On the rigidity of Riemannian manifolds

PhD defense

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Joint works with Yannick Guedes Bonthonneau, Sébastien Goüezel, Colin Guillarmou, Gerhard Knieper

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MATHEMATIQUES ORSAY

- (M, g_0) smooth closed (compact, $\partial M = \emptyset$) Riemannian manifold with negative sectional curvature $\rightarrow$ "chaotic" geodesic flow

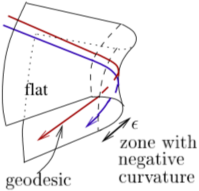

Figure: Image courtesy of Frédéric Faure

- **Question:** What are the geometric quantities which determine the Riemannian manifold (M, g_0)? In other words, can we find a quantity $A(g_0)$ such that if $A(g) = A(g_0)$, then $g \overset{\text{isom}}{\sim} g_0$?
- **Example:** On the topological side, an oriented surface is determined by a single number: its genus $g \in \mathbb{N}$.
- A first guess? The spectrum of the Laplacian $\{0 = \lambda_0 < \lambda_1 \leq ...\}$? **Milnor '55, Kac '66:** "Can one hear the shape of a drum?"

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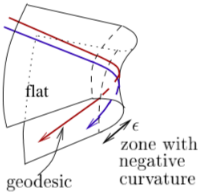

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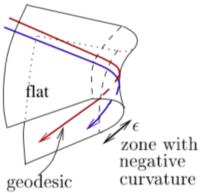

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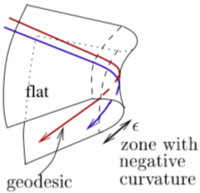

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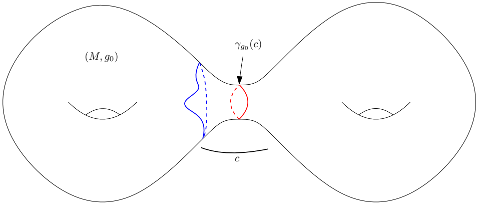
The marked length spectrum

- **Answer:** No! Counterexamples in constant curvature (**Vigneras '80**).
- The length spectrum i.e. the collection of lengths of closed geodesics is (under some mild assumptions) determined by the spectrum of the Laplacian. **Conclusion:** One needs a stronger notion to be able to determine the geometry of a manifold.
- $\mathcal{C}$ = set of free homotopy classes $\overset{\text{1-to-1}}{\leftrightarrow}$ closed g_0-geodesics (i.e. $\forall c \in \mathcal{C}, \exists! \gamma_{g_0}(c) \in c$)



The marked length spectrum

Definition (Marked length spectrum)

$$L_{g_0} : \mathcal{C} \to \mathbb{R}_+^*, \quad c \mapsto \ell_{g_0}(\gamma_c),$$

where $\ell_{g_0}(\gamma_c)$ Riemannian length computed with respect to g_0.

- This map is invariant by the action of $\mathrm{Diff}^0(M)$, the group of diffeomorphisms isotopic to the identity i.e. $L_{\phi^* g_0} = L_{g_0}$.

Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g_0 have negative sectional curvature, same marked length spectrum $L_g = L_{g_0}$, then $\exists\ \phi : M \to M$ smooth diffeomorphism isotopic to the identity such that $\phi^* g = g_0$.

Known results:

- **Guillemin-Kazhdan '80, Croke-Sharafutdinov '98**: proof of the infinitesimal version of the problem (for a deformation $(g_s)_{s\in(-1,1)}$ of the metric g_0): $L_{g_s} = L_{g_0} \implies \exists \phi_s, \quad \phi_s^* g_s = g_0$,
- **Croke '90**, **Otal '90**: proof for negatively-curved surfaces,
- **Katok '88**: proof for g conformal to g_0,
- **Besson-Courtois-Gallot '95**, **Hamenstädt '99**: proof when (M, g_0) is a locally symmetric space.

Theorem (Guillarmou-L. '18)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ such that: if $\|g - g_0\|_{C^k} < \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.

