ON SOME QUESTIONS OF GENERICITY FOR TOPOLOGICAL DYNAMICAL SYSTEMS

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Résumé. Ce mémoire porte sur quelques problèmes de généricité de systèmes dynamiques en topologie $\mathcal{C}^0$ sur des variétés connexes et compactes, à la fois dans les cas conservatif et dissipatif. Nous établissons de nouveaux résultats de nature topologique et ergodique, parmi lesquels la généricité de la propriété de pistage pour les homéomorphismes conservatifs, et énonçons une conjecture sur l'approximation de flots en topologie uniforme.

Mots clés : Homéomorphisme conservatif, propriété dynamique générique, ergodicité

Abstract. This memoir is concerned with some questions of genericity for dynamical systems on a compact connected manifold in the $\mathcal{C}^0$ topology, both in the conservative and in the dissipative case. In particular, we prove some new results of topological and ergodic nature - among them, the genericity of the shadowing property for conservative homeomorphisms - and state a conjecture on the approximation of flows in the uniform topology.

Keywords: Conservative homeomorphism, generic dynamical property, ergodicity

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Foreword

This memoir was redacted on the occasion of a four-month summer internship at the Instituto de Matemática of the Universidade Fereral do Rio de Janeiro (Brazil) under the supervision of Alexander Arbieto, and validates the *Programme d'Approfondissement de Mathématiques* of the École Polytechnique (France).

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Part 1. Prolegomena

1. Genericity in dynamical systems

1.1. What is genericity?

1.1.1. *A historical observation.* From the early developments of infinitesimal calculus, mathematicians and physicists realized the incredible complexity of differential equations and their profound inability to solve *exactly* the mathematical questions arising from physical systems. Since the ground-breaking works of H. Poincaré, *New Methods of Celestial Mechanics* (1892–1899) and *Lectures on Celestial Mechanics* (1905–1910), in which he proved, in particular, his famous theorem of recurrence, this incapacity has become in some sense the *leitmotiv* of dynamicians, who no longer aimed to compute the quantitative behaviour of a specific system but rather to evaluate the qualitative behaviour of a *class* of systems. In other words, if assessing the behaviour of one particular system – and among all physical examples, the Solar system played a crucial role in the development of this field – turned itself to be almost impossible without numerical tools, it appeared that understanding *typical systems*, or *systems in general*, acting on a given space for instance, was mathematically more fruitful.

1.1.2. *Definition.* Given a topological space M (most often a manifold), by *generic*, we mean a property which is satisfied on a countable intersection of dense open subsets of M. Note that if M is a Baire space – for example, if M is completely metrizable – a countable intersection of dense open subsets is still dense[1]. This is actually the topological definition of genericity, first introduced by René Thom in his theorems of transversality. Genericity is a quite naturel concept, when one tries to assess the *typical* properties of a transformation on the system M, namely the properties that a transformation should satisfy *in general*.

The word *transformation* here can be simply understood as an element of the set of homeomorphisms on M, or automorphism, *id est* one-to-one bi-measurable maps, if M is equipped with the Borel σ-algebra. Of course, no such tool as the Lebesgue measure exists on the set of homeomorphisms or automorphisms, which makes it difficult to define naturally an equivalent notion to “almost everywhere”.

At least, when trying to define these *general* transformations, one can require an intersection property, namely : if properties P_1 and P_2 hold in *general*, then property $P_1 \wedge P_2$ should hold in general too. Such a requirement is satisfied by the notion of genericity. However, one has to be aware that this notion is sometimes *weak* with regard to the notion of *metrical genericity*. Namely, given a continuous family of transformations $(f_\alpha)_{\alpha\in\mathbb{R}^p}$ depending on the parameter α, a topologically generic property P may only be verified on a set of Lebesgue-measure 0.

1.1.3. *What is it good for?* In order to understand the dynamics of a class of systems (homeomorphisms for example), the study of generic properties is extremely fruitful, and this for many reasons:

- In themselves, proofs of genericity produce mechanisms of perturbation which reveal the complexity of the situations which may occur for a general dynamics.

[1] We refer the reader to Appendix A for further details.

