A SURPRISING TOPOLOGY

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Abstract. In this paragraph, we present a surprising topological property of the measure space.

0.1. **The space $\mathcal{M}(X)$.** Let X be a topological space. We denote by $\mathcal{M}(X)$ the set of Borel probability measure on X. In this paragraph, we introduce the topology on $\mathcal{M}(X)$.

Definition 0.1 (Weak-* topology)**.** We say that a sequence $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}(X)^{\mathbb{N}}$ converges to μ for the weak-* topology if and only if $\mu_n(\phi) \to_{n\to\infty} \mu(\phi)$, for any $\phi \in \mathcal{C}(X)$. We denote this by $\mu_n \to^* \mu$.

Note that this actually defines the closed sets of the topology.

Proposition 0.1. *If X is compact, then $\mathcal{M}(X)$ is compact. In other words, given any sequence $(\mu_n)_{n\in\mathbb{N}}$, one can find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a measure μ such that $\mu_{n_k} \to^* \mu$.*

We assume now that X is a compact metric space. This implies, in particular, that the space $\mathcal{C}(X)$ of continuous functions endowed with the uniform topology is separable, that is, it admits a dense family $(f_n)_{n\in\mathbb{N}}$.

Definition 0.2. We define the following metric on $\mathcal{M}(X)$:

$$\forall \mu, \nu \in \mathcal{M}(X),\ d(\mu,\nu) = \sum_{n\in\mathbb{N}} 2^{-n} \cdot \min(1, |\mu(f_n) - \nu(f_n)|$$

Proposition 0.2. *The metric d is compatible with the weak-* topology.*

Given a continuous application $T : X \to T$, we denote by $\mathcal{M}_T(X)$ the set of Borel T-invariant probability measures.

Proposition 0.3. *The sets $\mathcal{M}(X)$ and $\mathcal{M}_T(X)$ are convex. The extremal points of $\mathcal{M}(X)$ are the Dirac measures. The extremal points of $\mathcal{M}_T(X)$ are the T-ergodic measures.*

0.2. **The specification property.** Given two integers n, m such that $n < m$, we denote $[|n,m|] = \{n, n+1, n+2, ..., m-1, m\}$.

Definition 0.3 (Specification)**.** A specification is a finite sequence of points $x_1, ..., x_n \in M$ and strings $A_1 = [|a_1, b_1|], ..., A_n = [|a_n, b_n|]$, such that $a_1 \leq b_1 < a_2 \leq b_2 < ... < a_n \leq b_n$.

Definition 0.4 (Periodic specification property)**.** We say that a homeomorphism $f : M \to M$ has the *periodic specification property* if for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that given any specification previously defined with $a_{i+1} - b_i >$

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$N(\varepsilon)$, for every $1 \leq i < n$, there exists a periodic point x of period p, for every $p > b_n - a_1 + N(\varepsilon)$, such that

$$\text{dist}(f^j(x), f^j(x_i)) < \varepsilon, \ \forall j \in A_i, \forall \ 1 \leq i \leq n$$

The *specification property* (without *periodic*) does not require the point x to be periodic. Morally, a homeomorphism has the periodic specification property if any finite number of pieces of orbits which are sufficiently time-spaced can be shadowed by the real orbit of a periodic point. The time $N(\varepsilon)$ corresponds to the time needed to switch from one orbit to another. In the case $n = 2$, the property is sometimes referred to as the *weak specification property.*

Such a definition may look extremely strong and, therefore, not satisfied by a wide class of homeomorphisms. And yet, as an application of the genericity of the periodic shadowing property, we will actually prove that the specification property is generic in the conservative case:

Theorem 0.5 (Guihéneuf, Lefeuvre)**.** *A generic homeomorphism in $\mathcal{H}(M, \mu)$ satisfies the specification property.*

This result is obtained by using the genericity of topologically mixing conservative homeomorphisms and the genericity of the shadowing property.

For some interesting examples of dynamical systems satisfying the specification property, we refer the reader to the paper [2] of K. Sigmund.

0.3. **Back to measures.** The specification property is a very strong property which substantially impacts the ergodic properties of a homeomorphism f. It allows to investigate the space $\mathcal{M}_f$ of f-invariant measures in greater details. For the proofs of the various results stated in this paragraph, we refer the reader to [1, Chapter 21].