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- Still holds in the more general setting of Anosov manifolds (i.e. manifolds on which the geodesic flow is uniformly hyperbolic), under an additional assumption of nonpositive curvature in dim ≥ 3.
- Proof relies on finding good stability estimates for the differential of the operator $g \mapsto \mathcal{L}(g) = L_g / L_{g_0}$:

$$d\mathcal{L}_{g_0} f = 1/2 \times I_2^{g_0} f : c \mapsto \frac{1}{\ell(\gamma_{g_0}(c))} \int_0^{\ell(\gamma_{g_0}(c))} f_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \mathrm{d}t,$$

with $\gamma_{g_0}(c)$ unique closed geodesic in $c \in \mathcal{C}$, that is:

$$\|f\|_{C^0} \leq C \|d\mathcal{L}_{g_0}(f)\|_{\ell^\infty}^{\theta} \|f\|_{C^1}^{1-\theta}, \quad \forall f \in \ker \delta$$

- Proof heavily relies on microlocal analysis and hyperbolic dynamical systems.

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with $\gamma_{g_0}(c)$ unique closed geodesic in $c \in \mathcal{C}$.

Theorem (Guillarmou-L. '18, Goüezel-L. '19)

For all $0 < \alpha < \beta$, there exists $C, \theta > 0$ such that:

$$\|f\|_{C^\alpha} \leq C \|I_2^{g_0}(f)\|_{\ell^\infty}^{\theta} \|f\|_{C^\beta}^{1-\theta}, \quad \forall f \in \ker \delta$$

Theorem (Guillarmou-Knieper-L. '19)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ such that if $\|g - g_0\|_{C^k} < \varepsilon$, there exists $\phi : M \to M$ such that:

$$\|\phi^* g - g_0\|_{H^{-1/2}} \leq C \limsup_{j \to +\infty} |\log L_g(c_j)/L_{g_0}(c_j)|^{1/2}.$$

- Proof relies on the notion of geodesic stretch (**Croke-Fathi '90, Knieper '95**) and the thermodynamic formalism (**Bowen, Ruelle '70s** ...)
- This can be seen as a distance on isometry classes.

Distances on Teichmüller space

$M = S$ is an oriented surface of genus $g \geq 2$, Teichmüller space $\mathcal{T} = \{\text{hyperbolic metrics}\}/\text{Diff}_0(S)$.

- Weil-Petersson/pressure metric: Given $g \in \mathcal{T}$, $T^*\mathcal{T} \equiv \{\text{holomorphic differentials}\}$. In local isothermal coordinates, if $g = \lambda|dz|^2$ and $\xi dz^2, \gamma dz^2 \in T^*\mathcal{T}$ are two holomorphic differentials:

$$\langle \xi dz^2, \gamma dz^2 \rangle_{WP} = \text{Re} \int_S \frac{\xi \overline{\gamma}}{\lambda} \mathrm{dLeb}$$

- Thurston's (asymmetric) distance:

$$d_T(g_1, g_2) = \limsup_{j \to +\infty} \log(L_{g_2}(c_j)/L_{g_1}(c_j))$$

It is also the "best" Lipschitz constant $\text{Lip}(F)$ when trying to find a quasi-isometry $(S, g_1) \xrightarrow{F} (S, g_2)$.

Pressure metric

Theorem (Guillarmou-Knieper-L. '19)

Let M be a smooth manifold. There exists a pressure metric G on $\mathcal{M} := Met_{<0}(M)/Diff_0(M)$ enjoying a uniform coercive estimate:

$$G_g(f, f) \geq C\|f\|^2_{H^{-1/2}}$$

If $M = S$ is a surface, this metric G restricts to (a multiple of) the Weil-Petersson metric on Teichmüller space.

Question: Geometry of $(\mathcal{M}, G)$? This is an infinite-dimensional manifold!

