FROM THE LOCAL TO THE GLOBAL INJECTIVITY OF THE X-RAY TRANSFORM

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Abstract. In this short note, we explain how the local injectivity of the X-ray transform proved by Uhlmann-Vasy [11] for manifolds with strictly convex boundary implies a global injectivity result, when the manifold admits a strictly convex function.

1. Local injectivity

Throughout this paper, we shall work in the smooth category, that is all the manifolds and coordinate charts are considered to be smooth.

Let (M, g) be a smooth compact n-dimensional manifold with strictly convex boundary. We recall that this means that given $p \in \partial M$, the second fundamental form

$$\Pi_p : \left| \begin{array}{l} T_p\partial M \times T_p\partial M \to \mathbb{R} \\ (v, w) \mapsto g(\nabla_v \nu, w) \end{array} \right. , \tag{1.1}$$

is strictly convex, where ν is the unit outward normal vector to the boundary at p, ∇ the Levi-Civita connection.

We denote by $\mathcal{G}$ the subset of geodesics on M which are not trapped (they enter and exit M in finite time). Given $f \in \mathcal{C}^\infty(M)$, we define its *X-ray transform* by:

$$If : \left| \begin{array}{l} \mathcal{G} \to \mathbb{R} \\ \gamma \mapsto If(\gamma) = \int_\gamma f \, d\gamma \end{array} \right.$$

We will denote by $\mathcal{U}_p$ a small neighborhood of p and by $\mathcal{G}_p$ the set of short geodesics contained in $\mathcal{U}_p$. This is an open subset of $\mathcal{G}$. The *local X-ray transform* is the restriction of If to $\mathcal{G}_p$.

Theorem 1.1 (Uhlmann-Vasy [11]). *For any $p \in \partial M$, there exists a small neighborhood $\mathcal{U}_p$ (and the associated set of geodesics $\mathcal{G}_p$) such that for all $s \geq 0$, there exists $C > 0$, such that:*

$$\forall f \in H^s_F(\mathcal{U}_p), \ ||f||_{H^{s-1}_F(\mathcal{U}_p)} \leq C||If|_{\mathcal{G}_p}||_{H^s(\mathcal{G}_p)}$$

Note that locally, there is a natural identification by a diffeomorphism of $\mathcal{G}_p$ and $\partial\mathcal{U}_p \times \partial\mathcal{U}_p \setminus \text{diag}$ and we will consider the Sobolev norm on $\mathcal{G}_p$ induced by this identification. Here, H^s_F is an exponentially-weighted Sobolev space. The open set $\mathcal{U}_p$ is actually expressed as the intersection $\{\rho \leq 0\} \cap \{\widetilde{x} \geq -c\}$, where $\widetilde{x}$ is a function constructed from ρ, which defines an artificial boundary which is less convex than the boundary of the manifold. The constant

C which appears is uniform in the point p chosen (and the "size" of $\mathcal{U}_p$, or more precisely its depth, is also uniform in p), which matters if one wants a global stability estimate or a reconstruction procedure (that is an inversion "formula" to recover the function from its X-ray transform). A natural question is to understand to what extent this result of local injectivity of the X-ray transform can imply a global result of injectivity.

The method of Uhlmann-Vasy actually provides a local reconstruction procedure, namely there exists an operator which we call I_p^{-1} such that $f|_{\mathcal{U}_p} = I_p^{-1}(If|_{\mathcal{G}_p})$. The expression of this operator is given in terms of a Neumann series.

2. From a local to a global result

2.1. A first condition. This first condition was formulated in the original paper. Assume there exists a boundary defining function $\rho : M \to \mathbb{R}_-$ such that:

(1) $\rho = 0, d\rho \neq 0$ on ∂M,
(2) $\rho \leq 0$ on M,
(3) $\Sigma_t := \rho^{-1}(\{t\})$ are strictly convex (when viewed from $\rho^{-1}((-\infty, t)))$ and $d\rho \neq 0$ on Σ_t (in other words $M_t := \rho^{-1}((-\infty, t])$ is a compact manifold with strictly convex boundary and $\rho_t := \rho - t$ is a boundary defining function),
(4) $M \setminus \cup_{t\in(T,0]}\Sigma_t = \rho^{-1}((-\infty, T])$ has either zero measure (i) or empty interior (ii) for some $T > 0$.

