Local rigidity of manifolds with hyperbolic cusps

Thibault Lefeuvre

Joint work with Yannick Guedes Bonthonneau

Institut de Mathématique d'Orsay

November 13th 2019

- (M, g_0) smooth closed (compact, $\partial M = \emptyset$) Riemannian manifold with negative sectional curvature.
- $\mathcal{C}$ = set of free homotopy classes $\overset{\text{1-to-1}}{\leftrightarrow}$ closed g_0-geodesics (i.e. $\forall c \in \mathcal{C}, \exists! \gamma_{g_0}(c) \in c$)

Definition (The marked length spectrum)

$$L_{g_0} : \left| \begin{array}{l} \mathcal{C} \to \mathbb{R}_+^* \\ c \mapsto \ell_{g_0}(\gamma_c), \end{array} \right.$$

$\ell_{g_0}(\gamma_c)$ Riemannian length computed with respect to g_0.

- (M, g_0) smooth closed (compact, $\partial M = \emptyset$) Riemannian manifold with negative sectional curvature.
- $\mathcal{C}$ = set of free homotopy classes $\overset{\text{1-to-1}}{\leftrightarrow}$ closed g_0-geodesics (i.e. $\forall c \in \mathcal{C}, \exists! \gamma_{g_0}(c) \in c$)

Definition (The marked length spectrum)

$$L_{g_0} : \left| \begin{array}{l} \mathcal{C} \to \mathbb{R}_+^* \\ c \mapsto \ell_{g_0}(\gamma_c), \end{array} \right.$$

$\ell_{g_0}(\gamma_c)$ Riemannian length computed with respect to g_0.

Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g_0 have negative sectional curvature, same marked length spectrum $L_g = L_{g_0}$, then $\exists \ \phi : M \to M$ smooth diffeomorphism isotopic to the identity such that $\phi^* g = g_0$.

- Analogue of Michel's conjecture of rigidity for simple manifolds with boundary (the boundary distance function should determine the metric up to isometries),
- Why the marked length spectrum ? The length spectrum (:= collection of lengths regardless of the homotopy) does not determine the metric (counterexamples by **Vigneras '80**)
- Conjecture can be generalized to Anosov manifolds i.e. manifolds on which the geodesic flow is uniformly hyperbolic.

Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g_0 have negative sectional curvature, same marked length spectrum $L_g = L_{g_0}$, then $\exists\ \phi : M \to M$ smooth diffeomorphism isotopic to the identity such that $\phi^* g = g_0$.

- Analogue of Michel's conjecture of rigidity for simple manifolds with boundary (the boundary distance function should determine the metric up to isometries),
- Why the marked length spectrum ? The length spectrum (:= collection of lengths regardless of the homotopy) does not determine the metric (counterexamples by **Vigneras '80**)
- Conjecture can be generalized to Anosov manifolds i.e. manifolds on which the geodesic flow is uniformly hyperbolic.

Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g_0 have negative sectional curvature, same marked length spectrum $L_g = L_{g_0}$, then $\exists \ \phi : M \to M$ smooth diffeomorphism isotopic to the identity such that $\phi^* g = g_0$.

