Generic dynamical properties of connections on vector bundles

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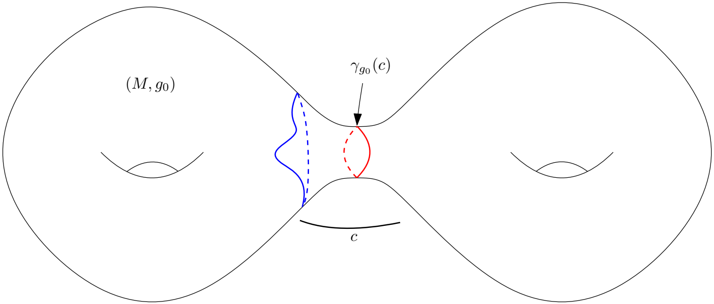
- (M, g) smooth closed (compact, $\partial M = \emptyset$) Riemannian manifold with negative sectional curvature (or Anosov manifold i.e. with Anosov geodesic flow on the unit tangent bundle).
- $SM = \{(x, v) \in TM \mid |v| = 1\}$ unit tangent bundle, $\varphi_t : SM \to SM$ geodesic flow and $X := d/dt(\varphi_t)|_{t=0}$ geodesic vector field.
- $\mathcal{C}$ = set of free homotopy classes $\overset{\text{1-to-1}}{\leftrightarrow}$ closed geodesics (i.e. $\forall c \in \mathcal{C}, \exists! \gamma_{g_0}(c) \in c$)



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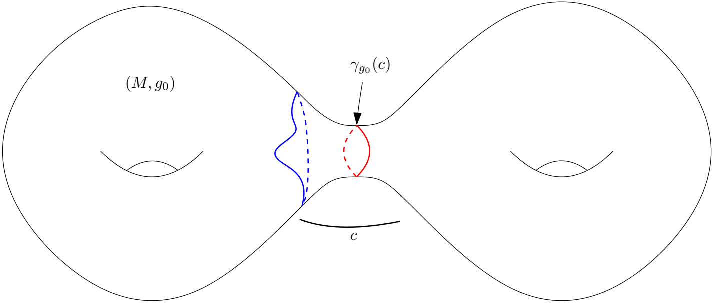

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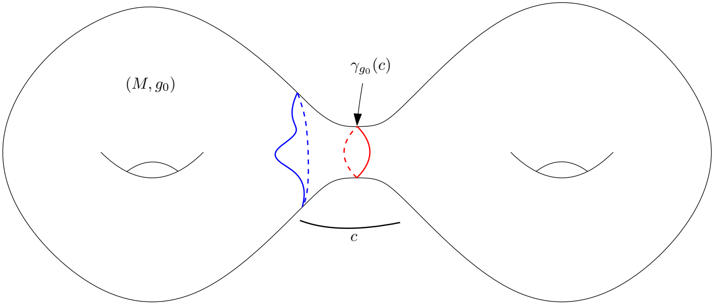

- Let $\mathcal{E} \to M$ be a smooth vector bundle over M equipped with a unitary connection $\nabla^{\mathcal{E}}$. Given an (oriented) geodesic $\gamma \subset M$, denote by $P_\gamma : \mathcal{E}_{x_-} \to \mathcal{E}_{x_+}$ the parallel transport along γ with respect to $\nabla^{\mathcal{E}}$, where x_-, x_+ are the two extremal points of γ.
- We want to study holonomy of connections along closed geodesics. For that, we introduce:

$$\mathrm{Hol}_{\nabla^{\mathcal{E}}} : \mathcal{C} \to \prod_{c \in \mathcal{C}} \mathrm{U}(\mathcal{E}_{x_c}), \quad c \mapsto P_{\gamma_g(c)},$$

where $x_c \in \gamma_g(c)$ is arbitrary.