Thurston's distance

Theorem (Guillarmou-Knieper-L. '19)

Let M be a smooth manifold. Let $\mathcal{E} = Met_{<0,h=1}(M)/Diff_0(M)$ be the subspace of metrics with topological entropy equal to 1. Then

$$d_T(g_1, g_2) := \limsup_{j \to +\infty} \log L_{g_2}(c_j)/L_{g_1}(c_j)$$

still defines a distance (like in Teichmüller space) in a neighborhood of the diagonal in $\mathcal{E} \times \mathcal{E}$.

- On Teichmüller space, Thurston proves that d_T is actually induced by an (asymmetric) Finsler norm:

$$\|f\|_F = \sup_{m \in \text{Mes}_{\mathbf{inv,erg}}} \int_{SM} f(v,v) dm(v)$$

- **Conjecture:** This distance is still induced by the same Finsler norm.
- This would actually solve the marked length spectrum rigidity conjecture.

Theorem (Guillarmou-L. '18, Goüezel-L. '19)

For all $0 < \alpha < \beta$, *there exists* $C, \theta > 0$ *such that:*

$$\|f\|_{C^\alpha} \leq C \|I_2^{g_0}(f)\|_{\ell^\infty}^{\theta} \|f\|_{C^\beta}^{1-\theta}, \quad \forall f \in \ker \delta$$

- The differential of the marked length spectrum is the X-ray transform

$$I_2^{g_0} : C^\infty(M, \mathrm{Sym}^2 T^*M) \to \ell^\infty(\mathcal{C}),$$

defined by

$$I_2^{g_0} f : c \mapsto \frac{1}{\ell(\gamma_{g_0}(c))} \int_0^{\ell(\gamma_{g_0}(c))} f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) \mathrm{d}t,$$

- The space $\ell^\infty(\mathcal{C})$ is not well-suited for analysis (the map $I_2^{g_0}$ does not seem to have closed range for instance). Somehow, we would like an operator which captures the information not only on closed geodesics but also on non-closed geodesics.
- **Question:** How to construct such an operator?

- A tensor $f \in C^\infty(M, \mathrm{Sym}^2 T^*M)$ can be identified to a function $\pi_2^* f \in C^\infty(SM)$ on the unit tangent bundle SM by the pullback map π_2^* defined as
$$\pi_2^* f(x,v) = f_x(v,v)$$
- Using the geodesic flow $\varphi_t^{g_0}$ on SM, the X-ray transform can be rewritten as
$$I_2^{g_0} f(c) = \frac{1}{\ell(\gamma_{g_0}(c))} \int_0^{\ell(\gamma_{g_0}(c))} e^{tX_0} \pi_2^* f(x,v) \mathrm{d}t,$$
where $e^{tX_0} u(x,v) = u(\varphi_t^{g_0}(x,v))$ is the propagator, X_{g_0} geodesic vector field.

- Instead of integrating on closed geodesics, we want to integrate on "any geodesics" to capture more information, i.e. we would like to define for any $(x, v) \in SM$ (unit tangent bundle) and $u \in C^\infty(SM)$ a map

$$\text{“}I^{g_0} u(x,v) = \int_0^{\ell(\gamma_{g_0}(x,v))} e^{tX_0} u(x,v) \mathrm{d}t\text{''}$$

Of course, $\ell(\gamma_{g_0}(x,v)) = +\infty$ "most of the time"!

- More generally, we want to make sense of the operator $\int_0^{+\infty} e^{tX_0} \mathrm{d}t$. A formal computation would yield

$$\int_0^{+\infty} e^{tX_0} \mathrm{d}t = -X_0^{-1}$$

- **Question:** What are e^{tX_0} and X_0^{-1} if X_0 is a (geodesic) vector field on a negatively-curved manifold? These operators exhibit the strong chaotic behaviour of the geodesic flow!

