The marked length spectrum of negatively-curved manifolds

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The marked length spectrum

- (M, g) smooth closed manifold with negative sectional curvature,
- $\mathcal{C}$ set of free homotopy classes: each class $c \in \mathcal{C}$ is represented by a unique closed geodesic $\gamma \in c$,

Definition (The marked length spectrum)

$$L_g : \left| \begin{array}{rl} \mathcal{C} & \to \mathbb{R}_+^* \\ c & \mapsto \ell_g(\gamma), \end{array} \right.$$

with $\gamma \in c$ unique closed geodesic and $\ell_g(\gamma)$ Riemannian length computed with respect to g.

- Definition still holds under the weaker assumption that the geodesic flow is Anosov (hyperbolic dynamics on the unit tangent bundle).

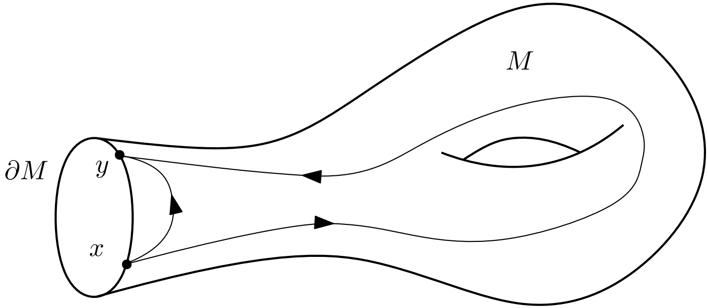
Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g' have negative sectional curvature, same marked length spectrum $L_g = L_{g'}$, then there exists $\phi : M \to M$ smooth diffeomorphism such that $\phi^* g' = g$.

- The action of diffeomorphisms is a natural obstruction one cannot avoid,
- Why the marked length spectrum ? The length spectrum (:= collection of lengths regardless of the homotopy) does not determine the metric (counterexamples by **Vigneras '80**)
- Conjecture still makes sense for Anosov geodesic flows.

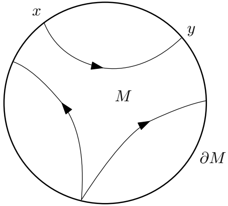
An analogy: manifolds with boundary

- On a negatively-curved manifold with strictly convex boundary, the equivalent notion is the marked boundary distance function (:= the Riemannian distance restricted to the boundary computed with respect to homotopy classes).



An analogy: manifolds with boundary

- If *M* is simply connected, this notion boils down to the boundary distance function : such manifolds are prototypes of simple manifolds (:= strictly convex boundary + no conjugate points + non-trapping)



Conjecture (Michel '81)

Simple manifolds are boundary distance rigid i.e. the boundary distance function determines the metric (up to isometries).

- Selected global positive results to Michel's conjecture: **Pestov-Uhlmann '03** in dimension 2, **Stefanov-Uhlmann-Vasy '17** in dimension > 2 if existence of a strictly convex foliation (guaranteed if (M, g) non-positively curved)

Positive results

Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g' have negative sectional curvature, same marked length spectrum $L_g = L_{g'}$, then there exists $\phi : M \to M$ smooth diffeomorphism such that $\phi^* g' = g$.

- **Croke '90, Otal '90**: proof of the Conjecture in dimension 2,
- **Katok '88**: proof if g' conformal to g,
- **Besson-Courtois-Gallot '95, Hamenstädt '99**: proof if g is a locally symmetric space.
- Conjecture remains open in full generality in dimension > 2 + Anosov case in any dimension ?

Our contribution

Theorem (Guillarmou-L. '18)

Let (M, g) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \mathcal{U}$ open C^k-neighborhood of g such that: if $g' \in \mathcal{U}$ and $L_{g'} = L_g$, then g' is isometric to g.

- It is a local version of Burns-Katok's Conjecture,
- The proof gives stability estimates quantifying how close are isometry classes of g and g' in terms of $L_g/L_{g'}$
- Theorem also holds if: i) $\dim(M) = 2$ and g has Anosov geodesic flow, ii) $\dim(M) > 2$, g has Anosov geodesic flow and is non-positively curved.

Our contribution

Corollary (Guillarmou-L. '18)

For all $a \in \mathbb{R}$ and for all bounded set $\mathcal{B} \subset \mathcal{C}^k$, $\exists$ at most finitely many isometry classes with same marked length spectrum, sectional curvature $K_g \leq -a^2 < 0$ and curvature tensor bounded in $\mathcal{B}$.

- The Corollary follows from the Theorem by using compactness results of **Hamilton '95**,
- First general results in dimension > 2.

Idea of proof

Our proof uses tools from different areas:

- Geometric inverse problems (X-ray transform on closed manifolds)
- Hyperbolic dynamics (classical chaos on compact sets)
- Microlocal analysis (recent progress made in the analytic study of flows: meromorphic extension of the resolvent $(-X \pm \lambda)^{-1}$ with X geodesic vector field, anisotropic Sobolev spaces, Pollicott-Ruelle resonances ...)