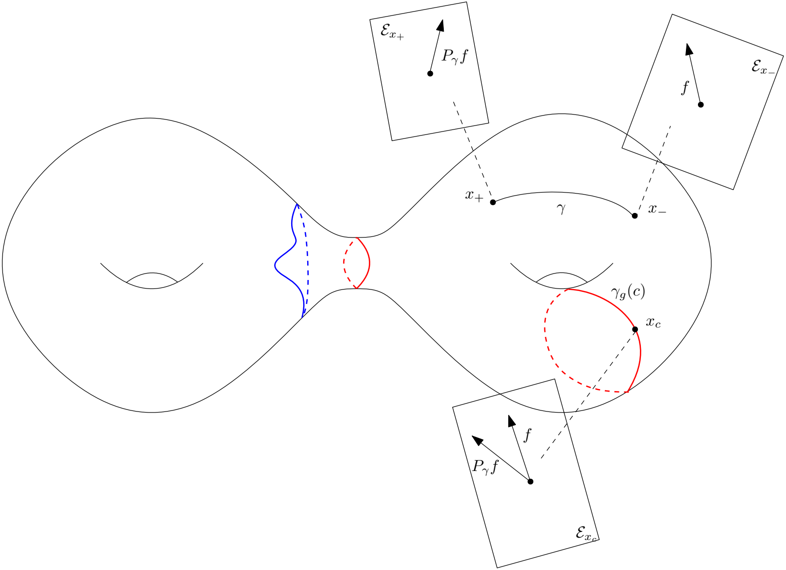

Figure: Parallel transport along geodesics.

Question: Does the holonomy of a connection along closed geodesics determine $\nabla^{\mathcal{E}}$ up to a gauge equivalent factor?

- Two connections $\nabla_1^{\mathcal{E}}$ and $\nabla_2^{\mathcal{E}}$ are gauge-equivalent if there exists $p \in C^\infty(M, \mathrm{U}(\mathcal{E}))$ such that $\nabla_1^{\mathcal{E}} f = p^{-1} \nabla_2^{\mathcal{E}}(pf)$.
- Similar question to the Marked Length Spectrum (MLS) Conjecture. The MLS is defined as the map

$$L_g : \mathcal{C} \to \mathbb{R}_+, \quad c \mapsto \ell_g(\gamma_g(c)).$$

It is conjectured (**Burns-Katok '85**) that in negative curvature the MLS should determine the metric g in the following sense:

Conjecture (Burns-Katok '85)

If $L_g = L_{g'}$, then g and g' are isometric i.e. there exists a diffeomorhism $\phi : M \to M$ isotopic to the identity such that $\phi^* g = g'$.

Answer 1: Yes for line bundles on Anosov manifolds (**Paternain '09-'13**).

- We can even produce stability estimates in the case of line bundles.
- Consider $\mathcal{L} \to M$ a line bundle. Then, for $c \in \mathcal{C}$, $\mathrm{Hol}_{\nabla^{\mathcal{L}}}(c) \in \mathbb{C}$ (and $\mathrm{U}(1)$ if $\nabla^{\mathcal{L}}$ unitary).

Theorem (Cekic-L. '20)

Assume (M, g) is Anosov. There exists $\alpha > 0$, $C > 0$ such that the following holds:

$$d(\nabla_1^{\mathcal{L}}, \nabla_2^{\mathcal{L}}) \leq C \sup_{c \in \mathcal{C}} \left(L_g(c)^{-1} |\mathrm{Hol}_{\nabla_1^{\mathcal{L}}}(c) \mathrm{Hol}_{\nabla_2^{\mathcal{L}}}^{-1}(c) - 1| \right)^{\alpha}$$

- $d(\nabla_1^{\mathcal{L}}, \nabla_2^{\mathcal{L}})$ is a natural distance on the space of connections which is 0 iff the connections are gauge-equivalent.
- Proof is based on an approximate Livsic Theorem for cocycles and the microlocal framework introduced in **Guillarmou '17, Guillarmou-L. '18, Gouëzel-L. '19**.

Answer 2: However, in higher rank, the situation is more complicated.

A first case to investigate is that of transparent connections:

Definition

We say that $\nabla^{\mathcal{E}}$ is transparent if the holonomy is trivial on every closed geodesics i.e. $\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c) = \mathbf{1}$ for all $c \in \mathcal{C}$.

