ON A TOPOLOGICAL LEMMA

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Abstract. In this paragraph, we present a useful topological lemma for $\mathcal{C}^0$ dynamics.

0.1. **Local Brouwer degree.** The reference for this paragraph is [1].

We consider a continuous map $f : \Omega \to \mathbb{R}^n$ and an open domain $\mathcal{U} \subset \Omega$. We assume that f is d-compact, that is $f^{-1}(0) \cap \mathcal{U}$ is compact. If $\overline{\mathcal{U}} \subset \Omega$ and $\overline{\mathcal{U}}$ is compact, this is equivalent to $0 \notin f(\partial D)$.

Definition 0.1. Given f defined as above, we define the following properties:

- Localization: Let $i : \mathcal{V} \to \mathcal{U}$ be the inclusion of an open subset satisfying $f^{-1}(0) \subset \mathcal{V}$. Then $\deg(f_{|\mathcal{V}}) = \deg(f)$.
- Units: Let $i : \mathcal{U} \to \mathbb{R}^n$ be the inclusion. Then:
$$\deg(i) = \begin{cases} 1, & \text{if } 0 \in \mathcal{U} \\ 0, & \text{if } 0 \notin \mathcal{U} \end{cases}$$
- Additivity: If $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$, $f_{|\mathcal{U}_1}, f_{|\mathcal{U}_2}$ are d-compact and $\mathcal{U}_1 \cap \mathcal{U}_2$ is disjoint from $f^{-1}(0)$, then:
$$\deg(f) = \deg(f_{|\mathcal{U}_1}) + \deg(f_{|\mathcal{U}_2})$$
- Homotopy invariance: Let $F : \mathcal{U} \times [0,1] \to \mathbb{R}^n$ be a d-compact homotopy, that is each $f_t = F(\cdot, t)$ is d-compact. Then:
$$\deg(f_0) = \deg(f_1)$$
- Multiplicativity: Let $f : \mathcal{U} \to \mathbb{R}^n, f' : \mathcal{U}' \to \mathbb{R}^p$ be two d-compact maps. We consider $F : \mathcal{U} \times \mathcal{U}' \to \mathbb{R}^n \times \mathbb{R}^p$. Then:
$$\deg(F) = \deg(f) \times \deg(f')$$

We are now able to define the topological degree of a continuous map. For a proof of the following result, we refer the reader to [1].

Theorem 0.2. *There exists a unique application* $\deg : (f, \mathcal{U}) \mapsto \deg(f) \in \mathbb{Z}$ *defined for d-compact maps and satisfying the five previous properties: localization, units, additivity, homotopy invariance and multiplicativity.*

We now present two useful properties of the degree.

Proposition 0.1. *Let* $f : \mathcal{U} \to \mathbb{R}^n$ *be a d-compact map. If* $\deg(f) \neq 0$*, then there exists a point* $x \in \mathcal{U}$ *such that* $f(x) = 0$.

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Proposition 0.2. *Assume* $f : \mathcal{U} \to \mathbb{R}^n$ *is a* $\mathcal{C}^1$ *map and* 0 *is a regular value for* f. *Then:*

$$\deg(f) = \sum_{x \in f^{-1}(0)} \operatorname{sgn}(\det(d_x f))$$

The theory of degree can easily been extended to the case of continuous applications between manifolds. The previous proposition is often used as a definition of the degree for differentiable applications. In order to extend it to the continuous applications, one has to use the fact that any continuous map is homotopic to a differentiable map, and that almost every point is regular for a differentiable application (Sard's theorem).

0.2. **Degree of a map $S^n \to S^n$.** We consider a continuous application $f : S^n \to S^n$, where S^n denotes the n-sphere. Recall that the n-th group of homology of the sphere $H^n(S^n)$ is isomorphic to $\mathbb{R}$, via the canonical map $u \mapsto \int_{S^n} u$. Therefore, the homomorphism $f^* : H^n(S^n) \to H^n(S^n)$ is of the form $f^*(u) = \alpha \cdot u$ for $\alpha \in \mathbb{R}$, for any $u \in H^n(S^n)$. The following proposition is a classical exercise in differentiable topology:

Proposition 0.3. *For a continuous map* $f : S^n \to S^n$, *the degree* $\deg(f)$ *defined in the previous paragraph coincides with the number* α.

Here B^n denotes the closed unit ball of $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$.