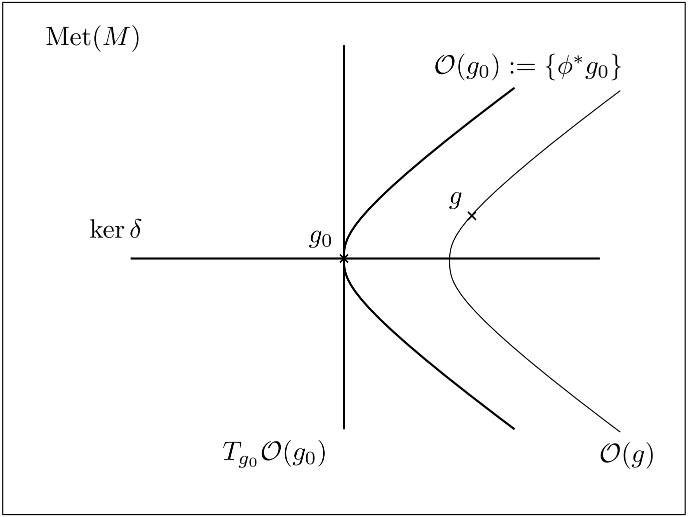
- Analogue of Michel's conjecture of rigidity for simple manifolds with boundary (the boundary distance function should determine the metric up to isometries),
- Why the marked length spectrum ? The length spectrum (:= collection of lengths regardless of the homotopy) does not determine the metric (counterexamples by **Vigneras '80**)
- Conjecture can be generalized to Anosov manifolds i.e. manifolds on which the geodesic flow is uniformly hyperbolic.

Known results:

- **Guillemin-Kazhdan '80, Croke-Sharafutdinov '98**: proof of the infinitesimal version of the problem (for a deformation $(g_s)_{s\in(-1,1)}$ of the metric g_0),
- **Croke '90**, **Otal '90**: proof for negatively-curved surfaces,
- **Katok '88**: proof for g conformal to g_0,
- **Besson-Courtois-Gallot '95**, **Hamenstädt '99**: proof when (M, g_0) is a locally symmetric space.

Theorem (Guillarmou-L. '18, Guillarmou-Knieper-L. '19)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ such that: if $\|g - g_0\|_{C^k} < \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.


$\mathrm{Met}(M)$
$\mathcal{O}(g_0) := \{\phi^* g_0\}$
g
$\ker \delta$
g_0
$T_{g_0}\mathcal{O}(g_0)$
$\mathcal{O}(g)$

- (M, g_0) is a cusp manifold i.e. a smooth non-compact Riemannian manifold with negative curvature s.t. $M = M_0 \cup_\ell Z_\ell$. The ends Z_ℓ are real hyperbolic cusps i.e. $Z_\ell \simeq [a, +\infty)_y \times (\mathbb{R}^d/\Lambda)_\theta$, where Λ is a unimodular lattice and

$$g|_{Z_\ell} \simeq \frac{dy^2 + d\theta^2}{y^2}$$

- $\mathcal{C}$ = set of hyperbolic free homotopy classes (in opposition to the parabolic ones wrapping exclusively around the cusps).

- (M, g_0) is a cusp manifold i.e. a smooth non-compact Riemannian manifold with negative curvature s.t. $M = M_0 \cup_\ell Z_\ell$. The ends Z_ℓ are real hyperbolic cusps i.e. $Z_\ell \simeq [a, +\infty)_y \times (\mathbb{R}^d/\Lambda)_\theta$, where Λ is a unimodular lattice and

$$g|_{Z_\ell} \simeq \frac{dy^2 + d\theta^2}{y^2}$$

- $\mathcal{C}$ = set of hyperbolic free homotopy classes (in opposition to the parabolic ones wrapping exclusively around the cusps).

Theorem (Guedes Bonthonneau-L. '19)

Let (M, g_0) be a cusp manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ and a codimension 1 submanifold $\mathcal{N}$ of the space of isometry classes such that: if $\mathcal{O}(g) \in \mathcal{N}$, $\|g - g_0\|_{y^{-k}C^k} < \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.

- Known results: proof for surfaces by **Cao '95**.

Idea of proof of

Theorem (Guillarmou-L. '18, Guillarmou-Knieper-L. '19)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ such that: if $\|g - g_0\|_{C^k} < \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.

