential operators on compact manifolds

We now move to the case of pseudodifferential operators on a smooth closed manifold M. There is no *intrinsic way* of defining pseudodifferential operators on compact manifolds (although some constructions may look more natural than others, there is always a part of choice in the definitions) but what is important is that the resulting class of operators $\Psi^m(M)$ obtained in the end *is independent* of all the choices made. Moreover, all the important features of the calculus (principal symbol, ellipticity) are independent of the choices made in the constructions.

We consider a cover of M by a finite number of open sets $M = \cup_i U_i$ such that there exists a smooth diffeomorphism $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{R}^{n+1}$. By assumption, since M is smooth, the transition maps $\phi_i \circ \phi_j^{-1}$ are smooth whenever they are defined. We consider a smooth partition of unity $\sum_i \Phi_i = \mathbf{1}$ subordinated to this cover of M and smooth functions

[13] V is an open conic neighborhood of (x_0, ξ_0) of $T^*\mathbb{R}^n \setminus \{0\}$ if it is open in $T^*\mathbb{R}^n \setminus \{0\}$ and contains for some $\varepsilon > 0$ small enough the set of points $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ such that $|x - x_0| < \varepsilon$ and $|\xi/|\xi| - \xi_0/|\xi_0|| < \varepsilon$.

Ψ_i supported in each patch U_i, defined such that $\Psi_i \equiv 1$ on the support of Φ_i. We call these elements $(U_i, \Phi_i, \Psi_i)_i$ a family of cutoff charts.

Definition A.3. A linear operator $P : C^\infty(M) \to C^\infty(M)$ is a pseudodifferential of order m on M if and only if there exists a family of cutoff charts $(U_i, \Phi_i, \Psi_i)_i$ such that, in the decomposition

$$P = \sum_i \Psi_i P \Phi_i + (1 - \Psi_i) P \Phi_i, \tag{A.3}$$

the operators $\Psi_i P \Phi_i$ can be written in coordinates

$$\Psi_i P \Phi_i f(\phi_i^{-1}(x)) = \psi_i \operatorname{Op}(p_i) \varphi_i f_i(x), \tag{A.4}$$

for some symbols $p_i \in S^m(\mathbb{R}^{n+1})$ (Op being the quantization (A.2) in Euclidean space), where $x \in \phi_i(U_i)$, $f_i := f \circ \phi_i^{-1}$ and $f \in C^\infty(M)$ is arbitrary, $\psi_i := \Psi_i \circ \phi_i^{-1}, \varphi_i := \Phi_i \circ \phi_i^{-1}$ and the operators $(1 - \Psi_i) P \Phi_i$ have smooth Schwartz kernel. We denote by $\Psi^m(M)$ the class of such operators.

Another formulation is the following: if one chooses a family of cutoff charts, given a symbol $p \in S^m(M)$, (A.4) provides a formula of quantization $\operatorname{Op}(p)$ (which depends on the choice of cutoff charts). Then the equality

$$\Psi^m(M) = \left\{ \operatorname{Op}(p) + R \mid p \in S^m(M), R \in \Psi^{-\infty}(M) \right\}$$

holds (here R is a smoothing operator, that is an operator with smooth Schwartz kernel), that is any other choice of cutoff charts produces the same class of operators. Note that *once a family of cutoff charts is chosen*, the decomposition (A.3) of P is unique and one can endow the Fréchet space $\Psi^m(M)$ with the semi-norms in local coordinates

$$\|P\|_{\alpha,\beta,\gamma} = \sum_i \|p_i\|_{\alpha,\beta} + \|(1 - \Psi_i) P \Phi_i\|_\gamma, \tag{A.5}$$

where $\|p_i\|_{\alpha,\beta}$ is given by (A.1) and, confusing $(1 - \Psi_i) P \Phi_i$ with its smooth Schwartz kernel, we define for $K \in C^\infty(M \times M)$ the semi-norms:

$$\|K\|_\gamma := \sup_{|j|+|k| \ \leq \gamma} \ \sup_{x,y \in M} |\partial_x^j \partial_y^k K(x, y)|$$

The principal symbol map $\sigma_m : \Psi^m(M) \to S^m(M)/S^{m-1}(M)$ is a well-defined map, independent of the quantization chosen. Let us recall some elementary properties of pseudodifferential operators:

Proposition A.4. *1. If $P \in \Psi^m(M)$, then $P : H^s(M) \to H^{s-m}(M)$ is bounded for all $s \in \mathbb{R}$,* *2. If $P_1 \in \Psi^{m_1}(M), P_2 \in \Psi^{m_2}(M)$, then $P_1 \circ P_2 \in \Psi^{m_1+m_2}(M)$ and $\sigma_{P_1 \circ P_2} = \sigma_{P_1}\sigma_{P_2}$,*

An operator $R \in \Psi^{-\infty}(M)$ is bounded and compact as a map $H^r(M) \to H^s(M)$, for all $s, r \in \mathbb{R}$. We now fix a smooth density $\mathrm{d}\mu$ on M. Every operator can be associated to a *formal adjoint* $P^* : C^\infty(M) \to C^\infty(M)$ which is also pseudodifferential and defined by the equality:

$$\langle Pf_1, f_2\rangle_{L^2(M,\mathrm{d}\mu)} = \langle f_1, P^* f_2\rangle_{L^2(M,\mathrm{d}\mu)}, \tag{A.6}$$

where $f_1, f_2 \in C^\infty(M)$. We say that P is *formally selfadjoint* when $P = P^*$. Note that the adjoint P^* depends on a choice of (smooth) density $\mathrm{d}\mu$. This necessary choice can be overcome by working with *half-densities* instead of functions but this will not be needed here.