The propagator e^{tX_0}

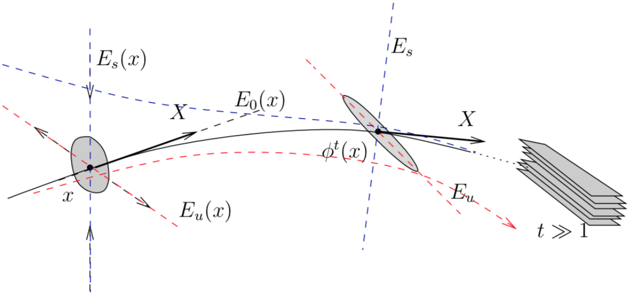
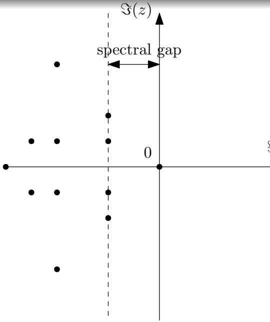


Figure: The evolution of the distribution u by the propagator e^{tX_0}. Image courtesy: Frédéric Faure.

Meromorphic extension of the resolvent $(X_0 \pm \lambda)^{-1}$



- We introduce the resolvents

$$R_\pm(\lambda) := (X_0 \pm \lambda)^{-1}$$

and we would like to define $R_\pm(0)$.

- They are initially defined on $\Re(\lambda) > 0$ and admit a meromorphic extension to $\mathbb{C}$ when acting on anisotropic Sobolev spaces with poles of finite ranks: the Pollicott-Ruelle resonances (**Liverani '04, Butterley-Liverani '07, Faure-Sjöstrand '11, Dyatlov-Zworski '13, Faure-Tsuji '13 '17**),
    - For the diffeo case, see **Blank-Keller-Liverani '02, Butterley-Liverani '07, Baladi-Tsuji '07 '08, Baladi '18**,
    - 0 is a pole of order 1 and $\mathrm{Res}_0((X \pm \lambda)^{-1}) = \mathbf{1} \otimes \mathbf{1}$,
    - Define (**Guillarmou '17**)

$$\Pi_2 := \pi_{2*}(R_+^{\mathrm{hol}}(0) - R_-^{\mathrm{hol}}(0))\pi_2^* + \pi_{2*}\mathbf{1} \otimes \mathbf{1}\pi_2^*$$

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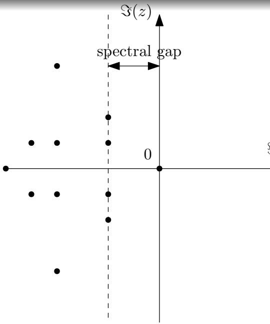

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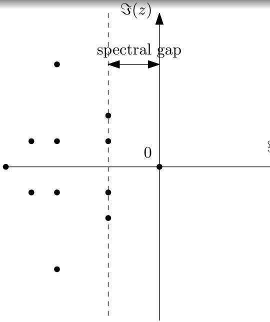

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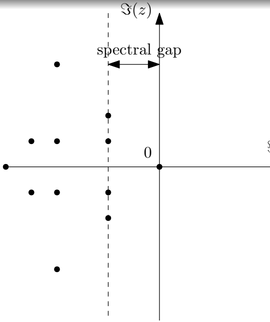

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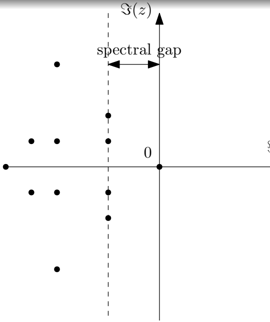

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Properties of Π_2

- Think of Π_2 as "$\pi_{2*} \circ \int_{\mathbb{R}} e^{tX} \mathrm{d}t \circ \pi_2^*$". This operator has also an expression in terms of the variance of the geodesic flow:

$$\langle \Pi_2 f, f \rangle_{L^2} = \mathrm{Var}^{X_0}_{\mu_{\mathrm{Liouville}}}(\pi_2^* f)$$

Theorem (Guillarmou '17, Guillarmou-L. '18, Gouëzel-L. '19)