1. Solenoidal reduction: there exists a diffeomorphism ϕ such that: $\delta(\phi^* g) = 0$. So, WLOG, we can assume $g - g_0 \in \ker \delta$.
2. Taylor expansion of the ratio of the length spectra:

$$\mathcal{L}(g) := L_g / L_{g_0} = \mathbf{1} + d\mathcal{L}_{g_0}(g - g_0) + \mathcal{O}(\|g - g_0\|_{C^3}^2)$$

3. If $L_g = L_{g_0}$, then $\|d\mathcal{L}_{g_0}(g - g_0)\|_{\ell^\infty} \leq C\|g - g_0\|_{C^3}^2$. Thus, if we have a stability estimate for $d\mathcal{L}_{g_0}$ on $\ker \delta$ like

$$\|f\|_{C^3} \leq C\|d\mathcal{L}_{g_0}(f)\|_{\ell^\infty},$$

we are done.

Idea of proof of

Theorem (Guillarmou-L. '18, Guillarmou-Knieper-L. '19)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \varepsilon > 0$ such that: if $\|g - g_0\|_{C^k} < \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.

1. Solenoidal reduction: there exists a diffeomorphism ϕ such that: $\delta(\phi^* g) = 0$. So, WLOG, we can assume $g - g_0 \in \ker \delta$.
2. Taylor expansion of the ratio of the length spectra:

$$\mathcal{L}(g) := L_g / L_{g_0} = \mathbf{1} + d\mathcal{L}_{g_0}(g - g_0) + \mathcal{O}(\|g - g_0\|^2_{C^3})$$

3. If $L_g = L_{g_0}$, then $\|d\mathcal{L}_{g_0}(g - g_0)\|_{\ell^\infty} \leq C\|g - g_0\|^2_{C^3}$. Thus, if we have a stability estimate for $d\mathcal{L}_{g_0}$ on $\ker \delta$ like

$$\|f\|_{C^2} \leq C\|d\mathcal{L}_{g_0}(f)\|^{\theta}_{\ell^\infty} \|f\|^{1-\theta}_{C^{1789}},$$

we are done (using some interpolation estimates).

Definition (Geodesic X-ray transform)

$$I_2^{g_0} : C^0(M, \otimes_S^2 T^*M) \to \ell^\infty(\mathcal{C}),$$

$$I_2^{g_0} f : \mathcal{C} \ni c \mapsto \frac{1}{\ell(\gamma_{g_0}(c))} \int_0^{\ell(\gamma_{g_0}(c))} f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) \mathrm{d}t,$$

with $\gamma_{g_0}(c)$ unique closed geodesic in c.

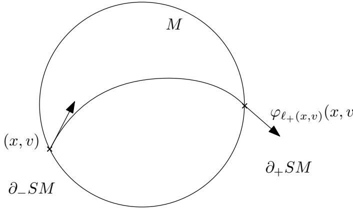
- $d\mathcal{L}_{g_0} = 1/2 \times I_2^{g_0}$,
- In negative curvature, $\ker I_2^{g_0} = T_{g_0}\mathcal{O}(g_0)$ (**Croke-Sharafutdinov '98**). In other words, $I_2^{g_0}$ is injective on $\ker \delta$.

Question: Stability estimates for the X-ray transform $I_2^{g_0}$?

Theorem (Guillarmou-L. '18, Goüzel-L. '19)

Let $0 < \alpha < \beta$. Then, $\exists C, \theta > 0$ such that:

$$\forall f \in C^\beta \cap \ker \delta, \qquad \|f\|_{C^\beta} \leq C \|I_2^{g_0} f\|_{\ell^\infty}^\theta \|f\|_{C^\alpha}^{1-\theta}$$



- On a simple manifold, for $(x, v) \in \partial_- SM$,

$$\begin{aligned} I_2 f(x,v) &= \int_0^{\ell_+(x,v)} f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\mathrm{d}t \\ &= \int_0^{+\infty} \pi_2^* f_{\mathrm{ext}}(\varphi_t(x,v))\mathrm{d}t \\ &= (I \circ \pi_2^*) f_{\mathrm{ext}}(x,v) \end{aligned}$$