Proposition A.5. *If $P \in \Psi^m(M)$ is globally elliptic, there exists $Q \in \Psi^{-m}(M)$ (also globally elliptic) such that*

$$PQ = \mathbb{1} + R_1, QP = \mathbb{1} + R_2,$$

where $R_1, R_2 \in \Psi^{-\infty}(M)$. Moreover $\ker(P) \subset C^\infty(M)$*, it is finite-dimensional and* $\operatorname{ran}(P|_{C^\infty(M)}) \subset C^\infty(M)$ *has finite codimension which coincides with that of* $\ker(P^*)$*. It is therefore Fredholm and the* Fredholm index *of P is the integer:*

$$\operatorname{ind}(P) := \dim\ker(P) - \dim\ker(P^*) < \infty$$

In particular, if P is formally selfadjoint, then $\operatorname{ind}(P) = 0$*.*

We will denote by $C^{-\infty}(M) := \cup_{s\in\mathbb{R}} H^s(M)$ the space of distributions. The following lemma on elliptic estimates is crucial:

Lemma A.6. *Let $P \in \Psi^m(M)$ be an elliptic pseudodifferential operator. For all $s, r \in \mathbb{R}$, there exists a constant $C := C(r, s)$ such that for all $f \in C^{-\infty}(M)$ such that $Pf \in H^{s-m}(M)$:*

$$\|f\|_{H^s} \leq C\left(\|Pf\|_{H^{s-m}} + \|f\|_{H^r}\right)$$

Moreover, if $P : H^s(M) \to H^{s-m}(M)$ is injective for some (and thus any) $s \in \mathbb{R}$, then:

$$\|f\|_{H^s} \leq C\|Pf\|_{H^{s-m}}$$

Proof. Let $Q \in \Psi^{-m}(M)$ be a parametrix for P, i.e. such that $QP = \mathbb{1} + R$, where $R \in \Psi^{-\infty}(M)$. Then:

$$\|f\|_{H^s} \lesssim \|QPf\|_{H^s} + \|Rf\|_{H^s} \lesssim \|Pf\|_{H^{s-m}} + \|f\|_{H^r},$$

since $R : H^r(M) \to H^s(M)$ is bounded and $Q : H^{s-m}(M) \to H^s(M)$ is bounded.

We now assume that P is invertible and we take $r = s$. Assume that the bound $\|f\|_{H^s} \lesssim \|Pf\|_{H^{s-m}}$ does not hold, so we can find a family of elements $f_n \in H^s(M)$ such that $\|f_n\|_{H^s} = 1$ and $\|f_n\|_{H^s} = 1 \geq n\|Pf_n\|_{H^{s-m}}$. So $Pf_n \to 0$ in $H^{s-m}(M)$. But $R : H^s(M) \to H^s(M)$ is compact and $(f_n)_{n\in\mathbb{N}}$ is bounded in $H^s(M)$ so we can assume (up to extraction) that $Rf_n \to v \in H^s(M)$. By the elliptic estimate

$$\|f_n\|_{H^s} \lesssim \|Pf_n\|_{H^{s-m}} + \|Rf_n\|_{H^s},$$

we obtain that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^s(M)$ which thus converges to $w \in H^s(M)$. But by continuity of P, $Pf_n \to 0 = Pw$ so $w \equiv 0$ since P is injective. This is contradicted by the fact that $\|w\|_{H^s} = 1$. □

Eventually, we recall Egorov's Theorem, in a weak version:

Lemma A.7 (Egorov's Theorem)**.** *Let $a \in S^m(M)$ and $F : M \to M$ be a smooth diffeomorphism. Let $\widetilde{F} : T^*M \to T^*M$ be the symplectic lift of F, defined by $\widetilde{F}(x,\xi) = (F(x), \mathrm{d}F(x)^{-\top} \cdot \xi)$, where $^{-\top}$ denotes the inverse transpose. Then:*

$$F^* \operatorname{Op}(a)(F^{-1})^* - \operatorname{Op}(a \circ \widetilde{F}) \in \Psi^{m-1}(M).$$

As usual, one can define pseudodifferential operators $P : C^\infty(M,E) \to C^\infty(M,F)$ acting on vector bundles $E, F \to M$ by taking local coordinates and matrix-valued pseudodifferential operators in these coordinates. All the results previously stated still hold in this general context. The principal symbol is then a map $\sigma_P : T^*M \to \operatorname{Hom}(E,F)$ and ellipticity is replaced by invertibility of $\sigma_P(x,\xi)$ (as a linear map $E_x \to F_x$) for large $|\xi| \to \infty$. When the vector bundles E and F have different ranks, ellipticity is replaced by injectivity of the principal symbol with a coercive estimate, that is

$$\|\sigma_P(x,\xi)\|_{E_x \to F_x} \geq C\langle\xi\rangle^m,$$

for $|\xi| > R, C > 0$. All the results also hold with very few changes when m has variable order.

A.3 Wavefront set of distributions

A.3.1 Definition

Definition A.8 (Wavefront set of a distribution)**.** Let $u \in C^{-\infty}(M)$. A point $(x_0,\xi_0) \in T^*M \setminus \{0\}$ *is not* in the wavefront set $\mathrm{WF}(u)$ of u, if there exists a conic neighborhood U of (x_0,ξ_0) such that for any smooth functions $\chi \in C_c^\infty(\pi(U))$ ($\pi : T^*M \to M$ being the projection), in any set of local coordinates, one has:

$$\forall N \in \mathbb{N}, \qquad \sup_{\xi \in U} |\widehat{\chi u}(\xi)||\xi|^N < \infty.$$

This is well-defined i.e. independent of the choice of coordinates. An equivalent definition is that $(x_0, \xi_0) \notin \mathrm{WF}(u)$ if and only if there exists a pseudodifferential operator A of order 0 microlocally supported in the conic neighborhood U, elliptic at (x_0, ξ_0) such that $Au \in C^\infty(M)$. By construction, the wavefront set of a distribution is *a conic set* in $T^*M \setminus \{0\}$. We will say that $u \in C^{-\infty}(M)$ is *smooth at* (x_0, ξ_0) if $(x_0, \xi_0) \notin \mathrm{WF}(u)$.

If $\imath : Y \to M$ is a smooth submanifold of M, then the *conormal* to Y is the set

$$N^*Y := \{(x, \xi) \in T^*M \mid \forall x \in Y, \forall Z \in T_xY, \langle \xi, Z \rangle = 0\} \subset T^*M$$

It is a smooth vector bundle over Y. We will say that a distribution $u \in C^{-\infty}(M)$ is *conormal to* Y if $\mathrm{WF}(u) \subset N^*Y$.

Example A.9 (Surface density)**.** Let $\imath : Y \to M$ be a submanifold. If σ is a smooth density on Y, then σ can be seen as a distribution $\overline{\sigma} \in C^{-\infty}(M)$ on M by setting $\langle \overline{\sigma}, f \rangle := \langle \sigma, f|_Y \rangle$, for $f \in C^\infty(M)$. Then $\mathrm{WF}(\overline{\sigma}) = N^*Y$, i.e. $\overline{\sigma}$ is conormal to Y.

