1. On the existence of local isothermal coordinates on a surface

The main reference for this section is [1]. In the following, we consider \((M, g)\) an \(n\)-dimensional smooth manifold endowed with a Riemannian metric \(g\).

1.1. Notations

Given a local chart \((U, \phi)\) on \(M\), we will denote by \((x_i)_{1 \leq i \leq n}\) the local coordinates and we write in these coordinates

\[
g = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j,
\]

where \(g_{ij}(x) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\). We denote by \((g^{ij})\) the coefficient of the inverse matrix. Given \(x \in M\), the norm of \(v \in T_x M\) is given by \(|v|_x = g_x(v, v)^{1/2}\), which will sometimes be denoted \(\langle v, v \rangle_x^{1/2}\). In the following, we will often drop the index \(x\). Note that if we are given other coordinates \((y_j)\), then one can check that in these new coordinates:

\[
(1.1) \quad g = \sum_{i,j,k,l=1}^{n} g_{ij} \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_l} dy^k dy^l
\]

We define the musical isomorphism at \(x \in M\) by

\[
b : T_x M \to T_x^* M \quad v \mapsto v^\flat = g(v, \cdot)
\]

This is an isomorphism between the two vector bundles \(TM\) and \(T^* M\) since they are equidimensional and \(g\) is symmetric definite positive, thus non-degenerate. Given an orthonormal basis \((e_i)\) of \(T_x M\), we will denote by \((e^i)\) the dual basis of \(T^*_x M\) which is, in other words, the image of the basis \((e_i)\) by the musical isomorphism.

If \(E\) is a vector bundle over \(M\), then the projection will be denoted \(\pi : E \to M\). We denote by \(\Gamma(M, E)\) the set of smooth sections of \(E\). \(\Gamma(M)\) denotes \(\Gamma(M, TM)\), the set of vector fields. We will denote by \(\Gamma(M, \otimes^m_S T^* M)\) the set of smooth symmetric covariant \(m\)-tensors on \(M\). On the cotangent bundle \(T^* M\), we denote by \(\omega\) the canonical symplectic form, which we write in coordinates \(\omega = \sum_{i=1}^{n} dp^i \wedge dx^i\). We recall that it is obtained as the differential of the canonical 1-form \(\lambda \in \Omega^1(T^* M)\), defined intrinsically as

\[
\lambda_{(x,p)}(\xi) = p\left( d\pi_{(x,p)}(\xi) \right)
\]
for a point \((x, p) \in T^*M, \xi \in T_{(x,p)}T^*M\) and where \(\pi : T^*M \to M\) denotes the projection.

Given a point \(x \in M\), if \((e_i)\) is an orthonormal basis of \(T_xM\), we define \(d\text{vol} = e^1 \wedge \ldots \wedge e^n\). In local coordinates, it is given by the formula:

\[
d\text{vol} = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^n
\]

We recall that the Laplacian in local coordinates is given by:

\[
\Delta f = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} g^{ij} \partial_j f \right)
\]

We denote by \(\nabla\) the Levi-Civita connection on \(TM\). In local coordinates, the Christoffel symbols are defined such that:

\[
\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k}
\]

They are given by the Koszul formula:

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left( \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right)
\]

We denote the torsion tensor \(T^\nabla\), which is defined as:

\[
T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]
\]

The curvature tensor is denoted by \(F^\nabla\) and defined as:

\[
F^\nabla(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z
\]

In particular, we clearly have \(F^\nabla(X,Y) = -F^\nabla(Y,X)\). We recall that the Levi-Civita connection is the unique torsion-free and \(g\)-metric connection, namely it satisfies for any \(X,Y,Z \in \Gamma(M)\):

\[
T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0
\]

\[
Z \cdot (g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)
\]

The sectional curvature \(K\) is given at \(x\) by

\[
K_x(e_1, e_2) = \langle F^\nabla(e_1, e_2)e_1, e_2 \rangle,
\]

where \(e_1, e_2 \in T_xM\) are orthogonal. In particular, in the case of a surface, which is what we will be mostly interested in, the sectional curvature is a real number referred to as the Gaussian curvature (or simply the curvature if the context is not ambiguous).
1.2. Isothermal coordinates

In this paragraph, we introduce isothermal coordinates, which will be widely used in the following.