Corollary 2.1. *Assume the previous conditions hold. Then if (i) holds, the global X-ray transform is injective on $L^2(M)$. If (ii) holds, it is injective on $H^s(M)$ for any $s > n/2$.*

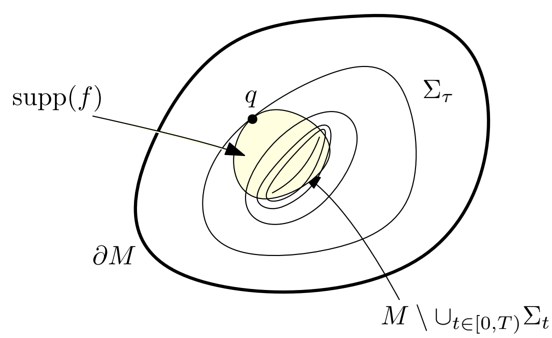

Figure 1. The foliation by strictly convex hypersurfaces

Proof. Let us prove (ii) for instance. Assume $f \in H^s(M)$ (with $s > n/2$) satisfies $If = 0$. By Sobolev injection, we know that f is continuous. If $f \neq 0, \text{supp}(f)$ has non-empty interior. We define $\tau := \sup_{x\in\text{supp}(f)} \rho(x)$. If $\tau \leq T$, then the conclusion is immediate since $M \setminus \cup_{t\in(T,0]}\Sigma_t$ has empty interior by assumption. We can assume that $\tau > T$. Thus, there exists a $q \in \text{supp}(f) \cap \partial M_\tau$. It is now sufficient to apply the theorem to the new manifold M_τ in a vicinity of q: we obtain that $f \equiv 0$ in a vicinity of q which is a contradiction. □

2.2. **The algorithm of reconstruction.** We assume that the previous condition holds and we explain the algorithm of reconstruction of Uhlmann-Vasy. It actually works in the more general context which will be described in the next paragraphs but the procedure is easier in this case. Under this assumption of foliation by strictly convex hypersurfaces given by ρ, one can actually take $\widetilde{x} = \rho$ that is, the local theorem actually holds for an annulus $\rho^{-1}((c,0])$ for some small $c<0$ (it is valid on a vicinity of the whole boundary and not only a single point). More generally, given $t > T$, there exists a $c_t < 0$ such that the theorem provides a stability estimate and a reconstruction procedure on $\rho^{-1}((c_t,t])$. Let us fix some $T' > T$. We can cover the interval $[T', 0]$ by a finite number $N \in \mathbb{N}$ of intervals $(c_k, t_k]$ (where $c_k := c_{t_k}$) such that $t_0 = 0$ and $c_N < T'$ and $t_{k+1} \in (c_k, t_k]$.

Assume that we are given If. By the local theorem, we know that we can recover f on $(c_0, t_0] = (c_0, 0]$. Then we take ϕ_0 such that $\phi_0 \equiv 1$ on $(-\infty, c_0]$ and is supported in $(-\infty, t_1)$. We write $f = (1-\phi_0)f + \phi_0 f$. By assumption, we know f in $(c_0, 0]$ which means that we actually know $(1-\phi_0)f$. Thus, we know If and $I(1-\phi_0)f$ so we can deduce $I\phi_0 f$. Now, we can apply the theorem to the interval $(c_1, t_1]$ and we recover $\phi_0 f$ and thus f on $(c_1, 0]$. By induction, one can recover step by step the function f on the whole interval $[T', 0]$ (there is a finite number of step).

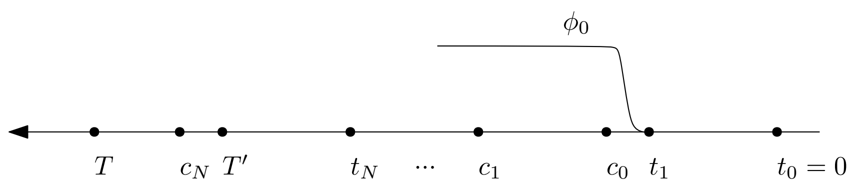

FIGURE 2. The procedure of reconstruction step by step

2.3. Examples.

2.3.1. *The foliation of the Earth.* At the beginning of the 20th century, Herglotz [4] and Wieckert and Zoeppritz [12] suggested a model of propagation of the sound in the Earth's crust (which is actually related to the propagation of seismic waves — they are called p-waves and s-waves) with a spherical symmetry and an isotropic property. In other words, they were considering the Earth as a round ball $B(0,R)$, endowed with a metric conformal to the euclidean metric (isotropy), where the conformal factor was only depending on the radius, that is on the distance from the Earth's kernel (spherical symmetry), i.e. $g = c^{-2}(r)g_{\text{euc}}$. They added the important condition that c had to satisfy