Proposition 0.4. *Let* $f : B^n \to \mathbb{R}^n$ *be a continuous map such that* $0 \notin f(S^{n-1})$. *We define*

$$s_f : \begin{cases} S^{n-1} \to S^{n-1} \\ x \mapsto \dfrac{f(x)}{||f(x)||} \end{cases}$$

Then:

$$\deg(f) = \deg(s_f)$$

In particular, for $n = 0$ and $S^0 = \{-1, 1\}$, the degree of $f : S^0 \to S^0$ is:

$$\deg(f) = \begin{cases} 1, & \text{if } f(1) = 1, f(-1) = -1, \\ -1, & \text{if } f(1) = -1, f(-1) = 1, \\ 0, & \text{otherwise} \end{cases}$$

0.3. **Markovian intersections.** The notion of *Markovian intersections* – and its related notion, the Smale's horseshoe – has been central in dynamical systems since its discovery, as it is one of the most striking example of a rich and complex dynamics.

Definition 0.3. We call *rectangle* a subset $R \subset M$ such that $R = f(I^n)$, where f is a homeomorphism. We call *faces* of R the image by f of the faces of I^n. We call *horizontal* the faces $f(I^{n-1} \times \{0\})$ and $f(I^{n-1} \times \{1\})$ and *vertical* the others. We say that a rectangle $R' \subset R$ is a strict horizontal (resp. vertical) subrectangle of R if the horizontal (resp. vertical) faces of R' are strictly disjoint of those those of R and the vertical (resp. horizontal) faces of R' are included in those of R.

Given $x \in \mathbb{R}^n$, we will denote by $\pi_1(x)$ its first coordinate. Following P. Zgliczynski and M. Gidea's article [3], we define Markovian intersections in the following way:

Definition 0.4. Let f be a homeomorphism of M, R_1 and R_2 two rectangles of M. We say that $f(R_1) \cap R_2$ is a *Markovian intersection* if there exists an horizontal subrectangle H of R_1 and a homeomorphism ϕ from a neighbourhood of $H \cup R_2$ to $\mathbb{R}^n$ such that:

- $\phi(R_2) = [-1, 1]^n$;
- either $\phi(f(H^+)) \subset \{x \mid \pi_1(x) > 1\}$ and $\phi(f(H^-)) \subset \{x \mid \pi_1(x) < -1\}$, or $\phi(f(H^-)) \subset \{x \mid \pi_1(x) > 1\}$ and $\phi(f(H^+)) \subset \{x \mid \pi_1(x) < -1\}$;
- $\phi(H) \subset \{x \mid \pi_1(x) < -1\} \cup [-1, 1]^n \cup \{x \mid \pi_1(x) > 1\}$.

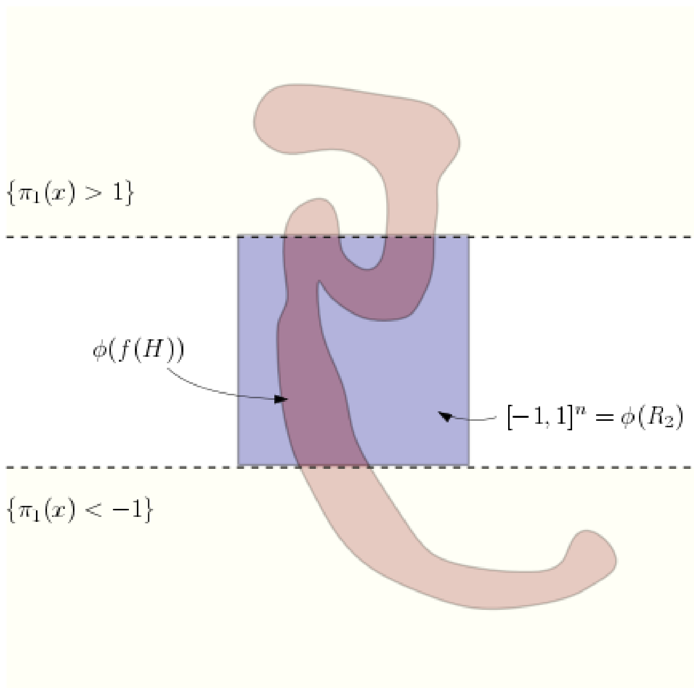

FIGURE 1. A Markovian intersection

Definition 0.5 (Smale's horseshoe)**.** For $k \in \mathbb{N}^*$, we say that a homeomorphism f contains a k-horseshoe if there exists a rectangle R such $f(R) \cap R$ contains at least k disjoint Markovian components.