- Π_2 *is a pseudodifferential of order* -1, *elliptic on tensors in* $\ker \delta$,
- *One has:* $\ker \Pi_2|_{\ker \delta} = \ker I_2|_{\ker \delta} = \{0\}$,
- *This implies the elliptic estimate:* $\|f\|_{H^s} \leq C\|\Pi_2 f\|_{H^{s+1}}, \forall f \in \ker \delta$

- Proof relies on microlocal tools developed by **Faure-Sjostrand '11, Dyatlov-Zworski '13**.
- **Problem:** Link between Π_2 and I_2? This is done via an approximate Livsic Theorem (**Goüezel-L '19, Guedes Bonthonneau-L '19**):

$$\|\Pi_2 f\|_{H^{s+1}} \leq C\|I_2 f\|^{\theta}_{\ell^\infty} \|f\|^{1-\theta}_{H^{s+1871}}$$

Approximate Livsic theorem

- Recall that

$$\Pi_2 := \pi_{2*} \underbrace{\left(R_+^{\text{hol}}(0) - R_-^{\text{hol}}(0) + \mathbf{1} \otimes \mathbf{1}\right)}_{=\Pi} \pi_2^*$$

- By construction Π does not see coboundaries namely $\Pi(Xu) = 0$ for all $u \in H^s(SM), s > 0$.

Theorem (Goüezel-L. '19)

There exists an orthogonal decomposition of functions

$$C^1(SM) \ni f = Xu + h, \qquad \|h\|_{H^s} \leq C \|If\|_{\ell^\infty}^{1-\theta} \|f\|_{C^1}^{1-\theta}$$

- Apply this to $\pi_2^* f = Xu + h$:

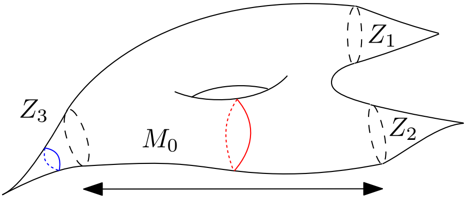
$$\begin{aligned} \|f\|_{H^{s-1}} \leq \|\Pi_2 f\|_{H^s} &= \|\pi_{2*}\Pi(\pi_2^* f)\|_{H^s} \\ &= \|\pi_{2*}\Pi(\cancel{Xu} + h)\|_{H^s} \\ &\leq \|\pi_{2*}\Pi h\|_{H^s} \leq \|h\|_{H^s} \leq C \|I_2 f\|_{\ell^\infty}^{1-\theta} \|f\|_{C^1}^{1-\theta} \end{aligned}$$

Manifolds with hyperbolic cusps

- (M, g_0) is a cusp manifold i.e. a smooth non-compact Riemannian manifold with negative curvature s.t. $M = M_0 \cup_\ell Z_\ell$. The ends Z_ℓ are real hyperbolic cusps i.e. $Z_\ell \simeq [a, +\infty)_y \times (\mathbb{R}^d/\Lambda)_\theta$, where Λ is a unimodular lattice and

$$g|_{Z_\ell} \simeq \frac{dy^2 + d\theta^2}{y^2}$$

- $\mathcal{C}$ = set of hyperbolic free homotopy classes (in opposition to the parabolic ones wrapping exclusively around the cusps).



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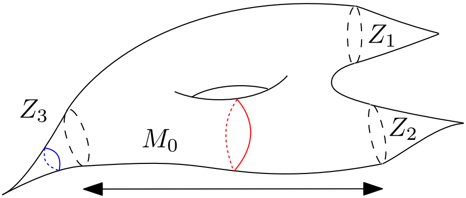
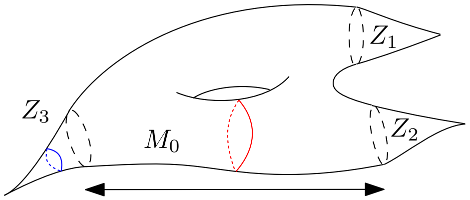