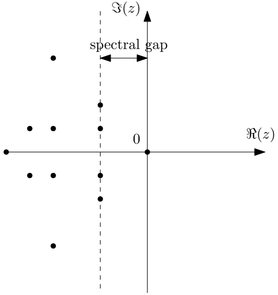
  - The normal operator $\Pi_2 := I_2^* I_2 : C^\infty(M, \otimes_S^2 T^*M) \circlearrowleft$ is
    - a ΨDO of order -1,
    - formally selfadjoint and nonnegative,
    - elliptic on $\ker \delta$.
  - One can write $\Pi_2 := {\pi_2}_* I^* I {\pi_2}^*$, with $I^* I = \int_{-\infty}^{+\infty} e^{tX} \mathrm{d}t$. If $R_\pm(\lambda) := (X \pm \lambda)^{-1}$ denotes the resolvent of the generator of the geodesic flow, then $I^* I = R_+(0) - R_-(0)$. Thus:

$$\Pi_2 = I_2^* I_2 = {\pi_2}_* (R_+(0) - R_-(0)) {\pi_2}^*$$

Meromorphic extension of the resolvent $(X \pm \lambda)^{-1}$

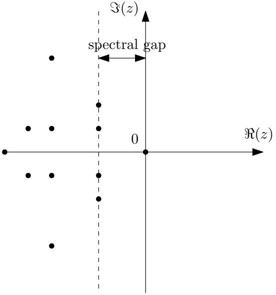
- Idea (**Guillarmou '17**): In the closed case, mimick the case of a simple manifold,
- $R_\pm(\lambda) := (X \pm \lambda)^{-1}$, initially defined on $\Re(\lambda) > 0$, admit a meromorphic extension to $\mathbb{C}$ when acting on anisotropic Sobolev spaces with poles of finite ranks: the Pollicott-Ruelle resonances (**Liverani '04, Butterley-Liverani '07, Faure-Sjöstrand '11, Dyatlov-Zworski '13, Faure-Tsuji '13 '17**),
  - 0 is a pole of order 1 and $\mathrm{Res}_0((X \pm \lambda)^{-1}) = \mathbf{1} \otimes \mathbf{1}$,
  - Define

$$\Pi_2 := \pi_{2*}(R_+^{\text{hol}}(0) - R_-^{\text{hol}}(0))\pi_2^* + \pi_{2*}\mathbf{1} \otimes \mathbf{1}\pi_2^*$$



Meromorphic extension of the resolvent $(X \pm \lambda)^{-1}$

- Idea (**Guillarmou '17**): In the closed case, mimick the case of a simple manifold,
- $R_\pm(\lambda) := (X \pm \lambda)^{-1}$, initially defined on $\Re(\lambda) > 0$, admit a meromorphic extension to $\mathbb{C}$ when acting on anisotropic Sobolev spaces with poles of finite ranks: the Pollicott-Ruelle resonances (**Liverani '04, Butterley-Liverani '07, Faure-Sjöstrand '11, Dyatlov-Zworski '13, Faure-Tsuji '13 '17**),



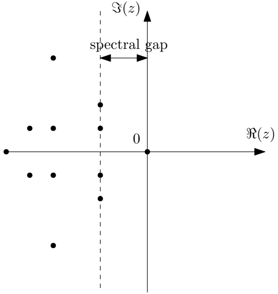
- 0 is a pole of order 1 and $\mathrm{Res}_0((X \pm \lambda)^{-1}) = \mathbf{1} \otimes \mathbf{1}$,
- Define

$$\Pi_2 := \pi_{2*}(R_+^{\mathrm{hol}}(0) - R_-^{\mathrm{hol}}(0))\pi_2^* + \pi_{2*}\mathbf{1} \otimes \mathbf{1}\pi_2^*$$