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0, \tag{2.1}$$

for $0 < r \le R$ (where the sphere $\mathbb{S}_R = \{|x| = R\}$ is considered to be the Earth's surface) which models the fact that the velocity of the sound increases as it gets deeper in the mantle. But actually, the condition (2.1) is equivalent to saying that the euclidean spheres $\mathbb{S}_r = \{|x| = r\}$, for $0 < r \leq R$, are strictly convex with respect to the metric g. The proof boils down to a computation which can be found in [9, Proposition 6.1]. Thus, in this case the function "radius" $x \mapsto r(x)$ satisfies the previous condition of Uhlmann-Vasy.

2.3.2. *Tubular neighborhood of a closed geodesic in negative curvature.* We assume that γ is a closed geodesic in (M, g) and that the sectional curvatures are negative in a neighborhood of γ. Then one can take a tubular neighborhood of γ (a cylinder) and the function $\rho(x) = d(x, \gamma)^2$ will satisfy the previous foliation condition (the arguments given in the following paragraphs will actually imply this result). But in this case, one can note that there are even trapped geodesics (geodesics which will wrap indefinitely around γ, i.e. for which γ is their ω-limit set.)

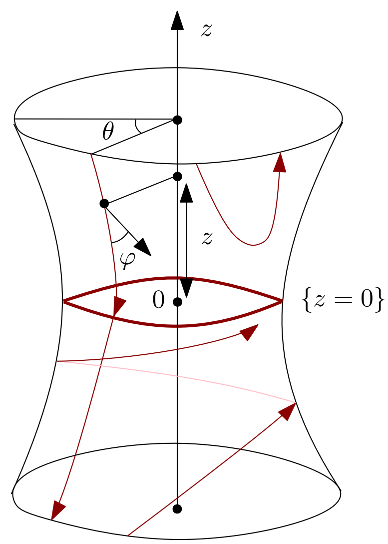

FIGURE 3. In red: the geodesics.

The two-dimensional case of a surface of revolution is probably easier to picture although it does not fit in the situation we are studying (because the argument of Uhlmann-Vasy only works in dimension ≥ 3 !). One takes the surface given in cylindrical coordinates by $(z, \psi(z), \theta)$ for z in a neighborhood of 0, where $\psi(z)$ is even and strictly convex (the curvature is given by $\kappa(z) = \frac{-\psi''(z)}{\psi(z)(\psi'^2(z)+1)^2}$). The geodesic in the middle (in bold) is trapped and if one takes a point and a direction such that $\psi(z)\ \sin\varphi = 1$, then by Clairaut's relation, one can prove that this geodesic will be trapped in the future.

2.4. **Strictly convex functions.** The following paragraphs are based on [5, Section 2].

2.4.1. *Statement.* The previous condition is not very satisfying insofar as it is not very clear whether a manifold can support such a function ρ. There is a natural generalization of this:

Definition 2.1. We say that $r : M \to \mathbb{R}$ is strictly convex at x if $\mathrm{Hess}_x(r) : T_xM \times T_xM \to \mathbb{R}$ is strictly positive (as a bilinear form).

Remark 2.1. We recall that the Hessian is defined by $\mathrm{Hess}_x(r)(v, w) := g(\nabla_v \nabla r(x), w)$. There is another equivalent definition. Consider the local geodesic γ generated by $\gamma(0) = x, \dot\gamma(0) = v$. Then:

$$\frac{d^2}{dt^2} r(\gamma(t))\bigg|_{t=0} = \mathrm{Hess}_x r(v, v)$$

Lemma 2.1. *Assume there exists a smooth function $r : M \to \mathbb{R}_-$ which is strictly convex. Then:*

(1) There exists at most one critical point in M, in which case it is also the minimum of r,
(2) The level sets $\Sigma_t := r^{-1}(t)$ are strictly convex from below (seen from $r^{-1}((-\infty, t])$)

In particular, the same exact proof as in the corollary works and I is injective on $L^2(M)$.