- As pointed out in [14], generic results may be used to prove the existence of non-trivial dynamics. For instance, it is hard to give an explicit example of a conservative homeomorphism on the unit square which is topologically mixing. However, one can show that a conservative homeomorphism on the unit square is actually generically topologically mixing.
- Eventually, generical properties are stable by a countable intersection. Indeed, given a finite number of generic properties, the set of homeomorphisms satisfying simultaneously all these properties is still a countable intersection of dense open sets, and is therefore dense. As a consequence, this allows us to talk about *generic* homeomorphisms and list their different properties.

However, in the C^0 topology, one must admit that the genericity can be considered as irrelevant from a practical viewpoint: a generic homeomorphism is nowhere differentiable, and exhibits wild behaviours such as Cantor sets of periodic points of a given period. Thus generic homeomorphisms badly represent most of real-world systems. However, results are usually much easier to obtain than in more regular topologies, and can constitute a first step to the studies of genericity in C^r topologies for greater numbers r.

1.1.4. *Development of the field.* The classical frame of the study of generic properties in literature is that of compact connected manifolds[2]. This is what we will consider in the following. The results on the generic properties of dynamical systems strongly depend on the kind of transformations considered : conservative or dissipative systems, automorphisms, homemorphisms or $\mathcal{C}^k$-diffeomorphisms for $k \geq 1$.

- $\mathcal{C}^0$ *topology, conservative case:* The generic properties of conservative homeomorphisms are now well understood, thanks to some fundamental results obtained in the 1940's by J. Oxtoby and S. Ulam, in the 1960's by A. Katok and A. Stepin and in the 1970's by S. Alpern, V. S. Prasad and P. Lax. In his survey [14], P.-A. Guihéneuf provides a general overview of the generic properties of conservative homeomorphisms, be they of topological or ergodic nature. We refer the reader to his work for any further details on this topic.
- $\mathcal{C}^0$ *topology, dissipative case:* The generic properties of dissipative homeomorphisms are well known too. E. Akin, M. Hurley and J. A. Kennedy published in 2004 an exhaustive survey (see [3]), mostly focused on the topological properties of generic homeomorphisms. Recently, M. Andersson and F. Abdenur completed it with a theorem on the ergodic behaviour of generic homeomorphisms (see [1]).
- $\mathcal{C}^1$ *topology:* The $\mathcal{C}^1$ topology is still very active, both in the conservative and the dissipative cases, and some important conjectures are still being

[2] Of course, there also exists many results for noncompact manifolds and even more generally, for metrizable topological spaces, but we will not mention them in this memoir.

investigated such as the Palis' conjecture[3]. We refer the reader to C. Bonatti's exhaustive survey [10] on the $\mathcal{C}^1$ topology for further details.

- $\mathcal{C}^k$ *topology,* $k \geq 2$*:* In higher regularity, the theory is not much developed due to the absence of real techniques of perturbation. This can be illustrated by the inexistence of a fundamental *closing lemma* in the $\mathcal{C}^k$ topologies (for $k \geq 2$), stating that any $\mathcal{C}^k$ diffeomorphism with a recurrent point x can be $\mathcal{C}^k$ approximated by a diffeomorphism g such that x is periodic for g.

1.2. **Reading guide.** In this memoir, we mainly focus on the $\mathcal{C}^0$ topology, although some of the results presented are discussed in higher regularities. The main goal is to explore some of the few remaining questions which are still open for continuous dynamical systems, both in the conservative and in the dissipative case.

The work exposed revolves around three main orientations which we briefly present in the following paragraphs. More precise definitions will be given in the concerned parts. Note that the Appendix presents some elementary topological results. As much as we could, we tried to recall fundamental notions in topological dynamics throughout this memoir. For the reader's convenience, we also joined a short Appendix to this work. Eventually, we refer the reader to A. Katok and B. Hasselblatt's reference [19] for further details.

1.2.1. *Genericity of the shadowing property for conservative homeomorphisms.* Most of practical dynamical systems are very complex and subject to exterior perturbations, thus limiting their modelling to a relatively rough approximation. Therefore, one can wonder if the small discrepancy between the real system and its model has big consequences from a dynamical viewpoint? It turns out that for dynamical systems possessing the *shadowing property* (see Definition **??**), the errors induced by the model do not destroy completely the dynamical behaviour.