Meromorphic extension of the resolvent $(X \pm \lambda)^{-1}$

- Idea (**Guillarmou '17**): In the closed case, mimick the case of a simple manifold,
- $R_\pm(\lambda) := (X \pm \lambda)^{-1}$, initially defined on $\Re(\lambda) > 0$, admit a meromorphic extension to $\mathbb{C}$ when acting on anisotropic Sobolev spaces with poles of finite ranks: the Pollicott-Ruelle resonances (**Liverani '04, Butterley-Liverani '07, Faure-Sjöstrand '11, Dyatlov-Zworski '13, Faure-Tsuji '13 '17**),
  - 0 is a pole of order 1 and $\mathrm{Res}_0((X \pm \lambda)^{-1}) = \mathbf{1} \otimes \mathbf{1}$,
- Define

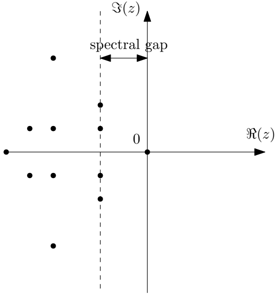
$$\Pi_2 := \pi_{2*}(R_+^{\mathrm{hol}}(0) - R_-^{\mathrm{hol}}(0))\pi_2^* + \pi_{2*}\mathbf{1} \otimes \mathbf{1}\pi_2^*$$



Meromorphic extension of the resolvent $(X \pm \lambda)^{-1}$

- Idea (**Guillarmou '17**): In the closed case, mimick the case of a simple manifold,
- $R_{\pm}(\lambda) := (X \pm \lambda)^{-1}$, initially defined on $\Re(\lambda) > 0$, admit a meromorphic extension to $\mathbb{C}$ when acting on anisotropic Sobolev spaces with poles of finite ranks: the Pollicott-Ruelle resonances (**Liverani '04, Butterley-Liverani '07, Faure-Sjöstrand '11, Dyatlov-Zworski '13, Faure-Tsuji '13 '17**),
  - 0 is a pole of order 1 and $\mathrm{Res}_0((X \pm \lambda)^{-1}) = \mathbf{1} \otimes \mathbf{1}$,
  - Define

$$\Pi_2 := \pi_{2*}(R_+^{\mathrm{hol}}(0) - R_-^{\mathrm{hol}}(0))\pi_2^* + \pi_{2*}\mathbf{1} \otimes \mathbf{1}\pi_2^*$$



Properties of Π_2

- A more "explicit" expression of Π_2: if $f_1, f_2 \in C^\infty(M, \otimes_S^2 T^*M)$ have 0-average, then

$$\langle \Pi_2 f_1, f_2 \rangle_{L^2} = \int_{-\infty}^{+\infty} \langle e^{tX} \pi_2^* f_1, \pi_2^* f_2 \rangle_{L^2(SM, d\mu_{\mathbf{Liouville}})} \mathrm{d}t$$

Think of Π_2 as "$\pi_{2*} \circ \int_{\mathbb{R}} e^{tX} \mathrm{d}t \circ \pi_2^*$".

Theorem (**Guillarmou '17**, **Guillarmou-L. '18**, **Gouëzel-L. '19**)

- Π_2 *is a* ΨDO *of order* -1, *elliptic on tensors in* $\ker \delta$,
- *One has:* $\ker \Pi_2 = \ker I_2 = T_{g_0}\mathcal{O}(g_0)$,
- *This implies the elliptic estimate:*

$$\|f\|_{H^s} \leq C \|\Pi_2 f\|_{H^{s+1}}, \quad \forall f \in \ker \delta$$

Question: link between Π_2 and I_2? We are looking for an estimate like:

$$\|\Pi_2 f\|_{H^{s+1}} \leq C \|I_2 f\|_{\ell^\infty}^{\theta} \|f\|_{H^{s+1776}}^{1-\theta}$$

Approximate Livsic theorem

- Recall that

$$\Pi_2 := \pi_{2*} \underbrace{\left(R_+^{\text{hol}}(0) - R_-^{\text{hol}}(0) + \mathbf{1} \otimes \mathbf{1}\right)}_{=\Pi} \pi_2^*$$