**Definition 1.1.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold. Isothermal coordinates are local coordinates such that the metric can be written $g = e^{2\lambda}(dx_1^2 + ... + dx_n^2)$, where $\lambda$ is a smooth function.

In dimension $n \geq 3$, in a neighborhood of a point, isothermal coordinates may not exist. However, in dimension $n = 2$, we have the

**Theorem 1.2.** Let $(M, g)$ be a Riemannian surface and $x \in M$. There exists isothermal coordinates in a neighborhood of $x$.

We provide a proof of this theorem in the next paragraph based on the resolution of a Dirichlet problem. The existence of isothermal coordinates on a surface is closely link to the existence of a Riemann structure (or holomorphic structure) on the surface, that is a covering by charts $\{U, \phi\}$ such that the transition maps are all holomorphic, as explained in the Appendix. A complex structure on $M$ is an endomorphism $J \in \text{End}(TM)$ such that $J^2 = -\text{id}$. In particular, the data of a given conformal class together with an orientation of the manifold is equivalent to that of a complex structure.

The Koszul formula allows to compute the Christoffel symbols in the isothermal coordinates. We obtain:

\[
\begin{array}{c|cc}
\Gamma_{ij}^1 & j = 1 & j = 2 \\
\hline
i = 1 & \partial_1 \lambda & \partial_2 \lambda \\
i = 2 & \partial_2 \lambda & -\partial_1 \lambda \\
\end{array}
\]

\[
\begin{array}{c|cc}
\Gamma_{ij}^2 & j = 1 & j = 2 \\
\hline
i = 1 & -\partial_2 \lambda & \partial_1 \lambda \\
i = 2 & \partial_1 \lambda & \partial_2 \lambda \\
\end{array}
\]

Note that in this case, the curvature of $M$ has a rather simple expression:

\[(1.9) \quad K = -e^{-2\lambda} \Delta \lambda\]

1.3. Proof

The reference for this part is [2].
We say that a smooth map $\varphi$ between two Riemannian manifolds $(M,g)$ and $(N,h)$ is **conformal** if there exists $\lambda \in C^\infty(M)$ such that $\varphi^* h = e^{2\lambda} g$. This is strictly equivalent to the fact that the application $\varphi$ preserves the angle. We want to show the following theorem

**Theorem 1.3.** — Let $(M, g)$ be a Riemannian surface and $p \in M$. There exists a local coordinate system around $p$ which is conformal, that is there exists a neighborhood $U \subset M$ of $p$, a chart $\varphi : U \to \varphi(U)$, and a function $\lambda \in C^\infty(\varphi(U))$ such that $\varphi^* (e^{2\lambda}(dx^2 + dy^2)) = g$.

We call this coordinate system an **isothermal** coordinate system. It is rather useful since it simplifies a lot of computations when carried out in local coordinates. The previous definition clearly shows that the composition of two conformal maps is still conformal. In particular, given two local conformal charts $(U, \varphi)$, $(V, \psi)$ on $M$ around $p$, we see that $\Phi := \psi \circ \varphi^{-1}$ is conformal where it is defined. Since an orientation-preserving conformal map between two oriented open sets of the plane is holomorphic when seen as a function of the complex variable (and note that the converse is also true), we conclude that $\Phi = f + ig$ is holomorphic. Now, this also immediately implies that $f$ and $g$ (which are respectively the real and the imaginary part of $\Phi$) are harmonic functions. As a consequence, the conformal map $\varphi : U \to \varphi(U)$ is harmonic (that is each of its coordinate is harmonic).

Now, let us see that there exists somehow a converse to this fact. Assume that we can find a map $f : U \to \mathbb{R}$ which is harmonic on $U$, and such that $df_p \neq 0$. We would like to find a function $g : U \to \mathbb{R}$ such that $\varphi = f + ig$ is holomorphic (given an orthonormal basis in $U$, it satisfies the Cauchy-Riemann equations) and $df_p$ and $dg_p$ are independent (and therefore $df$ and $dg$ will be independent in a vicinity of $p$). It is easy to see that $\varphi$ is conformal (i.e. holomorphic) if and only if $\ast df = dg$, where $\ast$ is the Hodge operator. This is just a rewriting of the Cauchy-Riemann equations. Since we can always assume $U$ to be simply connected (if not, we can always shrink $U$ so that it becomes true), the existence of $g$ is equivalent by Poincaré’s lemma to the fact that $d \ast df = 0$ But this is also equivalent to $\ast d \ast df = -\Delta f = 0$ and since $f$ is assumed to be harmonic, the existence of $g$ is guaranteed. Note that it is clear that $df_p$ and $dg_p$ are independent since $df = \ast dg$ and we assumed that $df_p \neq 0$. Thus, for a neighborhood $V \subset U$ small enough around $p$, $df_x$ and $dg_x$ will still be independent for $x \in V$. In other words, we have obtained that $(f, g)$ is a conformal coordinate system defined on $V$.