Informally, a homeomorphism f satisfies this property if any pseudo-orbit for f can be *shadowed by*[4] a real orbit for f. By pseudo-orbit, we mean a sequence of points $(x_k)_{k\in\mathbb{Z}}$ such that $f(x_k) \approx x_{k+1}$ (where the distance between $f(x_k)$ and x_{k+1} is uniformly controlled by a small constant $\delta > 0$).

This is actually what a computer would do when computing the iterates of a point x_0 by f, since a round-off error (which we can always assume to be controlled by some constant) may appear. Of course, an accumulation of the round-off errors may occur as $k \to \infty$, thus leading to a wrong computation of the trajectory of x. Precisely, this doesn't occur if f satisfies the shadowing property because, in that case, any pseudo-orbit stays close to the real orbit of a point (not obviously x_0 however).

In Part 2, we prove that the shadowing property is generic for conservative homeomorphisms. This result was obtained in collaboration with P.-A. Guihéneuf and consists of a submitted article. As a corollary, we prove the genericity of another strong property for conservative homeomorphisms, the *specification property.* Morally, a homeomorphism has the periodic specification property if any finite number of pieces of orbits which are sufficiently time-spaced can be shadowed by

[3]The Palis' conjecture states that if M is a compact connected manifold, then there exists a dense set in $\mathcal{D} \subset \mathrm{Diff}^r(M)$, such that the elements of $\mathcal{D}$ have a finite number of attractors.

[4]In other words: "stays close to".

the real orbit of a periodic point. We also obtain the genericity of the *average shadowing property* and the *asymptotic average shadowing property* .

1.2.2. *Ergodic properties in the dissipative case.* As mentioned previously, the generic topological properties of dissipative homeomorphisms are now well known. But understanding the ergodic properties of a general homeomorphism, that is finding an equivalent to the Oxtoby-Ulam theorem in the dissipative case, has long been an open question. Since F. Abdenur and M. Andersson's paper [1], we know that from an ergodic point of view, a generic dissipative homeomorphism f is *weird*. Informally, this means that given two distinct points $x, y \in M$, then almost surely their Birkhoff average will exist but will be distinct, thus revealing a chaotic behaviour. In particular, such a system does not possess any *physical measure*.

In Part 3, we investigate the ergodic behaviour that is defined in [1] as *wonderful*, namely: the homeomorphism f possesses a finite number of physical invariant measures, and their basins of attraction cover Lebesgue-almost all the manifold. In a certain way, this can be seen as the opposite behaviour to the weird property. We partially answer a question raised by C. Bonatti and E. Pujals, stated in [1], by showing that a weaker property – the δ-*wonderful* dynamics – is dense among homeomorphisms and conjecture a result.

1.2.3. *Genericity for $\mathcal{C}^0$ flows.* To our surprise, we discovered that the study of generic properties of continuous flows has not been addressed so far in literature. Part 4. deals with the inexistence of such a theory and tries to understand the main obstructions to it. This leads us to studying some approximation results of homeomorphisms and flows in the uniform topology.

In particular, we complete J.-C. Sikorav's proof (see [36]) about the approximation of a volume-preserving homeomorphism by a volume-preserving diffeomorphism and present the best results of approximation proved so far. We also state a conjecture about the approximation of continuous flows by smooth flows on a compact manifold, and provide a few elements of answer in the conservative case. Eventually, we present one – among various! – perturbation technique for smooth flows which could pave the way to some results for generic continuous flows if our conjecture appeared to be proven.

2. General setup

Throughout this memoir, we will consider M, a compact connected manifold with or without boundaries, of dimension $n \geq 2$, endowed with a distance dist.

2.1. **Conservative systems.**

2.1.1. *Good measures.*

Definition 2.1. A *good* Borel probability measure μ on M is a measure satisfying:

(1) $\forall p \in M, \mu(\{p\}) = 0$,
(2) $\forall \Omega \subset M$ non-empty open set, $\mu(\Omega) > 0$,
(3) $\mu(\partial M) = 0$.