Examples: (1) The trivial connection d on the trivial vector bundle $\mathbb{C}^r \times M \to M$ (don't worry, there are other examples!). (2) On a oriented Riemannian surface (M, g), the Levi-Civita connection is always transparent.

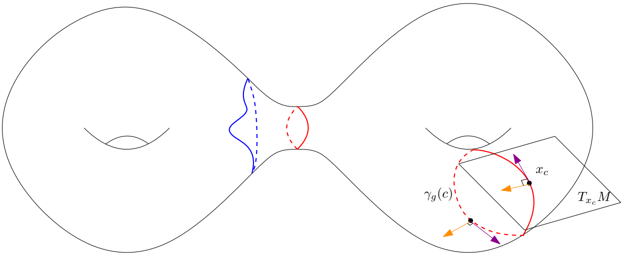

Figure: An oriented Riemannian surface is always transparent.

- Let $\pi : SM \to M$ be the projection. Consider $\pi^*\mathcal{E} \to SM$ equipped with the pullback connection $\pi^*\nabla^{\mathcal{E}}$. Parallel transport of sections of $\mathcal{E}$ along geodesics is equivalent to parallel transport of sections of $\pi^*\mathcal{E}$ along flowline of the geodesic flow.
- For $(x, v) \in SM$, denote by

$$P((x,v),t) : \mathcal{E}_x \to \mathcal{E}_{\pi(\varphi_t(x,v))},$$

the parallel transport map (this is a cocycle). Write $\mathbf{X} := (\pi^*\nabla^{\mathcal{E}})_X$ (generator of the cocycle).

- The connection $\nabla^{\mathcal{E}}$ is transparent if and only if $P((x,v),T) = \mathbf{1}$ for all T-periodic points $(x, v) \in SM$ of the geodesic flow.

Lemma (Folklore)

If $\nabla^{\mathcal{E}}$ is transparent, then $\pi^*\mathcal{E} \to SM$ is trivial and there exists a global basis $(e_1, ..., e_r)$ such that $e_i \in C^\infty(SM, \pi^*\mathcal{E})$ and $\mathbf{X}e_i = 0$.

Idea of proof: Consider $\mathcal{O}(x_0, v_0)$ a dense orbit for the geodesic flow. Consider $(e_1, ..., e_r)$ an orthonormal basis at $\mathcal{E}_{x_0,v_0}$ and parallel-transport this basis along the orbit.

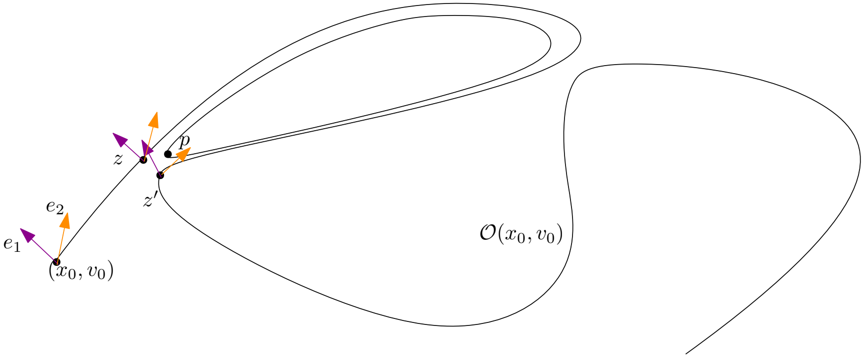

Figure: Parallel transport along the dense geodesic.

- We now want to study equations of the form $\mathbf{X}f = 0$, where $f \in C^\infty(SM, \pi^*\mathcal{E})$ or more generally $\mathbf{X}f = u$. These are called twisted cohomological equations.
- A first remark is: given $h \in C^\infty(SM)$, it can be decomposed in Fourier modes in the sphere fibers $h = \sum_{m \geq 0} h_m$, where

$$h_m \in \ker(\Delta^{\mathbb{V}}(x) + m(m+n-2)) =: \Omega_m(x)$$

is a spherical harmonics of degree $m \in \mathbb{N}$ and $\Delta^{\mathbb{V}}$ is the vertical Laplacian. ($\Omega_m \to M$ is a vector bundle over M.)