Thus, the proof of Theorem 1.3 reduces to proving the following
Proposition 1.4. — Given \( p \in M \), there exists a neighborhood \( U \subset M \) around \( p \) and a real valued function \( f \) such that \( \Delta f = 0 \) and \( df_p \neq 0 \).

Démonstration. — The proof mainly relies on a standard Dirichlet problem. Indeed, choose a centered coordinate system \( \varphi : x \mapsto (x_1, x_2) \) in a neighborhood \( U \) around \( p \). In these coordinates, we have, according to equation (1.3):

\[
\Delta f(x) = g^{ij}(x)\partial_j \partial_k f + b^k(x)\partial_k f,
\]

for some smooth functions \( b^k \). Now, let us choose a disk \( D \) small enough in \( \varphi(U) \) (of radius \( \varepsilon \)) centered at 0. We could define for instance \( f \) to be the solution of the Dirichlet problem \( \Delta f = 0 \) in \( D \) and such that \( f = x_1 \) on \( \partial D \), but this will not guarantee that \( df_0 = 0 \). In order to ensure this, we consider for \( \varepsilon > 0 \) the dilated coordinate system \( (x_1^\varepsilon, x_2^\varepsilon) = (x_1/\varepsilon, x_2/\varepsilon) \), sending the disk \( D_\varepsilon \) of the original coordinate system to the unit disk in the dilated coordinate system. Note that in this coordinate system, we have:

\[
\Delta_\varepsilon f(x) = g^{ij}(\varepsilon x)\partial_j \partial_k f + \varepsilon b^k(\varepsilon x)\partial_k f,
\]

Consider the function \( f_\varepsilon \) which is harmonic in the unit disk \( D_1 \) and equal to \( x_1^\varepsilon \) on the boundary (in the dilated coordinate system). We define the function \( v_\varepsilon \) such that \( \Delta_\varepsilon v_\varepsilon = b^1(\varepsilon x) = b^1_\varepsilon(\varepsilon x) \) on the unit disk and \( v_\varepsilon = 0 \) on its boundary. Then we remark that \( f_\varepsilon = x_1 - \varepsilon v_\varepsilon \) by unicity of the Dirichlet problem. Now let us see that for \( \varepsilon \) small enough, \( \partial_1 f_\varepsilon(0) = 1 - \varepsilon \partial_1 v_\varepsilon(0) \) is not zero. Since \( v_\varepsilon = 0 \) on \( \partial D_1 \), the Poincaré inequality provides:

\[
||v_\varepsilon||_{L^2(D_1)} \leq C||\nabla v_\varepsilon||_{L^2}^2 \leq C|\langle \Delta_\varepsilon v_\varepsilon, v_\varepsilon \rangle| = C||b^1_\varepsilon||_{L^2}||v_\varepsilon||_{L^2}^2 \leq C||v_\varepsilon||_{L^2}||b^1_\varepsilon||_{L^2}
\]

Since \( b^1_\varepsilon \) is bounded in \( L^2(D_1) \) (and in each \( H^k(D_1) \)) independently of \( \varepsilon \), we obtain : \( ||v_\varepsilon||_{L^2} \leq C \). Applying the global elliptic estimate (??), we thus obtain :

\[
||v||_{H^{k+1}} \leq C(||\Delta_\varepsilon v_\varepsilon||_{H^{k-1}}^2 + ||v_\varepsilon||_{L^2}^2) = C(||b^1_\varepsilon||_{H^{k-1}}^2 + ||v_\varepsilon||_{L^2}^2) \leq C
\]

Since \( H^k(D_1) \subset C^1(D_1) \) for \( k \) large enough, we obtain a uniform bound on the \( C^1 \) norm of \( v^\varepsilon \), independant of \( \varepsilon \), which concludes the proof.

\[\square\]

BIBLIOGRAPHY

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