Once for all, we fix a *good* Borel probability measure μ. These measures are also called OU (Oxtoby-Ulam) measures or Lebesgue-like measures in literature, according to the authors. The following result, due to J. Oxtoby and S. Ulam (see [30]) actually shows that they are all equivalent in some sense:

Theorem 2.2 (Oxtoby-Ulam)**.** *Let M be a compact manifold (it can be simply topological), and μ, ν two good measures on M. Then, there exists a homeomorphism h on M such that $h_*(\mu) = \nu$.*

As a consequence, with some abuses of notation, some authors in the literature on the $\mathcal{C}^0$ topology refer to this class of equivalence of measures (under the relation $\sim$ of being homeomorphic) as the "Lebesgue measure" on the manifold M.

2.1.2. *Volume forms.* Most of the time, in the rest of this memoir, by *conservative*, we will mean a system preserving a certain good measure. However, we will sometimes restrict ourselves to volume forms. In this section, we informally compare the two notions.

A volume form ω is an n-form which is nowhere equal to zero and is, in particular, a good measure in the sense of the previous definition. Some elementary properties of volume forms are their stability by multiplication by a smooth non-vanishing function f, that is to say that $f \cdot \omega$ is still a volume form, and the fact that any manifold possessing a volume form is orientable. Volume forms are only defined on differentiable manifolds whereas good measures can be defined on topological manifolds. For the sake of simplicity, in this section, we will only mention smooth volume forms and smooth manifolds, but the results naturally extends to the $\mathcal{C}^k$ regularity, $k \geq 1$.

In particular, in the Riemannian case, the metric tensor provides a canonical volume form. Indeed, if g is the matrix representing the metric tensor on the manifold, then one can consider in local coordinates:

$$\lambda = \sqrt{|\det g|} dx^1 \wedge ... \wedge dx^n$$

The following celebrated theorem on volume forms is due to Moser (see [23]):

Theorem 2.3 (Moser)**.** *Let M be a smooth compact manifold, and σ, τ two volume forms on M. Then, there exists a $\mathcal{C}^\infty$ diffeomorphism ϕ such that $\phi_*(\tau) = \sigma$.*

This theorem is of same nature as Theorem 2.2. In our case, this suggests that, even though the nature of the objects considered is quite different, one can morally consider good measures as the natural extension of volume forms to the frame of homeomorphisms.

Therefore, when one wants to study conservative dynamical systems, it is rather natural to consider good measures if the systems are only continuous, and volume forms if they are smooth (or at least $\mathcal{C}^1$).

However, we have missed one crucial point so far: are there good measures which are *not* volume forms ? Given a smooth orientable Riemannian manifold for instance, we can consider its canonical positive volume form λ, that we may call the Lebesgue measure. Take any good measure μ on M. By Radon-Nikodym's theorem, we can write

$$\mu = \mu_a + \mu_s, \ \mu_a \ll \lambda, \ \mu_s \perp \lambda,$$

where $a \ll b$ denotes the fact that a is absolutely continuous with respect to b, and $a \perp b$ that a and b are mutually singular (they have disjoint support). And, in particular, we have $d\mu_a = h \; d\lambda$, for some $h \in L^1(\lambda)$.

Therefore, to get examples of good measures which are not volume forms, one can consider measures of the form $d\mu = h\ d\lambda$, for a positive but not continuous density h, or measures which have a non-trivial positive singular part with respect to λ.

2.1.3. *Conservative versus dissipative.* The terminology *conservative*, *incompressible* and *volume-preserving* refers to the same notion. For an automorphism, homeomorphism or diffeomorphism, denoted by ψ, this means that for any Borel subset $B \subset M$, $\mu(\psi^{-1}(B)) = \mu(B)$ (which is equivalent to $\mu(\psi(B)) = \mu(B)$ if the map is bi-measurable). For a flow, this simply means that every element of the flow is volume-preserving. As far as vector fields are concerned, they are said to be conservative if the flow they generate is volume-preserving.

On the other hand, the word *dissipative* actually refers to the general case – and conservative transformations are only a particular case of dissipative transformations. As a consequence, this unhappy terminology can lead to intriguing formulations like : "a dissipative transformation can be conservative".