Indeed, by taking local coordinates, the computation actually boils down to considering the case $\sigma = \phi(x)\delta(x' = 0)$, with $x' \in \mathbb{R}^{n-k}, x \in \mathbb{R}^k$, where $M \simeq \mathbb{R}^n$ and $N \simeq \{x' = 0\}$, $\phi \in C^\infty(\mathbb{R}^k)$. But then, for $\chi \in C^\infty(\mathbb{R}^n)$ localized in a neighborhood of $(x, 0)$, and denoting $\eta = (\xi, \xi'), e_\eta : (x, x') \mapsto e^{i\eta \cdot (x, x')}$, one has:

$$\widehat{\chi\overline{\sigma}}(\xi, \xi') = \langle \overline{\sigma}, \chi e_\eta \rangle = \int_{\mathbb{R}^k} \phi(x)\chi(x, 0)e^{ix \cdot \xi} dx = \mathcal{O}(|\eta|^{-\infty})$$ [14]

by the non-stationary phase lemma, unless $\xi = 0$. Thus

$$\mathrm{WF}(\overline{\sigma}) = \left\{(0, \xi'), \xi' \in \mathbb{R}^{n-k} \setminus \{0\}\right\} = N^*\mathbb{R}^k$$

We can refine the definition of the wavefront set in order to evaluate the frequency behavior of the distribution at infinity:

Definition A.10 (H^s-wavefront set)**.** Let $u \in C^{-\infty}(M)$. A point $(x, \xi) \notin \mathrm{WF}_s(u)$ if there exists a conic neighborhood of (x, ξ) and a pseudodifferential operator A of order 0 microlocally supported in this conic neighborhood, elliptic at (x, ξ) such that $Au \in H^s(M)$. We will say that $u \in C^{-\infty}(M)$ is *microlocally* H^s *at* (x_0, ξ_0) if $(x_0, \xi_0) \notin \mathrm{WF}_s(u)$.

[14] By this, we mean that for all $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that the right-hand side is bounded by $C_N|\eta|^{-N}$

Example A.11. Let δ_0 be the Dirac mass at 0 in $\mathbb{R}^n$. Then

$$\mathrm{WF}_{-n/2}(\delta_0) = \{(0,\xi), \xi \in \mathbb{R}^n \setminus \{0\}\},$$

but $\mathrm{WF}_s(\delta_0) = \emptyset$ for all $s < -n/2$.

A.3.2 Elementary operations on distributions

We now introduce the **multiplication of distributions**. We will denote by $\mathrm{d}\,\mathrm{vol}$ the smooth Riemannian density on M. Given $u_1, u_2 \in C^\infty(M)$, the (complex) pairing

$$\langle u_1, u_2\rangle_{\mathbb{C}} := \int_M u_1(x)\overline{u_2(x)}\mathrm{d}\,\mathrm{vol}(x)$$

is always well-defined (note that M is compact). We want to understand to what extent this can be generalized to distributions $u_1, u_2 \in C^{-\infty}(M)$.

Lemma A.12. *Given $u_1, u_2 \in C^{-\infty}(M)$ such that $\mathrm{WF}(u_1) \cap \mathrm{WF}(u_2) = \emptyset$, there exists $A \in \Psi^0(M)$ such that*

$$\mathrm{WF}(u_1) \cap \mathrm{WF}(A)^{15} = \emptyset, \qquad \mathrm{WF}(u_2) \cap \mathrm{WF}(\mathbb{1} - A^*) = \emptyset.$$

Then:

$$\langle u_1, u_2\rangle_{\mathbb{C}} := \overline{\langle u_2, Au_1\rangle_{\mathbb{C}}} + \langle u_1, (\mathbb{1} - A^*)u_2\rangle_{\mathbb{C}}$$

is well-defined and independent of the choice of A, where the right-hand side is understood as the pairing of a distribution with a smooth function.

To construct A, one can take $A = \mathrm{Op}(a)$ for some $a \in S^0(M)$ supported in a conic neighborhood of $\mathrm{WF}(u_1)$ (in particular, $a \equiv 0$ on $\mathrm{WF}(u_2)$ since $\mathrm{WF}(u_1) \cap \mathrm{WF}(u_2) = \emptyset$) and such that $a \equiv 1$ on $\mathrm{WF}(u_1)$. We do not detail the proof which can be found in [Mel03, Proposition 4.9]. Then the real pairing is just $\langle u_1, u_2\rangle := \langle u_1, \overline{u_2}\rangle_{\mathbb{C}}$. Since

$$\mathrm{WF}(\overline{u}) = \{(x, -\xi) \mid (x,\xi) \in \mathrm{WF}(u)\},$$

it is defined as long as $\mathrm{WF}(u_1) \cap i(\mathrm{WF}(u_2)) = \emptyset$, where $i : T^*M \to T^*M$ stands for the involution $i(x,\xi) = (x, -\xi)$. This provides the

Lemma A.13. *Given $u_1, u_2 \in C^{-\infty}(M)$ such that $\mathrm{WF}(u_1) \cap i(\mathrm{WF}(u_2)) = \emptyset$, the multiplication $u_1 \times u_2 \in C^{-\infty}(M)$ is well-defined by*

$$\forall f \in C^\infty(M), \qquad \langle u_1 u_2, f\rangle := \langle u_1, f u_2\rangle = \langle f u_1, u_2\rangle$$

[15] See Example A.21 below for a definition of $\mathrm{WF}(A)$.

and coincides with the usual multiplication for $u_1, u_2 \in C^\infty(M)$*. Moreover:*

$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset &\{(x,\xi) \mid x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\}\\ &\cup \{(x,\xi) \mid x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_1)\}\\ &\cup \{(x,\xi) \mid \xi = \eta_1 + \eta_2, (x,\eta_i) \in \mathrm{WF}(u_i), i \in \{1,2\}\}\end{aligned}$$

The proof of the first part of the lemma simply follows from the previous computation. As to the wavefront set computation, it can be done directly in coordinates by using the definition.

We now introduce the **pushforward** of distributions. Let $\pi : M \times N \to M$ be the left-projection, where N is a smooth closed manifold[16]. We denote by (x, y) the coordinates on $M \times N$, $\mathrm{d}x$ and $\mathrm{d}y$ are smooth measures on M and N. The *pushforward* $\pi_* u$ of a distribution $u \in C^{-\infty}(M)$ is defined by duality according to the formula:

$$\forall f \in C^\infty(M), \qquad \langle \pi_* u, f\rangle := \langle u, \pi^* f\rangle,$$

where $\pi^* f := f \circ \pi$ is the pullback of f. In particular, if $u \in C^\infty(M \times N)$, this definition coincides with

$$\pi_* u(x) = \int_N u(x,y)\mathrm{d}y$$

The wavefront set of the pushforward is characterized by the following lemma:

Lemma A.14.

$$\mathrm{WF}(\pi_* u) \subset \{(x,\xi) \in T^*M \mid \exists y \in N, (x,\xi,y,0) \in T^*(M \times N)\}$$

We omit the proof, which can be done directly by using the characterization of the wavefront set with the Fourier transform. Morally, integration kills all the singularities except the ones which are *really conormal* to N i.e. the manifolds along which we integrate.