Remark 2.2. The previous situation does not *exactly* fit into this lemma. We would actually need to consider $r : U \to \mathbb{R}_-$ strictly convex with $U \subset M$ an open connected subset such that $M \setminus U$ has zero measure or empty interior. For the sake of simplicity, we avoid this technical complication. Also remark that the function r is not exactly the previous ρ. Indeed, the strict convexity of the sets Σ_t implies that $\mathrm{Hess}(\rho)(v, v) > 0$, for all $v \in T\Sigma_t \setminus \{0\}$ but it does not say anything on $\mathrm{Hess}(\rho)(n, n)$, where n is the unit inward normal vector to Σ_t. One has to look for r in the form $r = h(\rho)$ with a suitable function h (we will somehow use a similar argument in the following paragraph, so we do not detail it here and leave it for the reader as an exercise).

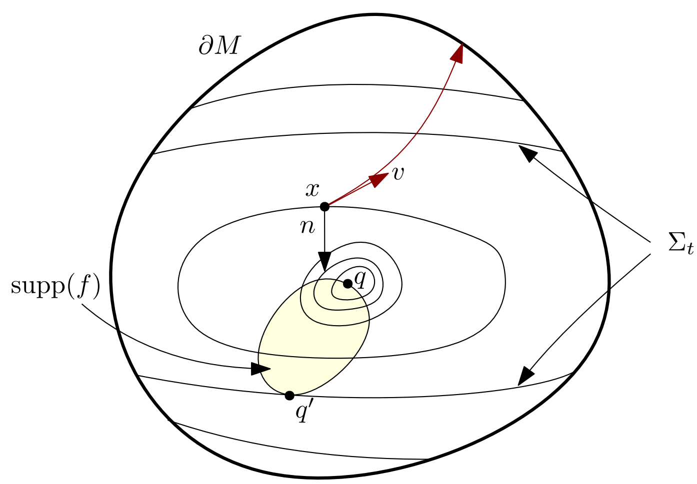

FIGURE 4. The foliation by level sets of r (here, the curves in black). In red, a geodesic.

2.4.2. *Proof.* Since M has strictly convex boundary, this implies in particular that between any pair of points $x_0, x_1 \in M$, there exists a unit-speed geodesic $\gamma : [0, T] \to M$ such that $\gamma(0) = x_0, \gamma(T) = x_1$ and $\gamma((0, T)) \subset \mathring{M}$. (In particular, if you add the condition that there are no pair of conjugate points on M, there is a unique geodesic in each homotopy class of curves joining x_0 to x_1.)

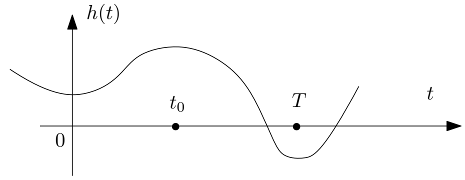

Figure 5. The function h

(1) Assume $r : M \to \mathbb{R}_-$ is strictly convex. If $x_0 \in M$ is a critical point, then by strict convexity, it is a local minimum. Assume there exists $x_0 \neq x_1 \in M$ which are two critical points. We choose $\gamma : [0, T] \to M$ a unit-speed geodesic joining x_0 to x_1. By convexity, $\gamma(t) \in \mathring{M}$ for all $t \in]0, T[$. We set $h(t) = r(\gamma(t))$ and thus $h''(t) = \text{Hess}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) > 0$. Now, there exists a $t_0 \in]0, T[$ such that $h(t_0) = \sup_{[0,T]} h$ but then $h'(t_0) = 0, h''(t_0) > 0$ which is a contradiction (see Figure ??). The same kind of argument also shows that the critical point, if it exists, has to be the global minimum.