2.2. **Notations.** We denote by $\mathcal{H}(M)$ the set of homeomorphisms on M and by $\mathcal{H}(M,\mu)$ the set of conservative homeomorphisms on M, namely the homeomorphisms on M preserving the measure μ. $\mathcal{H}$ will denote both spaces. Both are metrizable by the $\mathcal{C}^0$ distance $d(f,g) = \max_{x\in M} \operatorname{dist}(f(x), g(x))$, for $f,g \in \mathcal{H}$, or by the uniform distance for homeomorphisms $\delta(f,g) = d(f,g) + d(f^{-1}, g^{-1})$. Only the latter is complete, but one can check that if a sequence of homeomorphisms $\mathcal{C}^0$-converges to another homeomorphism, then it also converges for the uniform metric.

The set of (resp. conservative) flows is denoted $\mathcal{F}(M)$ (resp. $\mathcal{F}(M,\mu)$) and endowed with the complete metric:

$$d_{\mathcal{F}}(\phi,\psi) = \sup_{t\in[0,1]} d(\phi^t, \psi^t)$$

Note that the topologies obtained by the different metrics $\sup_{t\in[a,b]} d(\phi^t,\psi^t)$, for any $-\infty < a < b < +\infty$ are all equivalent.

The metrics considered here define the *compact-open topology.* Since the manifolds considered are all compact, the compact-open topology is equivalent to the *Whitney topology* (see [16] for more details).

We call G_δ a set of $\mathcal{H}$ which is a countable intersection of open sets (see Appendix A). Since $\mathcal{H}$ is a complete metric space, Baire's theorem states that a countable intersection of dense open sets is, in particular, a dense G_δ set. We call *residual* a dense G_δ set and we say that a property is *generic* in $\mathcal{H}$, if it is satisfied on a residual set.

2.3. **Main results and definitions in $\mathcal{H}$.** In this section, we recall the fundamental results which we will use in the rest of this memoir. This is largely inspired by P.-A. Guihéneuf's survey [14], to which we refer the reader for further details.

2.3.1. *Dyadic subdivisions.* λ denotes the Lebesgue measure and $I^n = [0,1]^n$ the unit cube in $\mathbb{R}^n$. One of the most fundamental results in the theory of conservative homeomorphisms is a combination of a theorem by Brown (see [9]) and by Oxtoby-Ulam (see Theorem 2.2 of this memoir, or the original paper [30]). A detailed proof of this result can be found in [4, Appendix 2]:

Theorem 2.4 (Oxtoby-Ulam-Brown). *Let μ be a good Borel probability measure on M. Then, there exists $\phi : I^n \to M$ continuous such that:*

(1) *ϕ is surjective*
(2) *$\phi|_{\overset{\circ}{I^n}}$ is a homeomorphism on its image*
(3) *$\phi(\partial I^n)$ is a closed subset of M, of empty interior, and disjoint of $\phi(\overset{\circ}{I^n})$*
(4) *$\mu(\phi(\partial I^n)) = 0$*
(5) *$\phi_*(\lambda) = \mu$*

Morally, this result asserts that the manifold (M, μ) is almost homeomorphic to (I^n, λ), upto a set of measure 0. Now, once for all, we fix such a $\phi : I^n \to M$. This allows us to introduce the notion of dyadic subdivision.

Definition 2.5 (Dyadic subdivision). A dyadic cube of order m of M is the image in M by ϕ of a cube $\prod_{i=1}^n [\frac{k_i}{2^m}, \frac{k_i+1}{2^m}]$, with $0 \le k_i \le 2^m - 1$. The dyadic subdivision $\mathcal{D}_m$ of order m of the manifold M is the collection of dyadic cubes of order m of M. In the following, $p_m = 2^{nm}$ will denote the number of cubes of the subdivision.

