ON BAIRE SPACES

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Abstract. In this paragraph, we recall the elementary definitions at the root of the theory of generic dynamics. In particular, we reproduce the proof of Baire's theorem.

The reference for this paragraph is [1].

Definition 0.1 (G_δ, F_σ sets)**.** Let X be a topological space. We call G_δ (resp. F_σ) a set of X which is a countable intersection (resp. union) of open (resp. closed) sets. We call *residual* (resp. meagre) a (resp. nowhere dense F_σ) dense G_δ set. We say that a property is *generic* in X, if it is satisfied on a residual set.

Definition 0.2 (Baire space)**.** A Baire space is a topological space in which a countable intersection of dense open sets is dense.

Definition 0.3 (Polish space)**.** A Polish space is a separable completely metrizable topological space.

In particular, a Polish space is a Baire space. Indeed, this is due to the celebrated Baire's theorem :

Theorem 0.4 (Baire)**.** *Every complete metric space is a Baire space.*

Proof. We denote by dist the distance on the complete metric space X. Let $V_1, V_2, ...$ be open dense set of X and W an open sets of X. We have to show that $\cap_n V_n \cap W \neq \emptyset$. We denote by $B(x,r)$ the ball centred in $x \in X$ of radius r.

Since V_1 is dense, $W \cap V_1$ is an non-empty open set so we can find x_1 and r_1 such that $\overline{B}(x_1, r_1) \subset W \cap V_1$ and $0 < r_1 < 1$. If $n \geq 2$ and x_{n-1} and r_{n-1} are chosen, the density of V_n shows that $V_n \cap B(x_{n-1}, r_{n-1})$ is non-empty and we can find x_n and r_n such that $\overline{B}(x_n, r_n) \subset V_n \cap B(x_{n-1}, r_{n-1})$ and $0 < r_n < 1/n$.

By recurrence, we construct a sequence $(x_n)_{n \geq 1}$ in X. Moreover, if $i, j > n$, x_i and x_j are both in $B(x_n, r_n)$ so that $\text{dist}(x_i, x_j) < 2/n$. The sequence has the Cauchy property and X is complete so it converges to a point x^* belonging to each V_n and to W by construction. This completes the proof. □

Some other examples of Baire spaces are given by locally compact Hausdorff spaces, or by open sets of a Baire space. The following result shows that in a Baire space without isolated points, a dense G_δ is a relatively "big" space:

Proposition 0.1. *If X is a Baire space without isolated points, no countable dense set is a G_δ.*

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Proof. Assume that there exists a countable dense set $E = \{x_k\}$ which is a G_δ. Therefore, $E = \cap_n V_n$, where the V_n are open and dense. If we set

$$W_n = V_n - \bigcup_{k=1}^{n} \{x_k\},$$

then the W_n are still open and dense (since X does not have isolated points). But $\cap_n W_n = \emptyset$, which is absurd. □

References

[1] Rudin, W., "Analyse réelle et complexe", Dunod, 3ème édition, 1987.

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