These notions are extremely fruitful in order to study generic dynamics of homeomorphisms. For instance, one can show that a k-horseshoe a topological entropy which is greater than $\log k$. In particular, using Smale's horseshoes, one can show that generically, a conservative homeomorphism:

- has an infinite topological entropy (see [2], Chapter 3),
- is strongly topologically mixing (see [2], Chapter 3).

0.4. **The lemma.** The following result extends the definition of Smale's horseshoe to the $\mathcal{C}^0$ case. In particular, one can check that it is robust to perturbation in the $\mathcal{C}^0$ topology. A reference for this paragraph is [3]:

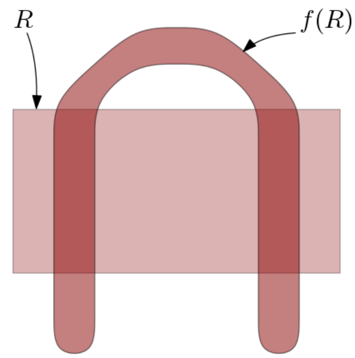

FIGURE 2. A 2-horseshoe

Proposition 0.5. *A Markovian intersection is $\mathcal{C}^0$ robust, namely if the intersection $f(R_1) \cap R_2$ has k Markovian components, then it is still true in a $\mathcal{C}^0$ neighbourhood of f.*

The following lemma shows how we will obtain periodic points for the periodic shadowing property. It is a simplified version of [3, Theorems 4, 16]. Their proof is based on an argument of homotopy and theory of degree, which we present below.

Lemma 0.6. *Let f be a homeomorphism and R be a rectangle such that $f(R) \cap R$ is a Markovian intersection. Then, there exists a fixed point for f in R.*

Proof. We assume $R = [-1, 1]^n \subset \mathbb{R}^n$. In the following, we denote by $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ the projection on the first coordinate and by $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection on the $(n-1)$-th last coordinates. We also define $R^+ = \{1\} \times [-1, 1]^{n-1}, R^- = \{-1\} \times [-1, 1]^{n-1}$.

By definition, there exists a strict horizontal subrectangle H of R such that $f(H)$ is a strict vertical subrectangle of R. Note that $f(H) \cap H$ is still a Markovian intersection, according to [2, Lemma 3.13]. Without loss of generality, we can assume that $f(H^+) \subset R^+$ and $f(H^-) \subset R^-$ (the other case being obtained by conjugating f to the hyperplanar symmetry whose plane of reflection is $\{\pi_1(x) = 0\}$).

We consider a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that: $\phi(H) = [-1, 1]^n$, $\phi(f(H^+)) \subset \{\pi_1(x) > 1\}$, $\phi(f(H^-)) \subset \{\pi_1(x) < -1\}$ and $\phi(f(H)) \subset \{|\pi_{n-1}(x)| < 1\}$. In the following, we will consider the application $\phi \circ f \circ \phi^{-1} : [-1, 1]^n \to \mathbb{R}^n$ and show that it has a fixed point. For the sake of simplicity, we will still denote this application f. Note that by hypothesis (see Figure 3):

(A) $f(R^-) \subset \{\pi_1(x) < -1\}$ and $f(R^+) \subset \{\pi_1(x) > 1\}$
(B) $f(R) \subset \{|\pi_{n-1}(x)| < 1\}$.

We consider a vertical segment $D_{y_0} = \{(\lambda, y_0), \lambda \in [-1, 1]\}$ for some $y_0 \in]-1, 1[^{n-1}$ and the following homotopy

$$h_t : [-1, 1]^n \to \mathbb{R}^n, x \mapsto f(\pi_1(x), ty_0 + (1-t)\pi_{n-1}(x)),$$

defined for $t \in [0, 1]$. Notice that $h_0 = f$ and $h_1 : x \mapsto f(\pi_1(x), y_0)$. By construction, $h_1(R) = f(D_{y_0})$, that is h_t retracts the image $f(R)$ onto the segment $f(D_{y_0})$ (see Figure 3).