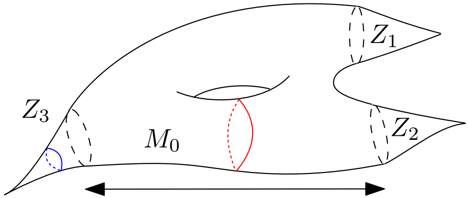
- By construction Π does not see coboundaries namely $\Pi(Xu) = 0$ for all $u \in H^s(SM), s > 0$. There exists an orthogonal decomposition of functions (**Gouëzel-L. '19**)

$$H^s(SM) \ni f = Xu + h, \qquad \|h\|_{H^s} \leq C \|If\|_{\ell^\infty}^{1-\theta} \|f\|_{C^1}^{1-\theta}$$

- Apply this to $\pi_2^* f = Xu + h$:

$$\begin{aligned} \|\Pi_2 f\|_{H^s} &= \|\pi_{2*}\Pi(\pi_2^* f)\|_{H^s} \\ &= \|\pi_{2*}\Pi(\cancel{Xu} + h)\|_{H^s} \\ &\leq \|\pi_{2*}\Pi h\|_{H^s} \\ &\leq \|h\|_{H^s} \leq C \|I_2 f\|_{\ell^\infty}^{1-\theta} \|f\|_{C^1}^{1-\theta} \end{aligned}$$

1. The marked length spectrum
   - Setting of the problem in the closed case
   - The case of manifolds with hyperbolic cusps
2. Ingredients of proof in the closed case
   - Taylor expansion of the marked length spectrum
   - The normal operator
3. What is new in the case of cusp manifolds?
   - Key ingredients
   - A geometric calculus


Z_1
Z_3
M_0
Z_2

Key ingredients of the previous proof

1. Meromorphic extension of $(X \pm \lambda)^{-1}$ to a strip $\{\Re(\lambda) > -1/1515\}$ to define Π_2,
2. Stability estimate $\|f\|_{H^s} \leq C\|\Pi_2 f\|_{H^{s+1}}$ for $f \in \ker \delta$,
3. Approximate Livsic Theorem.

Key ingredients of the previous proof

1. Meromorphic extension of $(X \pm \lambda)^{-1}$ to a strip $\{\Re(\lambda) > -1/1515\}$ to define Π_2,
2. Stability estimate $\|f\|_{H^s} \leq C\|\Pi_2 f\|_{H^{s+1}}$ for $f \in \ker \delta$,
3. Approximate Livsic Theorem.

Key ingredients of the previous proof

1. Meromorphic extension of $(X \pm \lambda)^{-1}$ to a strip $\{\Re(\lambda) > -1/1515\}$ to define Π_2,
2. Stability estimate $\|f\|_{H^s} \leq C\|\Pi_2 f\|_{H^{s+1}}$ for $f \in \ker \delta$,
3. Approximate Livsic Theorem.

Key ingredients of the previous proof

1. Meromorphic extension of $(X \pm \lambda)^{-1}$ to a strip $\{\Re(\lambda) > -1/1515\}$ to define Π_2,
2. Stability estimate $\|f\|_{H^s} \leq C\|\Pi_2 f\|_{H^{s+1}}$ for $f \in \ker \delta$,
3. Approximate Livsic Theorem.

Key ingredients of the previous proof

- Meromorphic extension of $(X \pm \lambda)^{-1}$ acting on anisotropic Sobolev spaces proved on a small strip $\{\Re(\lambda) > -\delta\}$ by **Guedes Bonthonneau-Weich '17**. This is done by combining the **Dyatlov-Zworski '13** approach (radial points estimates) with ideas inspired by **Melrose**'s b-calculus.
- The estimate $\|f\|_{H^s} \leq C\|\Pi_2 f\|_{H^{s+1}}$ for $f \in \ker \delta$ is based on a parametrix construction for Π_2 with a compact remainder.

Question: How to produce compact remainders?