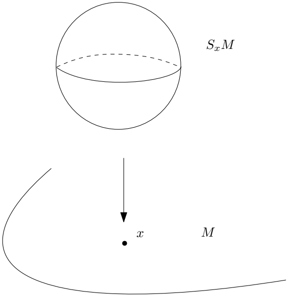

Figure: Sphere fibration.

- Let $M \ni x \mapsto (e_1(x), ..., e_r(x))$ be a local orthonormal basis of $\mathcal{E}$ around some point $x_0 \in M$. If $f \in C^\infty(SM, \pi^*\mathcal{E})$, then

$$f(x,v) = \sum_{k=1}^{r} f_k(x,v) e_k(x), \qquad f_k \in C^\infty(SM).$$

Each $f_k \in C^\infty(SM)$ can be pointwise decomposed in the sphere fibers into Fourier modes. Hence:

$$C^\infty(SM, \pi^*\mathcal{E}) = \oplus_{m \geq 0} C^\infty(M, \Omega_m \otimes \mathcal{E}).$$

We call degree $\deg(f) = N$, if $f = f_0 + ... + f_N$ and $f_N \neq 0$.

- We see $\mathbf{X} : C^\infty(SM, \pi^*\mathcal{E}) \to C^\infty(SM, \pi^*\mathcal{E})$ (recall $\mathbf{X} := (\pi^*\nabla^{\mathcal{E}})_X$) as a differential operator of order 1. One can show (**Guillemin-Kazhdan '80**):

$$\mathbf{X} : C^\infty(M, \Omega_m \otimes \mathcal{E}) \to C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E}),$$

thus $\mathbf{X} = \mathbf{X}_- + \mathbf{X}_+$. ($\mathbf{X}_+$ is of gradient-type, $\mathbf{X}_-$ of divergence-type.)

Question: If $\mathbf{X}f = u$ and $u = u_0 + ... + u_m \in C^\infty(SM, \pi^*\mathcal{E})$ has degree m, does f have degree $m-1$? (or 0 if $m = 0$)

Theorem (Guillarmou-Paternain-Salo-Uhlmann '16)

If (M, g) has negative curvature, then $\deg(f) < \infty$.

- Proof relies on an L^2-energy estimate called the (twisted) Pestov identity.
- Obstruction to having $\deg(f) = m - 1$ is the existence of twisted Conformal Killing Tensors, i.e. elements in $\ker \mathbf{X}_+|_{\Omega_m \otimes \mathcal{E}} \neq \{0\}$ for $m \geq 1$. Indeed, assume there are no CKTS of degree $m \geq 1$ and $\mathbf{X}f = 0$. Then $f = f_0 + ... + f_N$ (by [GPSU16]) and $\mathbf{X}f = 0$ implies $\mathbf{X}_+ f_N = 0$, hence $f_N = 0$ unless $N = 0$.
- In particular, if $\nabla^{\mathcal{E}}$ is transparent and has no CKTs, then $\mathbf{X}e_i = 0$ imply that $\deg(e_i) = 0$, i.e. $e_i \in C^\infty(M, \mathcal{E})$. Then $\mathbf{X}e_i = 0 = \nabla^{\mathcal{E}} e_i$. In other words, $(\mathcal{E}, \nabla^{\mathcal{E}})$ is isomorphic to the trivial bundle $(\mathbb{C}^r \times M, d)$ with trivial connection.

- Consider (M, g) smooth closed manifold (no curvature/Anosov assumption!), $\mathcal{E} \to M$ smooth vector bundle. Denote by $\mathcal{R}^{\mathcal{E}}$ the set of connections on $\mathcal{E}$ without CKTs.

Theorem (Cekic-L. '20)

Assume $\dim(M) \geq 3$. Then $\mathcal{R}^{\mathcal{E}}$ is residual (among all unitary connections of regularity C^k, $k \geq 2$).