(2) By convex, we mean that any geodesic which satisfies $\gamma(0) = x \in \Sigma_t := r^{-1}(\{t\})$ and $dr(\dot\gamma(0)) \geq 0$ stays in $r^{-1}([t, +\infty))$ in the future and reaches ∂M in finite time $T < \infty$. For the first part, it is immediate since for $h(t) = r(\gamma(t))$, one has:

$$h(t) = h(0) + \underbrace{h'(0)t}_{=dr(\dot\gamma(0)) \geq 0} + \frac{1}{2}\underbrace{h''(c(t))}_{>0} t^2 \geq h(0)$$

It remains to prove that $T < \infty$. If not, then by the same equality, using the strict convexity of r, $h(t) \geq h(0) + ct^2$ (for some positive constant c) and thus $h(t) \to +\infty$ which contradicts the fact that r is bounded.

Assume $f \in L^2(M)$ satisfies $If \equiv 0$. First, by the local theorem, we know that there exists a whole neighborhood of the boundary where f actually vanishes. Assume $f \neq 0$, then $\text{supp}(f)$ is a compact set in M and we can still define $\tau := \sup_{x \in \text{supp}(f)} r(x)$. Note that f has to be contained on one side (the lower side) of the level set Σ_τ, by construction. Take $q \in \Sigma_\tau \cap \text{supp}(f)$. If $dr \neq 0$, then the Theorem 1.1 applies and we get a contradiction, i.e. $q \notin \text{supp}(f)$. If not, it means that $\text{supp}(f)$ is actually reduced to a single point which attains the global minimum of r and this is not possible either.

3. Manifolds which admit a strictly convex function

3.1. **Statements.** We now prove that in some particular cases manifolds can support strictly convex functions. We denote by K the sectional curvature which is given for X and Y orthonormal by $K(X, Y) = g(\mathcal{R}(X, Y)X, Y)$, $\mathcal{R}$ being the curvature tensor which we define by the (unusual) convention (see [2] for instance):

$$\mathcal{R}(X, Y)Z := -\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)$$

Note that for a general free family, one has:

$$K(X, Y) = \frac{g(\mathcal{R}(X, Y)X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

We recall that a normal Jacobi vector field J along γ satisfies the Jacobi equation:

$$\nabla_{tt} J + \mathcal{R}(\dot\gamma, J)\dot\gamma = 0 \tag{3.1}$$

If we have $\gamma(0) = p$ and $\gamma(T) = \exp_p(v)$, then for $J(0) = 0, \nabla_t J(0) = w$, then $d(\exp_p)_v(Tw) = J(T)$. We also recall the

Definition 3.1. (M, g) has no focal points if for any geodesic $\gamma : [0, T] \to M$ and any non-trivial Jacobi vector field along γ with $J(0) = 0$, one has $\frac{d}{dt}|J(t)|^2 > 0$, for all $t > 0$. In particular, when there are no focal points, there are no conjugate points.

Proposition 3.1. *There exists a smooth strictly convex function $r : M \to \mathbb{R}$ if:*

(1) M is simply connected and $K \leq 0$,
(2) M is simply connected with no focal points,
(3) $K \geq 0$.

3.2. **Proof.** (2) $\implies$ (1) : A non-positively curved manifold has no focal points (and thus no conjugate points). Indeed, if J is a non-trivial normal Jacobi vector field along γ with $J(0) = 0$, then:

$$\frac{1}{2}\frac{d}{dt}\,|J(t)|^2\Big|_{t=0} = g(J(0), \nabla_t J(0)) = 0$$

And:

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\,|J(t)|^2 &= \frac{d}{dt} g(J(t), \nabla_t J(t)) \\
&= |\nabla_t J(t)|^2 + g(\nabla_{tt} J(t), J(t)) \\
&= |\nabla_t J|^2 - g(\mathcal{R}(\dot\gamma, J)\dot\gamma, J) && \text{since } \nabla_{tt} J + \mathcal{R}(\dot\gamma, J)\dot\gamma = 0 \\
&= |\nabla_t J|^2 - \underbrace{K(\dot\gamma, J)}_{\leq 0}(\underbrace{|\dot\gamma|^2}_{=1}\,|J|^2 - \underbrace{g(\dot\gamma, J)^2)}_{=0}) \\
&\geq |\nabla_t J|^2 \geq 0
\end{aligned}$$

In particular, one has : $\frac{1}{2}\frac{d^2}{dt^2}\,|J(t)|^2\Big|_{t=0} \geq |\nabla_t J(0)|^2 > 0$ (otherwise we would have $J \equiv 0$), which implies that $\frac{d}{dt}|J(t)|^2 > 0$, for all $t > 0$.