We now introduce the **restriction** of distributions. Let $\imath : Y \to M$ be the embedding of the smooth submanifold Y into M. Given $u \in C^{-\infty}(M)$, the pullback $\imath^* u$, that is the restriction of u to Y, is not always well-defined. We denote by δ_Y the smooth Riemannian density obtained by restricting the metric g to Y and then taking the Riemannian volume form induced. Morally, given $f \in C^\infty(Y)$, we want to define $\langle \imath^* u, f\rangle = \langle u \times \delta_Y, \tilde{f}\rangle$, where $\tilde{f}$ is any smooth extension in a neighborhood of Y (under the condition that the multiplication $u \times \delta_Y$ is defined). Note that by Example A.9, $\mathrm{WF}(\delta_Y) \subset N^*Y$.

[16] Once again, this can be generalized to the non-compact case, but then one has to consider distributions with compact support in the product.

Lemma A.15. *Assume $u \in C^{-\infty}(M)$ satisfies $\mathrm{WF}(u) \cap N^*Y = \emptyset$ (so u is not conormal* at all*). Then $u \times \delta_Y$ makes sense by Lemma A.13 and*

$$\forall f \in C^\infty(Y), \qquad \langle \imath^* u, f\rangle := \langle u \times \delta_Y, \tilde{f}\rangle,$$

is well-defined, independently of the extension $\tilde{f}$. Moreover,

$$\mathrm{WF}(\imath^* u) \subset \{(x,\xi) \in T^*Y \mid \exists \eta \in N_x^*Y, (x,(\xi,\eta)) \in \mathrm{WF}(u)\},$$

where (ξ,η) is seen as an element of T_x^*M.

It is actually not obvious that this definition is independent of the extension $\tilde{f}$ of f: the proof can be done by an approximation argument (see [Hö3, Theorem 8.2.3]).

We now introduce the **pullback** of distributions. Let $f : M \to N$ be a smooth map between the two smooth compact manifolds M and N[17]. The *normals of the map* (or the conormal to $f(M)$) is the set

$$N_f := N^* f(M) = \left\{(f(x),\xi) \in T^*N \mid x \in M, \mathrm{d}f^\top \xi = 0\right\}$$

The pullback f^*u of a distribution $u \in C^{-\infty}(N)$ is not always defined, whereas that of a smooth function is. If f is a diffeomorphism, then it is an elementary result that f^*u makes sense in a unique way: this amounts to saying that distributions are intrinsically defined i.e. are invariant by a change of coordinates. Moreover, the wavefront set of a distribution $u \in C^{-\infty}(N)$ is simply moved to

$$\mathrm{WF}(f^*u) \subset f^* \mathrm{WF}(u) = \left\{(x,\xi) \in T^*M \mid (f(x), \mathrm{d}f_x^{-\top}\xi) \in T^*N\right\},$$

where $\mathrm{d}f^{-\top}$ stands for the inverse transpose. But if f is no longer a diffeomorphism, if it maps spaces of different dimensions for instance, then the result may not be obvious.

We consider the graph

$$\mathrm{Graph}(f) := \{(x,y) \in M \times N \mid y = f(x)\} \xrightarrow{\imath} M \times N$$

which is an embedded submanifold of $M \times N$ (even if f is not a diffeomorphism!). We denote by $\pi_2 : M \times N \to N$ the right-projection and by $g : M \to \mathrm{Graph}(f)$ the diffeomorphism $g : x \mapsto (x, f(x))$. Then $f = \pi_2 \circ \imath \circ g$. For $u \in C^{-\infty}(N)$, we thus want to define f^*u by $g^* \circ \imath^* \circ \pi_2^* u$. So we have to study separately these three maps and understand under which

[17]If M and N are not compact, then one has to assume f is *proper*, i.e. the preimage of a compact subset is a compact subset.

conditions we can compose them. First, $\pi_2^* u = \mathbf{1} \otimes u$ is always defined and

$$\mathrm{WF}(\pi_2^* u) \subset \{(x, 0, y, \eta) \mid (y, \eta) \in \mathrm{WF}(u)\}$$

In the same fashion, the pullback of a distribution by g^* is always so one has to understand when the restriction $\imath^*$ is defined. But according to Lemma A.15, it is the case if $\mathrm{WF}(\pi_2^* u) \cap N^* \operatorname{Graph}(f) = \emptyset$. Note that

$$T \operatorname{Graph}(f) = \{(x, Z, f(x), \mathrm{d}f(Z)) \mid (x, Z) \in TM\} \subset T(M \times N).$$

Thus $N^* \operatorname{Graph}(f) = \{(x, 0, f(x), \eta) \mid (f(x), \eta) \in N_f\}$ so $\imath^* \circ \pi_2^* u$ is well-defined if $\mathrm{WF}(u) \cap N_f = \emptyset$.

Lemma A.16. *Let $u \in C^{-\infty}(N)$. If $\mathrm{WF}(u) \cap N_f = \emptyset$, then $f^* u := g^* \circ \imath^* \circ \pi_2^* u$ is well-defined and coincides for $u \in C^\infty(N)$ with $f^* u = u \circ f$. Moreover,*

$$\mathrm{WF}(f^* u) \subset f^* \mathrm{WF}(u) = \left\{(x, \mathrm{d}f^\top \xi) \mid (f(x), \xi) \in \mathrm{WF}(u)\right\}.$$

Example A.17. Let $\imath : M \to M \times M$ be the embedding $\imath : x \mapsto (x, x)$ of the diagonal $\imath(M) =: \Delta(M) \subset M \times M$. Note that $N^* \Delta(M) = \{(x, \xi, x, -\xi) \mid (x, \xi) \in T^* M\}$. Let $A : C^\infty(M) \to C^{-\infty}(M)$ be a linear operator with kernel K_A. Assume

$$\mathrm{WF}(K_A) \cap N^* \Delta(M) = \emptyset$$

Then $\imath^*(K_A) \in C^{-\infty}(M)$ is a well-defined distribution. We define the *flat trace* of A by

$$\mathrm{Tr}^\flat(A) := \langle \imath^*(K_A), \mathbf{1} \rangle.$$

One can prove that the flat trace is independent of the density chosen on M to define the Schwartz kernel. If $A \in \Psi^{-\infty}$, then A is a compact operator with smooth Schwartz kernel — in particular, it is trace class and its trace coincides with its flat trace.