- Generic absence of CKTs has other consequences. (It is more exactly the absence of CKTs for the induced connection on the endomorphism bundle.)

Theorem (Cekic-L. '20)

Assume (M, g) has negative curvature. Consider $\mathcal{E} \to M$ and a generic unitary connection $\nabla_0^{\mathcal{E}}$. Then, there exists $\varepsilon, \alpha, N, C > 0$ such that for all $\nabla^{\mathcal{E}}$ such that $\|\nabla_0^{\mathcal{E}} - \nabla^{\mathcal{E}}\|_{C^N} < \varepsilon$ the following inequality holds:

$$d(\nabla_0^{\mathcal{E}}, \nabla^{\mathcal{E}}) \leq C \sup_{c \in \mathcal{C}} \left(L_g(c)^{-1} \|\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c) \mathrm{Hol}_{\nabla^{\mathcal{E}}}^{-1}(c) - \mathbf{1}\| \right)^{\alpha}$$

Idea of proof:

- For fixed $m \in \mathbb{N}$, write $\mathcal{R}_m^{\mathcal{E}}$ for the set of connections such that $\ker \mathbf{X}_+|_{\Omega_m \otimes \mathcal{E}} = \{0\}$. Each $\mathcal{R}_m^{\mathcal{E}}$ is open and $\mathcal{R}^{\mathcal{E}} = \cap_{m \geq 0} \mathcal{R}_m^{\mathcal{E}}$. So it suffices to show that $\mathcal{R}_m^{\mathcal{E}}$ is dense.
- Fix a connection $\nabla^{\mathcal{E}}$. Introduce $\Delta_+ := (\mathbf{X}_+)^* \mathbf{X}_+ = -\mathbf{X}_- \mathbf{X}_+$. This is a Laplacian type operator. Then, $\nabla^{\mathcal{E}}$ has CKTs of degree m if and only if $0 \in \mathrm{Spec}(\Delta_+|_{\Omega \otimes \mathcal{E}})$.
- Hence, we want to perturb $\nabla^{\mathcal{E}}$ by $\nabla^{\mathcal{E}} + \Gamma$ (where $\Gamma \in C^\infty(M, T^*M \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E})))$ so that Δ_+^{Γ} has no eigenvalue at 0.

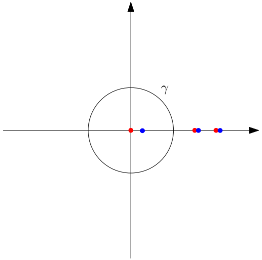

Figure: In red: eigenvalues of Δ_+. In blue: eigenvalues of Δ_+^{Γ}.

- Let $(u_1, ..., u_d)$ be an L^2-orthonormal basis of CKTs of degree $m \in \mathbb{N}$. Consider $\gamma \subset \mathbb{C}$, small circle around 0. Define:

$$\Pi^\Gamma := \frac{1}{2i\pi} \int_\gamma (z - \Delta_+^\Gamma)^{-1} \mathrm{d}z, \lambda^\Gamma = \mathrm{Tr}(\Delta_+^\Gamma \Pi^\Gamma)$$

  These correspond to the orthogonal projection on eigenstates inside the circle / the sum of eigenvalues inside the circle. We have $\lambda^{\Gamma=0} = 0$ and $\Pi^{\Gamma=0} = \Pi$ is the orthogonal projection on the CKTs of degree m of $\nabla^{\mathcal{E}}$.
- It suffices to produce Γ arbitrarily small such that $\lambda^\Gamma > 0$: indeed, this means that at least one of the CKTs was "ejected from 0". Hence the number of CKTs of degree m for $\nabla^{\mathcal{E}} + \Gamma$ is at most $d - 1$. Then iterate the process.
- Bad luck: $\mathrm{d}\lambda^{\Gamma=0} = 0$! What about the second derivative?