If f satisfies the specification property, then it can be easily shown that f has a positive topological entropy (see [1, Proposition 21.6]). This shows, in particular, that generic homeomorphisms in $\mathcal{H}(M, \mu)$ have positive topological entropy. However, this result does not have a great interest since it has been shown, as mentionned in the introduction to this memoir, that generic homeomorphisms in $\mathcal{H}(M, \mu)$ actually have infinite topological entropy (see [**?**] for instance).

In general, given any dynamical system f (not necessarily satisfying the specification property), the set of f-invariant measures with full support is either empty or a dense G_δ in $\mathcal{M}_f$. Since we only consider $f \in \mathcal{H}(M, \mu)$, we have obviously $\mu \in \mathcal{M}_f$ and therefore the set of f-invariant measures with full support is not empty, so it is a dense G_δ.

More specifically, when f satisfies the specification property, then one can show (see [1, Propositions 21.9, 21.10]) that the set of ergodic and nonatomic measures is a dense G_δ in $\mathcal{M}_f$.

Remark 0.6. This situation is quite peculiar, and impossible to picture. Recall that if $T : M \to M$ is a continuous application, then $\mathcal{M}_T(X)$ is a compact convex space (see Proposition 0.3). The extremal points of this set consist of the T-ergodic measures ... but these measures are dense and even uncountable (see Proposition **??**) in $\mathcal{M}_T(X)$!

We tried to sum up the situation Figure 0.3, where we suggest a cartography of the set $\mathcal{M} = \cup_{f\in\mathcal{H}(M,\mu)}\{f\}\times\mathcal{M}_f$. $\mathcal{R}\subset\mathcal{H}(M,\mu)$ denotes the residual set of μ-conservative homeomorphisms satisfying the specification property. Given, $f\in\mathcal{R}$, $\mathcal{R}_f\subset\mathcal{M}_f$ denotes the residual set of f-invariant, fully supported, non-atomic and ergodic measures. For $f\in\mathcal{H}(M,\mu)$, the sets $\mathcal{M}_f$ intersect at least in the point μ which is invariant by all $f\in\mathcal{H}(M,\mu)$.

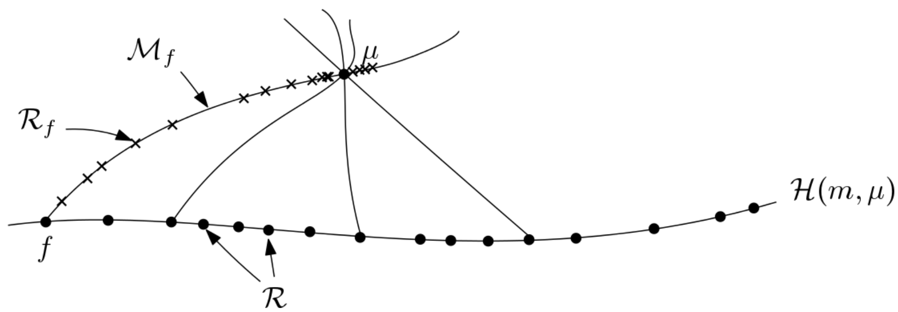

FIGURE 1. A cartography of $\mathcal{M}$

In 1941, J. Oxtoby and S. Ulam proved one of the first significant theorems in the theory of conservative homeomorphisms, namely that a generic conservative homeomorphism is ergodic. This result actually initiated the theory of generic dynamics. As an interesting remark, we thought that this cartography was surprisingly enough close to the Oxtoby-Ulam theorem in the sense that there exists a residual set of homeomorphisms f, such that μ can be approximated by non-atomic, fully supported, f-invariant and ergodic measures. In the case of the Oxtoby-Ulam theorem, the point μ would be contained in the sets $\mathcal{R}_f$, for $f\in\mathcal{R}$, a residual subset of $\mathcal{H}(M,\mu)$.

References

[1] DENKER, M., GRILLENBERGER, C., SIGMUND, K., "Ergodic Theory on Compact Spaces", Lecture Notes in Mathematics, Volume 527, Springer-Verlag, 1976.

[2] SIGMUND, K., *On dynamical systems with the specification property,* Trans. Amer. Math. Soc. 190, 285-299, 1974.

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