This last example is very important to us:

Example A.18. Let X be a smooth vector field generating a flow $(\varphi_t)_{t \in \mathbb{R}}$ on the manifold M and consider the propagator $U(t) = e^{-tX}$. It acts on functions by pullback, namely $e^{-tX} f(\cdot) = f(\varphi_{-t}(\cdot))$. The flow $(\varphi_t)_{t \in \mathbb{R}}$ generates a Hamiltonian flow $(\Phi_t)_{t \in \mathbb{R}}$ on $T^* M$ given by $\Phi_t(x, \xi) = (\varphi_t(x), \mathrm{d}\varphi_t^{-\top}(\xi))$, where $A^{-\top}$ stands for the inverse transpose. Note that Φ_t is the flow induced by the Hamiltonian vector field $\mathbf{H}$ obtained from the Hamiltonian $p(x, \xi) := \langle X(x), \xi \rangle$, which is ($i$ times) the principal symbol of X. As a consequence, Lemma A.16 describes its wavefront set:

$$\mathrm{WF}(e^{-tX} f) = \{\Phi_t(x, \xi) \mid (x, \xi) \in \mathrm{WF}(f)\}.$$

Using Lemma A.14, we obtain that for all $\chi \in C_c^\infty(\mathbb{R})$, if $A := \int_{-\infty}^{+\infty} \chi(t) e^{-tX} \mathrm{d}t$, then:

$$\mathrm{WF}'(A) \subset \{(\Phi_t(x,\xi),(x,\xi)) \mid (x,\xi) \in \Sigma, t \in \mathrm{supp}(\chi)\}$$

In other words, the operator A is smoothing in the flow-direction (since it is obtained by integration in this direction) and propagates forward singularities (by the Hamiltonian flow $(\Phi_t)_{t\in\mathbb{R}}$) in the orthogonal directions to the flow. The operator Π introduced in this manuscript is morally the operator A with $\chi \equiv 1$ on $\mathbb{R}$. This is no longer a FIO: indeed Π not only propagates forward the singularities in the orthogonal directions to the flow, but it also creates (from scratch) singularities in the stable and unstable bundles $E_s^* \cup E_u^*$.

A.4 The canonical relation

If $A : C^\infty(M) \to C^{-\infty}(M)$ is a linear operator, we denote by $K_A \in C^{-\infty}(M \times M)$ its Schwartz kernel. We define the *canonical relation* $\mathrm{WF}'(A)$ of A (also denoted by C_A) by

$$\mathrm{WF}'(A) := \{(x,\xi,y,\eta) \mid (x,\xi,y,-\eta) \in \mathrm{WF}(K_A)\}$$

Given $f \in C^\infty(M)$, using the Schwartz kernel theorem, we know that

$$Au(x) = \langle K_A(x,\cdot), u\rangle = \int_M K_A(x,y)u(y)dy,$$

where this equality has to be understood in a formal sense. By the previous operations introduced, we can rewrite this as $\pi_{2*}(K_A \times \pi_2^* u)$, where $\pi_2 : M \times M \to M$ is the projection on the second coordinate. If we want to extend A to $C^{-\infty}(M)$, then we have to understand this decomposition of A in light of the elementary operations seen so far. Recall that $\pi_2^* f = \mathbf{1} \otimes f$ has wavefront set included in $\{(x,0,y,\eta) \mid (y,\eta) \in \mathrm{WF}(u)\}$. As a consequence, $K_A \times \pi_2^* u$ makes sense as a distribution if

$$\mathrm{WF}(K_A) \cap \{(x,0,y,-\eta) \mid (y,\eta) \in \mathrm{WF}(u)\} = \emptyset,$$

and by Lemma A.13:

$$\begin{aligned}\mathrm{WF}(K_A \times \pi_2^* u) \subset \{(x,\xi,y,\eta) \mid y \in \mathrm{supp}(u), (x,\xi,y,\eta) \in \mathrm{WF}(K_A)\} \\ \cup \{(x,0,y,\eta) \mid (x,y) \in \mathrm{supp}(K_A), (y,\eta) \in \mathrm{WF}(u)\} \\ \{(x,\xi,y,\eta) \mid y \in \mathrm{supp}(u), (x,\xi,y,\eta) \in \mathrm{WF}(K_A)\}\end{aligned} \tag{A.7}$$

By Lemma A.14, we know that:

$$\mathrm{WF}(\pi_{2*}(K_A \times \pi_2^* u) \subset \{(x,\xi) \mid \exists y \in M, (x,\xi,y,0) \in \mathrm{WF}(K_A \times \pi_2^* u)\}$$

As a consequence, in (A.7), the first set in the union of the right-hand side is immediately ruled-out. We obtain:

$$\begin{aligned}\mathrm{WF}(\pi_{2*}(K_A \times \pi_2^* u) \subset & \{(x,\xi) \mid \exists y \in \mathrm{supp}(u), (x,\xi,y,0) \in \mathrm{WF}(K_A)\} \\ & \cup \{(x,\xi) \mid \exists (y,\eta) \in T^*M, (x,\xi,y,-\eta) \in \mathrm{WF}(K_A), (y,\eta) \in \mathrm{WF}(u)\}\end{aligned}$$

We introduce the compact notation

$$\mathrm{WF}'(A) \circ \mathrm{WF}(u) := \{(x,\xi) \mid \exists (y,\eta) \in \mathrm{WF}(u), (x,\xi,y,\eta) \in \mathrm{WF}'(A)\}$$

Note that this is precisely the last set on the right-hand side of the previous formula. We write

$$\mathrm{WF}(K_A, u)_1 := \{(x,\xi) \mid \exists y \in \mathrm{supp}(u), (x,\xi,y,0) \in \mathrm{WF}(K_A)\}.$$

These points are the singularities created by A, no matter the regularity of u. In other words, if $u \in C^\infty(M)$, then $\mathrm{WF}(Au) \subset \mathrm{WF}(K_A, u)_1$. We sum up this discussion in the

Lemma A.19. *Let $A : C^\infty(M) \to C^{-\infty}(M)$ be a linear operator. Then A extends by continuity to a linear map*

$$A : \left\{u \in C^{-\infty}(M) \mid \mathrm{WF}(K_A) \cap \{(x,0,y,-\eta) \mid (y,\eta) \in \mathrm{WF}(u)\} = \emptyset\right\} \to C^{-\infty}(M)$$

and $\mathrm{WF}(Au) \subset \mathrm{WF}(K_A, u)_1 \cup \mathrm{WF}'(A) \circ \mathrm{WF}(u)$.

As we will see, given a general operator A, there is no practical way to characterize its Schwartz kernel by testing it against well-chosen distributions (unless we are given other informations on A). To do this, one has to resort to semiclassical analysis which we do not want to introduce here.