- Recall that $\mathbf{X}_+ : C^\infty(M, \Omega_m \otimes \mathcal{E}) \to C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})$. The operator $\mathbf{X}_+$ is of *gradient type* (its principal symbol is injective). Moreover:

$$\mathbf{X}_+^* = -\mathbf{X}_- : C^\infty(M, \Omega_{m+1} \otimes \mathcal{E}) \to C^\infty(M, \Omega_m \otimes \mathcal{E}).$$

  Hence:

$$C^\infty(M, \Omega_{m+1} \otimes \mathcal{E}) = \operatorname{ran}(\mathbf{X}_+) \oplus^\perp \ker(\mathbf{X}_-).$$

- If $A \in C^\infty(M, T^*M \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E})) \simeq C^\infty(M, \Omega_1 \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$, and $f \in C^\infty(M, \Omega_m \otimes \mathcal{E})$, then

$$Af \in C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E}),$$

  i.e. $Af = A_- f + A_+ f$.

Lemma

$\forall A \in C^\infty(M, T^*M \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$, $\mathrm{d}^2\lambda^{\Gamma=0}(A, A) = \sum_{i=1}^d \|\pi_{\ker \mathbf{X}_-} A_+ u_i\|_{L^2}^2$

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- We want to show that $\mathrm{d}^2\lambda^{\Gamma=0}(A,A) > 0$ for some A. We argue by contradiction. Assume that this is always 0. Then, $A_+ u_1 \in \mathrm{ran}(\mathbf{X}_+)$ for all A. Since $\mathrm{ran}(\mathbf{X}_+) = \ker(\mathbf{X}_-)^\perp$, this implies that $\forall w \in \ker(\mathbf{X}_-|_{C^\infty(M,\Omega_{m+1}\otimes\mathcal{E})}), \forall A$:

$$\langle A_+ u_1, w\rangle_{L^2} = 0 = \int_M \langle A_+ u_1, w\rangle_x \mathrm{d}\,\mathrm{vol}(x).$$

- Since A is arbitrary, it can be localized near any point $x \in M$ and implies the equality pointwise in x:

$$\langle A_+ u_1, w\rangle_x = 0, \forall A, \forall w \in \ker(\mathbf{X}_-).$$

  We want to show that this implies $u_1 \equiv 0$ (which is a contradiction since $\|u_1\|_{L^2} = 1$).

Question: At a given point $x \in M$, what are the values $w(x)$ that can be achieved by elements of $\ker(\mathbf{X}_-)$?

- Let $E, F \to M$ be two vector bundles with $\mathrm{rank}(E) > \mathrm{rank}(F)$. Let $P : C^\infty(M, E) \to C^\infty(M, F)$ be an operator of divergence type i.e. its principal symbol

$$\sigma_P(x, \xi) \in \mathrm{Hom}(E_x, F_x)$$

is surjective for every $(x, \xi) \in T^*M \setminus \{0\}$.

- For $x \in M$, consider the evaluation map:

$$\mathrm{ev}_x : \ker(P) \to E_x, \quad w \mapsto w(x).$$

Proposition (Cekic-L. '20)

We have the following (sharp) lower bound:

$$\Sigma_{\xi \in T_x^*M, |\xi|=1} \ker(\sigma_P(x, \xi)) \subset \mathrm{ran}(\mathrm{ev}_x) \subset E_x.$$

- In particular, we say that P is of uniform divergence type if we have the equality:

$$\Sigma_{\xi \in T_x^*M, |\xi|=1} \ker(\sigma_P(x, \xi)) = \mathrm{ran}(\mathrm{ev}_x) = E_x$$

- Recall that
$$\langle A_+ u_1, w\rangle_x = 0, \forall A, \forall w \in \ker(\mathbf{X}_-).$$
- As A_+ is injective for $A \neq 0$, it suffices to show that $\mathbf{X}_-$ is of uniform divergence type.
- For that, we can forget about the twist (i.e. take $\mathcal{E} = \mathbb{C}$) and consider
$$X_- : C^\infty(M, \Omega_{m+1}) \to C^\infty(M, \Omega_m).$$
We need to show that
$$\Sigma_{\xi \in T_x^* M, |\xi|=1} \ker(\sigma_{X_-}(x,\xi)) = \Omega_m(x)$$
- There is a pointwise (in $x \in M$) identification of trace-free symmetric m-tensors and spherical harmonics of degree m:
$$\pi_m^* : \otimes_S^m T_x^* M|_{0-\mathrm{Tr}} \to \Omega_m(x), \quad \pi_m^* f(x,v) := f_x(v, ..., v).$$
Using this identification, $\sigma_{X_-}(x,\xi) = \imath_{\xi^\sharp}$.