The following homotopy

$$i_t : x \mapsto (\pi_1(h_1(x)), (1-t)\pi_{n-1}(h_1(x))),$$

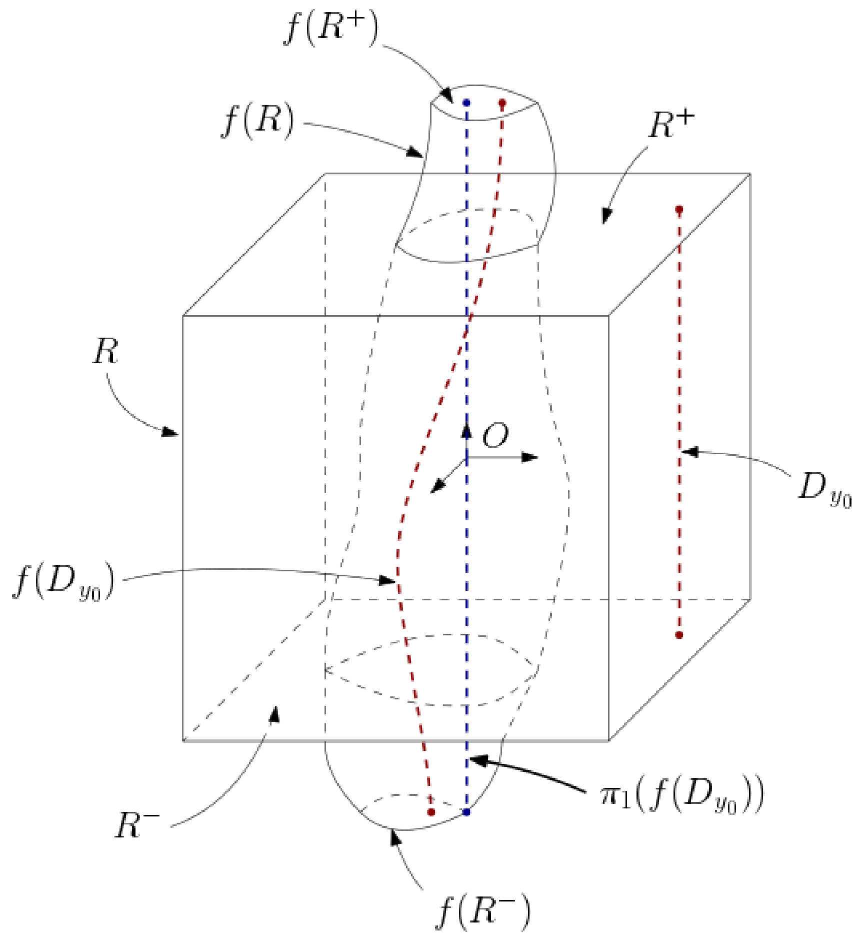

FIGURE 3. Illustration of the proof of Lemma 0.6

is such that $i_0 = h_1$ and $i_1 : x \mapsto (\pi_1(h_1(x)), 0)$. In other words, the homotopy i_t retracts the segment $f(D_{y_0})$ onto $(\pi_1(f(D_{y_0})), 0)$, its projection on the line $\{\pi_{n-1}(x) = 0\}$ (see Figure 3). Note that there exists a continuous function $\beta : [-1, 1] \to \mathbb{R}$ such that for all $x \in R$, $i_1(x) = (\beta(\pi_1(x)), 0)$. Therefore, the map $t \mapsto (i \cdot h)_t$[1] homotopes the application $(f - Id)$ to $((\beta \circ \pi_1, 0) - Id)$.

Moreover:

$$\deg((\beta, 0) - Id_{|\mathbb{R}^n}, R, 0) = (-1)^{n-1} \deg(\beta - Id_{|\mathbb{R}}, [-1, 1], 0)$$

By (A), we immediately have that $\deg(\beta - Id_{|\mathbb{R}}, [-1, 1], 0) = 1$, so $\deg((\beta, 0) - Id_{|\mathbb{R}^n}, R, 0) \neq 0$.

Now to conclude, it suffices to notice that there are no fixed points for $(i \cdot h)_t$ on the boundary of R, for any $t \in [0, 1]$. Indeed, by (B) the vertical faces are all sent strictly inside $\{-1 < \pi_{n-1}(x) < 1\}$ and by (A) the horizontal top and bottom

[1] $i \cdot h$ denotes the concatenation of the homotopies h then i.

faces are respectively sent in $\{\pi_1(x) > 1\}$ and $\{\pi_1(x) < -1\}$. Therefore, $\deg(f - Id, R, 0) \neq 0$ and f has a fixed point. □

References

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