Example A.20. Let

$$\Lambda \subset T^*(M \times M) \setminus \{0\} \tag{A.8}$$

be a conic Lagrangian submanifold (i.e. such that the canonical symplectic form $\omega \oplus -\omega$ vanishes on Λ). We say that $K \in C^{-\infty}(M \times M)$ is *Lagrangian* with respect to Λ if $\mathrm{WF}(K) \subset \Lambda$. The Fourier Integral Operators (FIOs) are the operators having Lagrangian distribution kernels with Lagrangian included in $T^*M \setminus \{0\} \times T^*M \setminus \{0\}$[18] (and an order condition on the symbol of their quantification, see [Hö3, Chapter XXV]). In particular, if Λ is the Lagrangian of a FIO A, then

$$\mathrm{WF}(K_A)_1 := \{(x,\xi) \mid \exists y \in M, (x,\xi,y,0) \in \mathrm{WF}(K_A)\} = \emptyset$$

[18]Note that this is stronger than (A.8).

As a consequence, the wavefront set relation in Lemma A.19 is simply: $\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u)$. Here $\mathrm{WF}'(A) = \{(x, \xi, y, -\eta) \mid (x, \xi, y, \eta) \in \Lambda\}$ is the canonical relation. In other words, a FIO does not create singularities from scratch. It may only kill or duplicate (and propagate) already existing singularities.

Example A.21. If P is a pseudodifferential operator on M, then K_P is a distribution which is conormal to the diagonal $\Delta(M) \subset M \times M$, i.e. $\mathrm{WF}(K_P) \subset N^*\Delta(M)$. In other words, its canonical relation $\mathrm{WF}'(P)$ satisfies

$$\mathrm{WF}'(P) \subset \Delta(T^*M \setminus \{0\})$$

We can define the *wavefront set of* P by

$$\mathrm{WF}(P) := \{(x, \xi) \in T^*M \setminus \{0\} \mid (x, \xi, x, \xi) \in \mathrm{WF}'(P)\}$$

This has to be understood in the following way: the operator P is smoothing outside its wavefront set $\mathrm{WF}(P)$. The wavefront set $\mathrm{WF}(P)$ is also called the *essential support* of P or the *microlocal support*. If $P = \mathrm{Op}(p)$ is a quantization of $p \in C^\infty(T^*M)$, then $\mathrm{WF}(P)$ coincides with the *cone support* of p, namely the complementary of the set of directions in T^*M for which p, as well as all its derivatives (both in the x and ξ variables), decays like $\mathcal{O}(|\xi|^{-\infty})$.

Index

References

[Abe12] Helmut Abels. *Pseudodifferential and singular integral operators*. De Gruyter Graduate Lectures. De Gruyter, Berlin, 2012. An introduction with applications.

[Ahl62] Lars V. Ahlfors. Curvature properties of Teichmüller's space. *J. Analyse Math.*, 9:161–176, 1961/62.

[AKL18] Artur Avila, Alejandro Kocsard, and Xiao-Chuan Liu. Livšic theorem for diffeomorphism cocycles. *Geom. Funct. Anal.*, 28(4):943–964, 2018.

[Ano67] D. V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90:209, 1967.

[Bal18] Viviane Baladi. *Dynamical zeta functions and dynamical determinants for hyperbolic maps*, volume 68 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2018. A functional approach.

[BCG95] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.*, 5(5):731–799, 1995.

[BCLS15] Martin Bridgeman, Richard Canary, François Labourie, and Andres Sambarino. The pressure metric for Anosov representations. *Geom. Funct. Anal.*, 25(4):1089–1179, 2015.

[Ber53] Marcel Berger. Sur les groupes d'holonomie des variétés riemanniennes. *C. R. Acad. Sci. Paris*, 237:472–473, 1953.

[Ber03] Marcel Berger. *A panoramic view of Riemannian geometry*. Springer-Verlag, Berlin, 2003.

[BI10] Dmitri Burago and Sergei Ivanov. Boundary rigidity and filling volume minimality of metrics close to a flat one. *Ann. of Math. (2)*, 171(2):1183–1211, 2010.

[BK85] K. Burns and A. Katok. Manifolds with nonpositive curvature. *Ergodic Theory Dynam. Systems*, 5(2):307–317, 1985.

[BL07] Oliver Butterley and Carlangelo Liverani. Smooth Anosov flows: correlation spectra and stability. *J. Mod. Dyn.*, 1(2):301–322, 2007.

[Bry93] Jean-Luc Brylinski. *Loop spaces, characteristic classes and geometric quantization*, volume 107 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[CDS00] Christopher B. Croke, Nurlan S. Dairbekov, and Vladimir A. Sharafutdinov. Local boundary rigidity of a compact Riemannian manifold with curvature bounded above. *Trans. Amer. Math. Soc.*, 352(9):3937–3956, 2000.

[CEG20] Dong Chen, Alena Erchenko, and Andrey Gogolev. Riemannian Anosov extension and applications. *arXiv e-prints*, page arXiv:2009.13665, September 2020.

[CLa] Mihajlo Cekić and Thibault Lefeuvre. The holonomy inverse problem on Anosov manifolds. *in preparation*.

[CLb] Mihajlo Cekić and Thibault Lefeuvre. On transparent manifolds. *in preparation*.

[CL20] Mihajlo Cekić and Thibault Lefeuvre. Generic dynamical properties of connections on vector bundles. *arXiv e-prints*, page arXiv:2008.09191, August 2020.

[Con92] Gonzalo Contreras. Regularity of topological and metric entropy of hyperbolic flows. *Math. Z.*, 210(1):97–111, 1992.

[Cro90] Christopher B. Croke. Rigidity for surfaces of nonpositive curvature. *Comment. Math. Helv.*, 65(1):150–169, 1990.

[CS98] Christopher B. Croke and Vladimir A. Sharafutdinov. Spectral rigidity of a compact negatively curved manifold. *Topology*, 37(6):1265–1273, 1998.

[DG75] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[dlL01] Rafael de la Llave. Remarks on Sobolev regularity in Anosov systems. *Ergodic Theory Dynam. Systems*, 21(4):1139–1180, 2001.

[dlLMM86] R. de la Llave, J. M. Marco, and R. Moriyón. Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation. *Ann. of Math. (2)*, 123(3):537–611, 1986.

[dlLW10] Rafael de la Llave and Alistair Windsor. Livšic theorems for non-commutative groups including diffeomorphism groups and results on the existence of conformal structures for Anosov systems. *Ergodic Theory Dynam. Systems*, 30(4):1055–1100, 2010.