A geometric calculus

- **Guedes Bonthonneau '16** introduced on cusps a "geometric" calculus $\cup_{m\in\mathbb{R}}\Psi^m$ in which Π_2 will fit. It is an extension of the algebra of differential operators generated by the orthonormal vectors

$$y\partial_y, \qquad y\partial_\theta$$

- An elliptic ΨDO $P \in \Psi^m$ can be inverted in this calculus: $QP = \mathbf{1} + R$, with R smoothing i.e. $R : H^{-s} \to H^s$ bounded for all $s \in \mathbb{R}$. But R is not compact! because the inclusion $H^{s_1} \hookrightarrow H^{s_2}$ for $s_1 > s_2$ is no longer compact. However $y^{\rho-\epsilon}H^{s_1} \hookrightarrow y^\rho H^{s_2}$ is compact ($\epsilon > 0$).

Two important remarks:

- The lack of compactness in Kato-Rellich comes from θ-independant functions in the cusp. In other words, for $s_1 > s_2$, $H^{s_1}_\perp \hookrightarrow H^{s_2}_\perp$ is compact, where the $\perp$-subscript denotes functions f such that

$$\forall y, \in [a, +\infty), \qquad \int_{(\mathbb{R}/\Lambda)^d} f(y,\theta) d\theta = 0$$

- The elliptic operators P we are interested in (like Π_2) are geometric and thus "commute" with ∂_θ in the sense that $[P, \partial_\theta] = \mathrm{compact}$. In other words, they act diagonally on Fourier modes in the θ-variable (modulo compact junk).

Conclusion: In order to invert a geometric elliptic ΨDO P modulo compact remainder, one needs to invert it exactly on θ-independent functions i.e. construct Q', R' with R' smoothing such that $Q'P = \mathbf{1} + R'$ where given $f \in H^s$, the θ-independent component of $R'f$ is ≈ 0 (i.e. fast decay at infinity).

Conclusion: In order to invert a geometric elliptic ΨDO P modulo compact remainder, one needs to invert it exactly on θ-independent functions i.e. construct Q', R' with R' smoothing such that $Q'P = \mathbf{1} + R'$ where given $f \in H^s$, the θ-independent component of $R'f$ is ≈ 0 (i.e. fast decay at infinity).

- Given such a geometric operator P, it also commutes with the generator of the dilation $y\partial_y$ on θ-independent functions i.e. $[P, y\partial_y] = \mathrm{compact}$ on such functions. In the $r = \log y$ variable, $[P, \partial_r] = \mathrm{compact}$.
- Thus, modulo compact junk, on θ-independent functions and sufficiently high in the cusp, P looks like a Fourier multiplier. In other words, for $\xi \in \mathbb{R}$, $P(e^{i\xi r}) \approx I_P(i\xi)e^{i\xi r}$, with $I_P(i\xi) \in \mathbb{C}$. More generally, for $\lambda = \rho + i\xi \in \mathbb{C}$,

$$P(e^{\lambda r}) = P(e^{\rho r} e^{i\xi r}) \approx I_P(\lambda)e^{\lambda r}$$

  Here $\rho \in \mathbb{R}$ is a weight and corresponds to looking at the operator P on the spaces $y^{d/2} y^{\rho} H^s$.

$$P(e^{\lambda r}) = P(e^{\rho r} e^{i\xi r}) \approx I_P(\lambda) e^{\lambda r}$$

- We call $\mathbb{C} \ni \lambda \mapsto I_P(\lambda) \in \mathbb{C}$ the indicial operator associated to P, it is a holomorphic function of λ. Like in b-calculus, the inversion of P modulo compact remainder on the spaces $y^{d/2} y^\rho H^s$ requires

$$I_P(\rho + i\xi) \neq 0, \quad \forall \xi \in \mathbb{R}$$

.