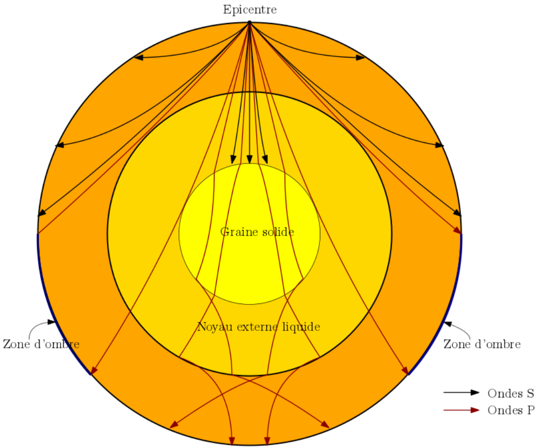
Manifolds with hyperbolic cusps

Theorem (Guedes Bonthonneau-L. '19)

Let (M, g_0) be a cusp manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ and a codimension 1 submanifold $\mathcal{N}$ of the space of isometry classes such that: if $\mathcal{O}(g) \in \mathcal{N}$, $\|g - g_0\|_{y^{-k}C^k} < \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.



Manifolds with boundary



- **Herglotz 1905, Wiechert-Zoeppritz 1907**

- A simple manifold (M, g_0) is a manifold with strictly convex boundary, no conjugate points and no trapped set (the exponential map is a diffeomorphism at each point). In particular, between each pair of points on the boundary $(x, y) \in \partial M \times \partial M$, there exists a unique geodesic $\gamma_{x,y}$.
- The boundary distance function is the map

$$d_g : \partial M \times \partial M \to \mathbb{R}_+, (x, y) \mapsto \ell_{g_0}(\gamma_{x,y}).$$

- The map $g \mapsto d_g$ is invariant by the action of the group of diffeomorphisms $\phi : M \to M$ such that $\phi|_{\partial M} = \mathrm{id}$.

Conjecture (Michel '81)

The boundary distance function determines the metric i.e. if g and g_0 are simple and $d_g = d_{g_0}$, there exists a diffeomorphism $\phi : M \to M$ such that $\phi|_{\partial M} = \mathrm{id}$ and $\phi^* g = g_0$.

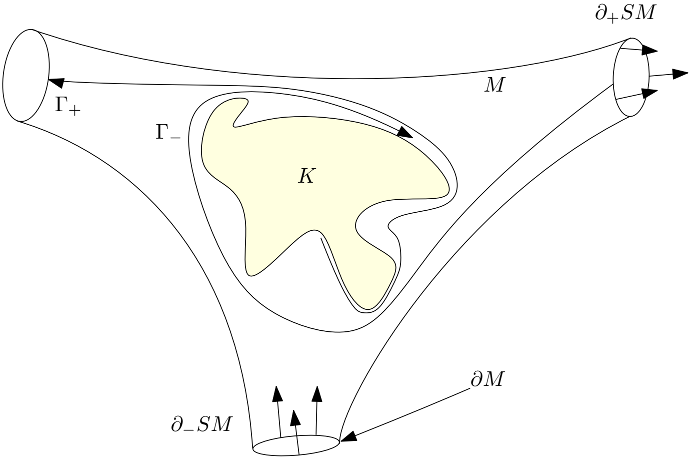
Manifolds with boundary

Known results:

- **Otal '89**: proof for surfaces of negative curvature.
- **Croke-Dairbekov-Sharafutdinov '00, Stefanov-Uhlmann '04**: local rigidity results.
- **Pestov-Uhlmann '05**: proof for arbitrary simple surfaces.
- **Burago-Ivanov '10**: metrics close to the euclidean one.
- **Stefanov-Uhlmann-Vasy-17**: proof for manifolds admitting a foliation by strictly convex hypersurfaces.

Manifolds with boundary

- We assume that (M, g_0) has strictly convex boundary, no conjugate points and a hyperbolic trapped set.