(2) We consider any $p \in M$ and define $r(x) = \frac{1}{2} d(x, p)^2$. M has strictly convex boundary and no conjugate points (since it has no focal points) so $\exp_p : \overline{\Omega} \subset T_pM \to M$ ($\overline{\Omega}$ is star-shaped, thus simply connected, and compact) is a surjective covering map and since M is simply connected, it is a diffeomorphism. In the usual terminology, we say that M is simple. We define the (non-smooth, it degenerates at $x = p$) function $f : x \mapsto d(x, p)$. We compute the Hessian with Jacobi vector fields. Let $x_0 \in M$ ($x_0 \neq p$) and $\gamma : [0, T] \to M$ be the unique geodesic joining p to x_0. First, $\nabla f(x_0) = \dot\gamma(T)$. Indeed, ∇f is orthogonal

to the hyperspheres $d(x,p) = \text{cst} = d(x_0,p)$ and so is $\dot\gamma(T) = d(\exp_p)_u(\frac{u}{|u|})$ by Gauss lemma ($g(u,w) = g(d(\exp_p)_u(u), d(\exp_p)_u(w))$). Thus $\nabla f(x_0)$ and $\dot\gamma(T)$ are colinear and $|\nabla f| = 1 = |\dot\gamma(T)|$ which implies that $\nabla f(x_0) = \dot\gamma(T)$. Now, for $v \in S_{x_0}M, v \perp \nabla f(x_0)$, we can find a Jacobi vector field J such that $J(0) = 0, J(T) = v$ (because $\exp_p$ is a diffeomorphism). We consider a variation of geodesic

$$\gamma : \left| \begin{array}{l} [0,T] \ \times (-\varepsilon, \varepsilon) \to M \\ (t,s) \mapsto \gamma(t,s) \end{array} \right. ,$$

such that $\gamma(0,s) = p, \gamma(T,0) = x_0$, $\partial_s \gamma(T,s) \mid_{s=0} = J(T) = v$.

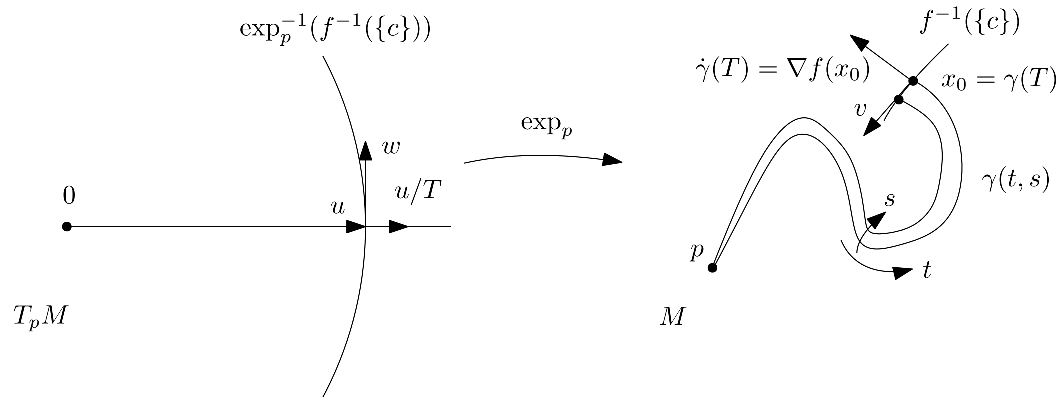

FIGURE 6. A picture of the situation. Note that, as usual, we identify $T_u(T_pM) \simeq T_pM$.

Then J is a normal Jacobi vector field and:

$$\begin{aligned}
\text{Hess}_{x_0} f(v,v) &= g(\nabla_v \nabla f(x_0), v) \\
&= g(\nabla_s \nabla_t \gamma(t,s)|_{t=T,s=0}, J(T)) \\
&= g(\nabla_t \underbrace{\nabla_s \gamma(t,s)|_{s=0,t=T}}_{=J(t)}, J(T)) \\
&= g(\nabla_t J(T), J(T)) \\
&= \frac{1}{2} \frac{d}{dt} |J(t)|^2 \Big|_{t=T} > 0,
\end{aligned}$$

since there are no focal points. Thus, for $x_0 \neq p$, $v \perp \nabla f$ and $v \neq 0$, we have $\text{Hess}_{x_0}(f)(v,v) > 0$. But remark that for all $Z \in T_{x_0}M$,

$$\text{Hess}(f)(Z, \nabla f) = g(\nabla_Z \nabla f, \nabla f) = \frac{1}{2} \ \mathcal{L}_Z \underbrace{|\nabla f|^2}_{=1} = 0$$