- If P acts on a vector bundle $E \to Z$, $I(\lambda)$ is matrix-valued. This is the case for $P = \Pi_2$.
- P may also act on a product manifold $F \times Z$, in which case $I(\lambda)$ takes values in (pseudo)differential operators acting on $C^\infty(F)$. This is the case for the geodesic vector field X acting on SM, unit tangent bundle of a cusp surface. In the (y, θ, ϕ) coordinates,

$$X = \cos\phi y \partial_y + \sin\phi y \partial_\theta + \sin\phi \partial_\phi$$

Thus: $I_X(\lambda) = \lambda \cos\phi + \sin\phi \partial_\phi \in \mathrm{Diff}^1(\mathbb{S}^1)$.

Back to Π_2!

- $\mathbb{C} \ni \lambda \mapsto I_{\Pi_2}(\lambda)$ is a matrix-valued holomorphic function of λ.
  Question: What are its indicial roots i.e. for which values is it invertible? We need to look its action on symmetric 2-tensors of the form
  $$f = a\frac{dy^2}{y^2} + b^i \frac{dy \otimes d\theta_i + d\theta_i \otimes dy}{2y^2} + c^{ij}\frac{d\theta^2_{ij}}{y^2}$$
  i.e. compute $I_{\Pi_2}(\lambda)f = y^{-\lambda}\Pi_2(y^\lambda f)$.
- However, we are only interested in Π_2 acting on $\ker \delta$. This implies the linear relations $b_i = 0, a(\lambda - 1) + \mathrm{Tr}(c) = 0$ for f. Moreover, it is sufficient to compute $\langle I_{\Pi_2}(\lambda)f, f\rangle$ and show that this is $\neq 0$ when $f \neq 0$. This implies a "lower bound" on the indicial roots of Π_2.

- We obtain:

$$\langle I_{\Pi_2}(\lambda) f, f\rangle = \frac{\mathrm{vol}(\mathbb{S}^{d-1})\pi}{(\lambda+1)(d+1-\lambda)} \frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{d-\lambda}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{d+1-\lambda}{2}\right)}$$

$$\times\left[|a|^2\left(1+\frac{|d-\lambda|^2}{d}+\frac{\lambda(d-\lambda)}{d}+|d-\lambda|^2\frac{\lambda(d-\lambda)}{d(d+2)}\right)\right.$$

$$\left.+2\,\mathrm{Tr}\,|c|^2\frac{\lambda(d-\lambda)}{d(d+2)}\right]$$

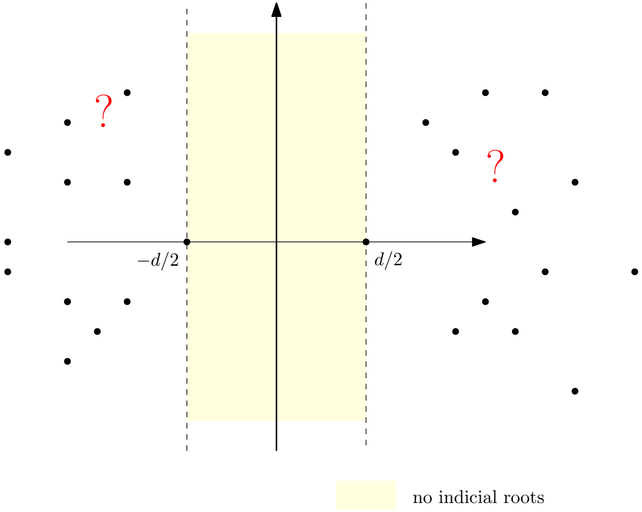


Figure: Lower bound on the indicial roots of $I_{\Pi_2}(\lambda + d/2)$.

Theorem (Guedes Bonthonneau-L. '19)

For $s > 0$ small enough, there exists $C, \theta > 0$ such that:

$$\forall f \in C^1 \cap \ker \delta, \qquad \|f\|_{H^{-1-s}} \leq C \|I_2 f\|_{\ell^\infty}^{\theta} \|f\|_{C^1}^{1-\theta}$$

Thank you for your attention!