- Define:

$$W(x) := \Sigma_{\xi \in T_x^* M, |\xi|=1} \ker(\imath_{\xi^\sharp}) \subset \Omega_m(x).$$

  We want to show $W(x) = \Omega_m(x)$.

- There is a natural $\mathrm{SO}(n)$ action on symmetric tensors: f is a symmetric m-tensor, $A \in \mathrm{SO}(n)$, then $A^* f = f(A\cdot, ..., A\cdot)$.

Lemma

$W(x)$ is invariant by the natural $\mathrm{SO}(n)$-action.

- As $\Omega_m(x)$ is an irreducible representation of $\mathrm{SO}(n)$ (since $n = \dim(M) \geq 3$), this implies that $W(x) = \Omega_m(x)$. This ends the proof.

Definition

We say that (M, g) is transparent is the tangent bundle equipped with the Levi-Civita connection ∇ is transparent.

- As we saw, any oriented Riemannian surface is transparent.

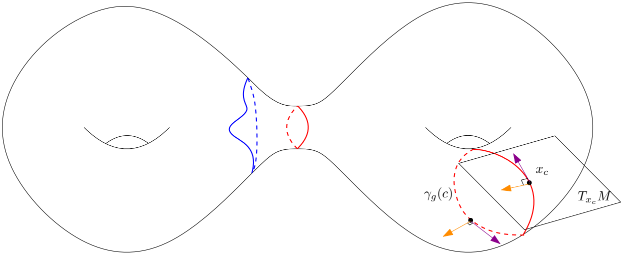

Figure: An oriented Riemannian surface is always transparent.

Question: Are there transparent negatively-curved manifolds of dimension ≥ 3? transparent Anosov manifolds of dimension ≥ 3?

Answer: We do not know! Although we conjecture this should never happen.

Theorem (Cekic-L. '20)

Assume (M, g) is negatively-curved and transparent. Then $\dim(M) = 2, 4$ or 8. Moreover, hyperbolic metrics are never transparent (except in dimension 2).

Idea of proof:

- If (M, g) is transparent, then $\pi^* TM \to SM$ is trivialized by $e_1, ..., e_n$ such that $\mathbf{X} e_i := (\pi^* \nabla)_X e_i = 0$. Observe that the tautological section $s(x, v) := v$ always satisfies $\mathbf{X} s = 0$. Moreover, one can choose $e_2, ..., e_n \in C^\infty(SM, \pi^* TM)$ to be pointwise (in (x, v)) orthogonal to s. A short argument shows that this forces the sphere $S_x M$ to be parallelizable, hence of dimension $1, 3$ or 7.

- On hyperbolic manifolds, one can show that the Levi-Civita connection has no CKTs of degree ≥ 2. Hence, $\mathbf{X}e_i = 0$ implies that the e_i are of degree 1 i.e.

$$e_i(x, v) = \sum_{kj=1}^{n} \alpha_{jk}^{(i)}(x) v_j e_k(x).$$

  Hence, they can be written $e_i(x, v) = R_i(x)v$, where $R_i \in C^\infty(M, \mathrm{End}(TM))$.
- Using elements of Clifford algebra theory, one shows that the R_i are parallel almost complex structures i.e. $R_i^2 = -\mathbf{1}$ and $\nabla R_i = 0$.
- Thus (R_2, R_3, R_4) endows M with a hyperkähler structure. This forces (M, g) to be Ricci-flat. It can thus not be negatively-curved.

Thank you for your attention !