We need to consider $r := h(f)$ with $h(t) = \frac{1}{2}t^2$, that is $r : x \mapsto \frac{1}{2}d(x,p)^2$. This is a smooth function. Then:

$$\mathrm{Hess}(r) = \underbrace{h'(f)}_{=f}\mathrm{Hess}(f) + \underbrace{h''(f)}_{=2} df \otimes df$$

This implies by the previous arguments that $\mathrm{Hess}_x(r) > 0$ for all $x \neq p$. The last step is to prove that $\mathrm{Hess}_p(r) = g > 0$ to conclude. Let γ be a unit-speed geodesic with $\gamma(0) = p, \dot{\gamma}(0) = v$. Then $\lim_{t\to 0} d(\gamma(t), p)/t = |v|$ (this can be seen in coordinates for instance) and thus $d(p, \gamma(t)) = t|v| + \mathcal{O}(t^2)$ and $r(\gamma(t)) = \frac{1}{2}t^2|v|^2 + \mathcal{O}(t^3)$ which implies the sought result.

(3) The function we are going to consider will be a modification of $f : x \mapsto d(x, \partial M)$. This function is smooth in a vicinity of the boundary. Actually, for any $x \in M$, there exists a $z \in \partial M$ such that $x = \exp_z(tn)$, where n is the unit inward-pointing normal vector to the boundary at z and $t = d(x, \partial M)$. We set:

$$\tau_{\partial M}(z) := \sup\left\{t, \gamma_z|_{[0,t]} : s \mapsto \exp_z(sn) \text{ is the unique shortest geodesic from } z \text{ to } \exp_z(tn)\right\}$$

And $\omega_{\partial M} := \{\exp_z(\tau_{\partial M}(z)n), z \in \partial M\}$, which is the boundary cut locus. It has zero measure and f is smooth on the open set $M \setminus \omega_{\partial M}$. Moreover, f is a distance function, that is $|\nabla f| = 1$ and $\nabla f(\exp_z(tn)) = \dfrac{d}{dt}\exp_z(tn) = d(\exp_z)_{tn}(n)$. Let us now estimate the Hessian of f. On the boundary, for $z \in \partial M, v \in T_z\partial M$:

$$\mathrm{Hess}_z(f)(v,v) = g(\nabla_v \underbrace{\nabla f}_{=n}, v) = -\Pi(v,v) \leq -\lambda|v|^2,$$

for some constant $\lambda > 0$, since the boundary is strictly convex (which amounts to saying that the second fundamental form Π is strictly positive and thus admits a uniform lower bound on the compact ∂M). If we take for $z \in \partial M$ an orthonormal basis $\{E_1, ..., E_{n-1}\}$ of $T_z\partial M$ (with $E_n = n$) and parallel-transport it along $\gamma_z|_{[0,\tau_{\partial M}(z))}$ and consider the symmetric matrix $A(t) := (\mathrm{Hess}(r)_{\gamma(t)}(E_i(t), E_j(t))_{1\leq i,j\leq n-1}$, then using the Jacobi equation, one can prove that:

$$A'(t) + A(t)^2 + R(t) = 0, \tag{3.2}$$

with $R(t)_{ij} = g(\mathcal{R}(E_i, \dot{\gamma})E_j, \dot{\gamma})$. Indeed, if we set $S := \mathrm{Hess}(r)$, which we see as a $(1,1)$-tensor (that is $S(X) = \nabla_X \nabla r$), then it satisfies the curvature equation (see [6, Theorem 2, Section 4.2]):

$$\nabla_{\nabla_r} S + S^2 + R_{\nabla_r} = 0,$$

which is exactly (3.2) restricted to the geodesic[1]. We know that $A(0) \leq -\lambda \mathrm{Id}$ and $R(t) \geq 0 \geq -\lambda^2 \mathrm{Id}$. There exists a comparison principle for Riccati matrix equations like (3.2). We denote by $B(t)$ the solution to the equation