- Infinitesimal (or "linear") version of the problem: if $(g_s)_{s\in(-1,1)}$ is a smooth deformation of the metric g such that $L_{g_s} = L_g$, is there a smooth isotopy $(\phi_s)_{s\in(-1,1)}$ such that $\phi_s^* g_s = g$?

- By differentiating L_{g_s}/L_g, this amounts to studying the s-injectivity of the X-ray transform over symmetric 2-tensors i.e. proving that

$$I_2^g : \left| \begin{array}{l} \mathcal{C}^0(M, S^2T^*M) \to \ell^\infty(\mathcal{C}) \\ f \mapsto \left(c \mapsto \dfrac{1}{\ell_g(\gamma)} \int_0^{\ell_g(\gamma)} f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) dt \right) \end{array} \right.$$

($\gamma \in c$ unique closed geodesic) has kernel reduced to

$$\ker I_2^g = \left\{ \mathcal{L}_V g, V \in \mathcal{C}^1(M, TM) \right\}$$

(These elements are called potential tensors.)

Positive results for the linear problem

- **Guillemin-Kazhdan '80**: pioneer work, s-injectivity of I_2^g for (M, g) negatively-curved surface,
- **Croke-Sharafutdinov '98**: s-inj. of I_2^g for (M, g) negatively-curved manifold,
- **Paternain-Salo-Uhlmann '14**: s-inj. of I_2^g for (M, g) surface with Anosov geodesic flow.

One may try to pass from the linear to the local problem by using the inverse function theorem. Is the differential a bounded isomorphism ?
Problem : surjectivity of

$$\hat{I}_2^g : C^0(M, S^2 T^* M)/\ker\,(I_2^g) \to \ell^\infty(\mathcal{C})?$$

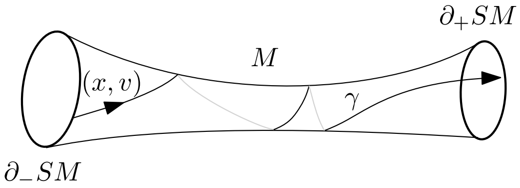
Need another approach: let us go back one moment to the case of manifolds with boundary ...

An analogy: manifolds with boundary

- In the study of marked boundary rigidity, it is convenient to look at the normal operator $N_2 := {I_2^g}^* I_2^g$, where

$$I_2^g(f) \underbrace{(x,v)}_{\in \partial_- SM} = \int_0^{l(x,v)} f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt,$$

It is a pseudodifferential operator of order -1.



The operator Π_2

- Idea (**Guillarmou '15**): In the case of a closed manifold with Anosov geodesic flow, there is a natural selfadjoint operator $\Pi : H^s(SM) \to H^{-s}(SM)$ on the unit tangent bundle SM obtained by weak limit of damped correlation (see also **Faure-Sjöstrand '11**):

$$\langle \Pi f, \psi \rangle_{L^2(SM)} = \lim_{\lambda \to 0^+} \int_{\mathbb{R}} e^{-\lambda |t|} \langle f \circ \varphi_t, \psi \rangle_{L^2(SM)}, \quad f, \psi \in \mathcal{C}^\infty(SM)$$

(φ_t geodesic flow). Moreover, if X is the geodesic vector field, $X \circ \Pi \equiv 0, \Pi \circ X \equiv 0$.

- Morally, one has to think that "$\Pi f(x,v) = \int_{-\infty}^{+\infty} f_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt$"
- By microlocal techniques, **Dyatlov-Zworski '16** computed the wavefront set of the Schwarz kernel of Π. If $\pi_2 : \underbrace{(x,v)}_{\in SM} \mapsto \underbrace{(x, \otimes^2 v)}_{\in \otimes^2 T^* M}$, this allows to prove that $\Pi_2 := {\pi_2}_* \Pi \pi_2^*$ is a ΨDO of order -1 on symmetric 2-tensors, just like N_2. It is elliptic on solenoidal tensors, which are the L^2-orthogonal of potential tensors ($= \ker I_2^g$).

The operator Π_2

- Problem: there is (a priori) no explicit link between Π_2 and I_2^g (contrary to the case with boundary where the relation between N_2 and I_2^g is explicit). Idea: the relation is obtained by a positive version of Livcic theorem due to **Lopes-Thieullen '03** for Anosov flows on closed manifolds.
- Then: s-injectivity of I_2^g + ellipticity of Π_2 provides stability estimate for f, solenoidal tensor:

$$\|f\|_{H^{-s-1}(M)} \lesssim \|\Pi \pi_2^* f\|_{H^{-s}(M)}$$

- Combined with geometrical estimates (some due to **Croke-Dairbekov-Sharafutdinov '03**), the proof then boils down to a rather simple sequence of inequalities, similar to the analogous problem of marked boundary rigidity on manifolds with boundary (L. **'18**).

Perspectives

- It seems hopeless to obtain a global result with this technique: morally, we control the differential I_2^g of the marked length spectrum with tame estimates (loss of derivatives) like in the Nash-Moser Theorem $\implies$ local result.
- Lower the assumptions in the corollary on the finiteness number of isometry classes with same marked length spectrum ?

Thank you for your attention !