[DS03] Nurlan S. Dairbekov and Vladimir A. Sharafutdinov. Some problems of integral geometry on Anosov manifolds. *Ergodic Theory Dynam. Systems*, 23(1):59–74, 2003.

[DS10] N. S. Dairbekov and V. A. Sharafutdinov. Conformal Killing symmetric tensor fields on Riemannian manifolds. *Mat. Tr.*, 13(1):85–145, 2010.

[DZ16] Semyon Dyatlov and Maciej Zworski. Dynamical zeta functions for Anosov flows via microlocal analysis. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(3):543–577, 2016.

[DZ17] Semyon Dyatlov and Maciej Zworski. Ruelle zeta function at zero for surfaces. *Invent. Math.*, 210(1):211–229, 2017.

[Ebe73] Patrick Eberlein. When is a geodesic flow of Anosov type? I,II. *J. Differential Geometry*, 8:437–463; ibid. 8 (1973), 565–577, 1973.

[Ebi68] David G. Ebin. On the space of Riemannian metrics. *Bull. Amer. Math. Soc.*, 74:1001–1003, 1968.

[FRS08] Frédéric Faure, Nicolas Roy, and Johannes Sjöstrand. Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances. *Open Math. J.*, 1:35–81, 2008.

[FS11] Frédéric Faure and Johannes Sjöstrand. Upper bound on the density of Ruelle resonances for Anosov flows. *Comm. Math. Phys.*, 308(2):325–364, 2011.

[FT13] Frédéric Faure and Masato Tsujii. Band structure of the Ruelle spectrum of contact Anosov flows. *C. R. Math. Acad. Sci. Paris*, 351(9-10):385–391, 2013.

[FT17] Frédéric Faure and Masato Tsujii. The semiclassical zeta function for geodesic flows on negatively curved manifolds. *Invent. Math.*, 208(3):851–998, 2017.

[GC20] Colin Guillarmou and Mihajlo Cekic. First band of Ruelle resonances for contact Anosov flows in dimension 3. *arXiv e-prints*, page arXiv:2011.05959, November 2020.

[GGHW20] Yannick Guedes Bonthonneau, Colin Guillarmou, Joachim Hilgert, and Tobias Weich. Ruelle-Taylor resonances of Anosov actions. *arXiv e-prints*, page arXiv:2007.14275, July 2020.

[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, third edition, 2004.

[GK80a] V. Guillemin and D. Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology*, 19(3):301–312, 1980.

[GK80b] Victor Guillemin and David Kazhdan. Some inverse spectral results for negatively curved n-manifolds. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 153–180. Amer. Math. Soc., Providence, R.I., 1980.

[GKL19] Colin Guillarmou, Gerhard Knieper, and Thibault Lefeuvre. Geodesic stretch, pressure metric and marked length spectrum rigidity. *arXiv e-prints*, page arXiv:1909.08666, Sep 2019.

[GL] Yannick Guedes Bonthonneau and Thibault Lefeuvre. Microlocal regularity of solutions to cohomological equations.

[GL06] Sébastien Gouëzel and Carlangelo Liverani. Banach spaces adapted to Anosov systems. *Ergodic Theory Dynam. Systems*, 26(1):189–217, 2006.

[GL19a] Sébastien Gouëzel and Thibault Lefeuvre. Classical and microlocal analysis of the X-ray transform on Anosov manifolds. *arXiv e-prints*, Apr 2019.

[GL19b] Yannick Guedes Bonthonneau and Thibault Lefeuvre. Local rigidity of manifolds with hyperbolic cusps I. Linear theory and microlocal tools. *arXiv e-prints*, page arXiv:1907.01809, Jul 2019.

[GL19c] Yannick Guedes Bonthonneau and Thibault Lefeuvre. Local rigidity of manifolds with hyperbolic cusps II. Nonlinear theory. Oct 2019.

[GL19d] Colin Guillarmou and Thibault Lefeuvre. The marked length spectrum of Anosov manifolds. *Ann. of Math. (2)*, 190(1):321–344, 2019.

[GM18] Colin Guillarmou and Marco Mazzucchelli. Marked boundary rigidity for surfaces. *Ergodic Theory Dynam. Systems*, 38(4):1459–1478, 2018.

[GPSU16] Colin Guillarmou, Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The X-ray transform for connections in negative curvature. *Comm. Math. Phys.*, 343(1):83–127, 2016.

[Gro83] Mikhael Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[Gui77] Victor Guillemin. Lectures on spectral theory of elliptic operators. *Duke Math. J.*, 44(3):485–517, 1977.

[Gui17a] Colin Guillarmou. Invariant distributions and X-ray transform for Anosov flows. *J. Differential Geom.*, 105(2):177–208, 2017.

[Gui17b] Colin Guillarmou. Lens rigidity for manifolds with hyperbolic trapped sets. *J. Amer. Math. Soc.*, 30(2):561–599, 2017.

[GW17] Y. Guedes Bonthonneau and T. Weich. Ruelle-Pollicott Resonances for Manifolds with Hyperbolic Cusps. *ArXiv e-prints*, December 2017.

[GWW92] C. Gordon, D. Webb, and S. Wolpert. Isospectral plane domains and surfaces via Riemannian orbifolds. *Invent. Math.*, 110(1):1–22, 1992.

[HÖ3] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[Ham99] U. Hamenstädt. Cocycles, symplectic structures and intersection. *Geom. Funct. Anal.*, 9(1):90–140, 1999.

[HF] Boris Hasselblatt and Todd Fisher. *Hyperbolic flows.* In preparation, *.

[HZ19] Hamid Hezari and Steve Zelditch. One can hear the shape of ellipses of small eccentricity. *arXiv e-prints*, page arXiv:1907.03882, July 2019.

[IM18] J. Ilmavirta and F. Monard. Integral geometry on manifolds with boundary and applications. *ArXiv e-prints*, June 2018.

[Jou86] Jean-Lin Journé. On a regularity problem occurring in connection with Anosov diffeomorphisms. *Comm. Math. Phys.*, 106(2):345–351, 1986.

[Jé19] Malo Jézéquel. Global trace formula for ultra-differentiable Anosov flows. *arXiv e-prints*, page arXiv:1901.09576, January 2019.

[Kac66] Mark Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[Kal11] Boris Kalinin. Livšic theorem for matrix cocycles. *Ann. of Math. (2)*, 173(2):1025–1042, 2011.