Manifolds with boundary

- The marked length spectrum is replaced by a similar quantity : the marked boundary distance function d_g. This map assigns to each pair of points $(x, y) \in \partial M \times \partial M$ and each free homotopy class $[\gamma]$ of curves with endpoints x and y, the length of the unique geodesic joining x to y. (**Guillarmou '17, Guillarmou-Mazzucchelli '18**)

Theorem (L. '19)

Let (M, g_0) be such a manifold and further assume that it has negative curvature if $\dim(M) \geq 3$. Then, there exists $\varepsilon > 0, k \in \mathbb{N}^*$ such that: if $\|g - g_0\|_{C^k} < \varepsilon$ and $d_g = d_{g'}$, then $\exists \phi : M \to M$ such that $\phi|_{\partial M} = \mathrm{id}$ and $\phi^* g = g_0$.

Asymptotically hyperbolic surfaces

- An AH surface (M, g_0) is a conformally compact Riemannian manifold such that near $\partial\overline{M}$, there exists a boundary defining function $y : M \to \mathbb{R}_+$ s.t.

$$g_0 = \frac{dy^2 + h(y,x)dx^2}{y^2}$$

- **Example**: any deformation with compact support of the hyperbolic plane $\mathbb{H}^2$, hyperbolic surface with three funnels (the infinite pair of pants), ...
- A notion of renormalized marked boundary distance D_g between pair of points on the boundary at infinity can be defined (**Graham-Guillarmou-Stefanov-Uhlmann '17**).

Theorem (L' 19)

If g and g_0 are AH and $D_g = D_{g_0}$, then g is isometric to g_0 by a diffeomorphism fixing the boundary $\partial\overline{M}$.

Perspectives

On this topic:

- The global conjecture of Burns-Katok (who knows ...).
- Investigate the generalized Thurston's distance d_T in variable curvature. Maybe something can be done on surfaces using the theory of laminations. Also, investigate the geometry of $\mathrm{Met}/\mathrm{Diff}_0$ endowed with the pressure metric (generalized Weil-Petersson metric).
- Prove a local rigidity result for the unmarked length spectrum. This is linked to a conjecture due to Sarnak on the finiteness of isospectral isometry classes.
- Investigate the strictly convex foliation assumption of Stefanov-Uhlmann-Vasy: can simple manifolds be foliated? This would solve Michel's conjecture.

Broader questions:

- Spectral/microlocal study of non-uniformly hyperbolic/parabolic flows: description of the spectral measure on the real line, study of the resolvent, mixing properties for the flow, ...

Thank you for your attention!

References (I)

- *Local rigidity of manifolds with hyperbolic cusps II. Nonlinear theory*, with Yannick Guedes Bonthonneau, preprint
- *Geodesic stretch and marked length spectrum rigidity* (https://arxiv.org/abs/1909.08666), with Colin Guillarmou and Gerhard Knieper, preprint
- *Local rigidity of manifolds with hyperbolic cusps I. Linear theory and pseudodifferential calculus* (https://arxiv.org/abs/1907.01809), with Yannick Guedes Bonthonneau, preprint
- *Classical and microlocal analysis of the X-ray transform on Anosov manifolds* (https://arxiv.org/abs/1904.12290), with Sébastien Gouëzel, to appear in **Analysis and PDE**

References (II)

- *The marked length spectrum of Anosov manifolds* (https://arxiv.org/abs/1806.04218), with Colin Guillarmou, **Annals of Mathematics** (2), 190(1):321–344, 2019
- *Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces* (https://arxiv.org/abs/1805.05155), to appear in **Commentarii Mathematici Helvetici**
- *Local marked boundary rigidity under hyperbolic trapping assumptions* (https://arxiv.org/abs/1804.02143), to appear in **Journal of Geometric Analysis**
- *On the s-injectivity of the X-ray transform for manifolds with hyperbolic trapped set* (https://arxiv.org/abs/1807.03680), **Nonlinearity**, vol. 32, no. 4 (2019), 1275–1295