$$B'(t) + B^2(t) + \widetilde{R}(t) = 0,$$

with $B(0) = -\lambda \mathrm{Id}$ and $\widetilde{R}(t) = -\lambda^2 \mathrm{Id} \leq R(t)$. Then, one has $A(t) \leq B(t)$ (see [1]). But here, it is immediate that $B(t) = -\lambda \mathrm{Id}$, that is $A(t) \leq -\lambda \mathrm{Id}$ which gives for $x \in M \setminus \omega_{\partial M}, v \in S_x M, v \perp \nabla r$: $\mathrm{Hess}_x(f)(v,v) \leq -\lambda |v|^2$. Since f is a distance function, it cannot be strictly convex because $\mathrm{Hess}(f)(Z, \nabla f) = \dfrac{1}{2}\mathcal{L}_Z \underbrace{|\nabla f|^2}_{=1} = 0$. We introduce $R = \sup_{x \in M} d(x, \partial M)$ and $r = h(f)$ with $h(t) = -t + \frac{1}{4R}t^2$. Then:

$$\mathrm{Hess}(r) = h'(f)\mathrm{Hess}(f) + h''(f) df \otimes df = (-1 + \frac{f}{2R})\mathrm{Hess}(f) + \frac{1}{2R} df \otimes df$$

Since $f \leq R$, we obtain by the previous arguments the existence of a constant $c > 0$ such that for all $x \in M \setminus \omega_{\partial M}, v \in S_x M$, $\mathrm{Hess}_x(r)(v,v) \geq c|v|^2$.

Now, r is continuous on M, strictly convex (with constant $c > 0$) on $M \setminus \omega_{\partial M}$ and for $x \in \omega_{\partial M}$ it is not hard to construct a smooth $\overline{r} \leq r$ in a neighborhood of x such that $\overline{r}(x) = r(x)$ and $\mathrm{Hess}_x(\overline{r})(v,v) \geq \varepsilon |v|^2$, where $\varepsilon < c$ can be chosen arbitrarily close to c. In the usual terminology, this means that the continuous function r is c-convex on M. Then there exists a smoothing procedure (see [3, Theorem 2], it is rather technical and we do not detail it here) which yields a smooth strictly convex function in M. Moreover, this new smooth function can be chosen arbitrarily close to r in the $\mathcal{C}^0$ topology.

4. To finish

The existence of a strictly convex function on a Riemannian manifold with boundary is actually a very difficult question. Using some of the techniques introduced in the paper [11], Stefanov-Uhlmann-Vasy [10] solved the *boundary distance problem* on manifolds which admit such a function, namely: if $d_g = d_{g'}$ on $\partial M \times \partial M$ (the distance functions coincide on the boundary), then there exists a diffeomorphism $\phi : M \to M$ which fixes the boundary such that $\phi^* g' = g$. A famous conjecture of Michel [8] states that *simple*

[1]If X is a vector field, this equality follows from:

$$\begin{aligned}
(\nabla_{\nabla_r} S)(X) + S^2(X) &= \nabla_{\nabla_r}(S(X)) - S(\nabla_{\nabla_r} X) + \nabla_{\nabla_X \nabla r} \nabla r \\
&= \nabla_{\nabla_r} \nabla_X \nabla r - \nabla_{\nabla_{\nabla_r} X} \nabla r + \nabla_{\nabla_X \nabla r} \nabla r \\
&= \nabla_{\nabla_r} \nabla_X \nabla r - \nabla_{(\nabla_{\nabla_r} X - \nabla_X \nabla r)} \nabla r \\
&= \nabla_{\nabla_r} \nabla_X \nabla r - \nabla_{[\nabla r, X]} && \text{(since the connection is torsion-free)} \\
&= -R(\nabla r, X)\nabla r + \underbrace{\nabla_X \nabla_{\nabla r} \nabla r}_{=0} && \text{(by the definition of the curvature)}
\end{aligned}$$

manifolds (compact manifolds with boundary such that $\exp_p$ is a diffeomorphism at each point) are boundary rigid. It was proved by Pestov-Uhlmann [7] in dimension 2 but the problem remains open in dimension ≥ 3. As a consequence, if one was able to prove that simple manifolds admit a strictly convex function, the conjecture of Michel would be solved.

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