[Kat88] Anatole Katok. Four applications of conformal equivalence to geometry and dynamics. *Ergodic Theory Dynam. Systems*, 8*(Charles Conley Memorial Issue):139–152, 1988.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[Kli74] Wilhelm Klingenberg. Riemannian manifolds with geodesic flow of Anosov type. *Ann. of Math. (2)*, 99:1–13, 1974.

[KN69] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II.* Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[Kni02] G. Knieper. *Hyperbolic dynamical systems, in Handbook of Dynamical Systems Vol 1A*, pages 239–319. Elsevier, Amsterdam, 2002.

[Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. 1970.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Lef18] Thibault Lefeuvre. Local marked boundary rigidity under hyperbolic trapping assumptions. *arXiv e-prints*, page arXiv:1804.02143, Apr 2018.

[Lef19] Thibault Lefeuvre. On the s-injectivity of the x-ray transform on manifolds with hyperbolic trapped set. *Nonlinearity*, 32(4):1275–1295, 2019.

[Liv72] A. N. Livšic. Cohomology of dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:1296–1320, 1972.

[Liv04] Carlangelo Liverani. On contact Anosov flows. *Ann. of Math. (2)*, 159(3):1275–1312, 2004.

[LT05] A. O. Lopes and Ph. Thieullen. Sub-actions for Anosov flows. *Ergodic Theory Dynam. Systems*, 25(2):605–628, 2005.

[Mel03] Richard B. Melrose. *Introduction to Microlocal Analysis*. 203.

[Mic82] René Michel. Sur la rigidité imposée par la longueur des géodésiques. *Invent. Math.*, 65(1):71–83, 1981/82.

[Mil64] J. Milnor. Eigenvalues of the Laplace operator on certain manifolds. *Proc. Nat. Acad. Sci. U.S.A.*, 51:542, 1964.

[MP] Will Merry and Gabriel P. Paternain. *Inverse Problems in Geometry and Dynamics*. Lecture notes, **.

[NP99] M. Nicol and M. Pollicott. Measurable cocycle rigidity for some non-compact groups. *Bull. London Math. Soc.*, 31(5):592–600, 1999.

[NT95] Viorel Niţică and Andrei Török. Cohomology of dynamical systems and rigidity of partially hyperbolic actions of higher-rank lattices. *Duke Math. J.*, 79(3):751–810, 1995.

[NT98] Viorel Niţică and Andrei Török. Regularity of the transfer map for cohomologous cocycles. *Ergodic Theory Dynam. Systems*, 18(5):1187–1209, 1998.

[NZ15] Stéphane Nonnenmacher and Maciej Zworski. Decay of correlations for normally hyperbolic trapping. *Invent. Math.*, 200(2):345–438, 2015.

[Ota90] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Ann. of Math. (2)*, 131(1):151–162, 1990.

[Par99] William Parry. The Livšic periodic point theorem for non-abelian cocycles. *Ergodic Theory Dynam. Systems*, 19(3):687–701, 1999.

[Pat99] Gabriel P. Paternain. *Geodesic flows*, volume 180 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[Pat09] Gabriel P. Paternain. Transparent connections over negatively curved surfaces. *J. Mod. Dyn.*, 3(2):311–333, 2009.

[Pat11] Gabriel P. Paternain. Bäcklund transformations for transparent connections. *J. Reine Angew. Math.*, 658:27–37, 2011.

[Pat12] Gabriel P. Paternain. Transparent pairs. *J. Geom. Anal.*, 22(4):1211–1235, 2012.

[Pat13] Gabriel P. Paternain. Inverse problems for connections. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 369–409. Cambridge Univ. Press, Cambridge, 2013.

[PS18] Gabriel P. Paternain and Mikko Salo. Carleman estimates for geodesic X-ray transforms. *arXiv e-prints*, page arXiv:1805.02163, May 2018.

[PSU14] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Spectral rigidity and invariant distributions on Anosov surfaces. *J. Differential Geom.*, 98(1):147–181, 2014.

[PSU15] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Math. Ann.*, 363(1-2):305–362, 2015.

[PSU21] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Geometric inverse problems in dimension two. *in preparation*, 2021.

[PU05] Leonid Pestov and Gunther Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math. (2)*, 161(2):1093–1110, 2005.

[PW01] M. Pollicott and C. P. Walkden. Livšic theorems for connected Lie groups. *Trans. Amer. Math. Soc.*, 353(7):2879–2895, 2001.

[PZ16] Gabriel P. Paternain and Hanming Zhou. Invariant distributions and the geodesic ray transform. *Anal. PDE*, 9(8):1903–1930, 2016.

[Sad13] Victoria Sadovskaya. Cohomology of $GL(2,\mathbb{R})$-valued cocycles over hyperbolic systems. *Discrete Contin. Dyn. Syst.*, 33(5):2085–2104, 2013.

[Sad17] Victoria Sadovskaya. Fiber bunching and cohomology for Banach cocycles over hyperbolic systems. *Discrete Contin. Dyn. Syst.*, 37(9):4959–4972, 2017.

[Sch99] Klaus Schmidt. Remarks on Livšic' theory for nonabelian cocycles. *Ergodic Theory Dynam. Systems*, 19(3):703–721, 1999.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[Shu01] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[SU04] Plamen Stefanov and Gunther Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.

[SUV17] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge. *ArXiv e-prints*, page arXiv:1702.03638, February 2017.

[Tay11] Michael E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.

[Tro92] Anthony J. Tromba. *Teichmüller theory in Riemannian geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992. Lecture notes prepared by Jochen Denzler.

[TZ20] Masato Tsujii and Zhiyuan Zhang. Smooth mixing Anosov flows in dimension three are exponential mixing. *arXiv e-prints*, page arXiv:2006.04293, June 2020.

[UV16] Gunther Uhlmann and András Vasy. The inverse problem for the local geodesic ray transform. *Invent. Math.*, 205(1):83–120, 2016.

[Vig80] Marie-France Vignéras. Variétés riemanniennes isospectrales et non isométriques. *Ann. of Math. (2)*, 112(1):21–32, 1980.

[Wal00] C. P. Walkden. Livšic theorems for hyperbolic flows. *Trans. Amer. Math. Soc.*, 352(3):1299–1313, 2000.

[Wil14] Amie Wilkinson. Lectures on marked length spectrum rigidity. In *Geometric group theory*, volume 21 of *IAS/Park City Math. Ser.*, pages 283–324. Amer. Math. Soc., Providence, RI, 2014.

[Zel04] Steve Zelditch. The inverse spectral problem. In *Surveys in differential geometry. Vol. IX*, volume 9 of *Surv. Differ. Geom.*, pages 401–467. Int. Press, Somerville, MA, 2004. With an appendix by Johannes Sjöstrand and Maciej Zworski.

[Zwo12] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.