For a dyadic subdivision $\mathcal{D}_m = (C_i)_{1 \le i \le p_m}$, we will denote by $\chi(\mathcal{D}_m)$ the maximum diameter of its cubes, namely:

$$\chi(\mathcal{D}_m) = \max_{1 \le i \le p_m} \operatorname{diam} C_i$$

These dyadic subdivisions satisfy some good properties:

- Each cube is connected and obtained as the closure of the open cube $\prod_{i=1}^n]\frac{k_i}{2^m}, \frac{k_i+1}{2^m}[$.
- For all m, $\mathcal{D}_m$ is a cover of M by a finite number of cubes of same measure and whose interior are two by two disjoint.
- For all m, $\mathcal{D}_{m+1}$ is a refinement of the subdivision $\mathcal{D}_m$.
- The measure of the cubes as well as the maximum of the diameter of the cubes of $\mathcal{D}_m$ tend to 0 as $m \to \infty$. The measure of the boundary of the cube is zero.

Note that the image of the dyadic subdivision by any $f \in \mathcal{H}(M, \mu)$ is still a dyadic subdivision satisfying the same properties.

2.3.2. *Perturbation lemmas in the $\mathcal{C}^0$ topology.* The results presented in this paragraph are extensively used to prove most of the generic topological properties of (conservative) homeomorphisms. They are naturally easier to prove in the dissipative frame. Lemmas 2.6 (extension of finite maps) and 2.8 (local modification) will be used in Part 2. For the reader's convenience, we included a theorem by P. Lax and S. Alpern which will not be used in the rest of this memoir, but has a great importance in the theory of conservative homeomorphisms. Below, we give an outlook on the generic properties that these results pave the way to in the conservative case.

Lemma 2.6 (Extension of finite maps). *Let $x_1, ..., x_n$ be n different points of $M \setminus \partial M$ and $\Phi : \{x_1, ..., x_n\} \to M$ an injective map such that $d(\Phi, Id) < \delta$. Then, there exists $\varphi \in \mathcal{H}$ such that $\varphi(x_i) = \Phi(x_i)$, for all $i \in \{x_1, ..., x_n\}$ and $d(\varphi, Id) < \delta$. Moreover, given n injective continuous paths γ_i joining x_i to $\Phi(x_i)$, the support of*

φ can be chosen as small as wanted, and contained in a neighbourhood of the union of the paths γ_i.

The local modification theorem relies on the *annulus* theorem. The latter states that the region of $\mathbb{R}^n$ located between two *bicollared* spheres is homeomorphic to an annulus.

Definition 2.7. An embedding i of a manifold Σ into an manifold M is said to be *bicollared* if there exists an embedding $j : [-1, 1] \times \Sigma \to M$ such that $j_{\{0\}\times\Sigma} = i$.

This requirement of bicollared spheres avoids pathological cases such as Alexander horned sphere, which can be seen as a counterexample proving that no analogue of Jordan's theorem holds in dimension greater than 2. A first proof of the *annulus* theorem was given by T. Radó in 1924 in dimension 2. Then, in 1952, E. Moise proved it in dimension 3, R. Kirby extended the result to the dimensions bigger than 5 in 1969, and eventually F. Quinn gave a proof for dimension 4. Surprisingly enough, we will see in Part 4 another example of a problem which has been solved in any dimension, except dimension 4.

Lemma 2.8 (Local modification)**.** *Let $\sigma_1, \sigma_2, \tau_1, \tau_2$ be four bicollared embeddings of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, such that σ_1 is in the bounded connected component of σ_2 and τ_1 in the bounded connected component of τ_2. Let A_1 be the bounded connected component of $\mathbb{R}^n - \sigma_1$ and B_1 be the bounded connected component of $\mathbb{R}^n - \tau_1$, Σ the connected component of $\mathbb{R}^n - (\sigma_1 \cup \sigma_2)$ with boundaries $\sigma_1 \cup \sigma_2$ and Λ the connected component of $R^n - (\tau_1 \cup \tau_2)$ with boundaries $\tau_1 \cup \tau_2$, A_2 the unbounded connected component of $R^n - \sigma_2$ and B_2 the unbounded connected component of $R^n - \tau_2$.*

Consider two homeomorphisms $f_i : A_i \to B_i$, such that they both preserve or reverse the orientation. Then, there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = f_1$ on A_1 and $f = f_2$ on A_2.

Moreover, if we assume $\lambda(A_1) = \lambda(B_1)$ and $\lambda(\Sigma) = \lambda(\Lambda)$ and the homeomorphisms f_i conservative, then f can be chosen conservative too.

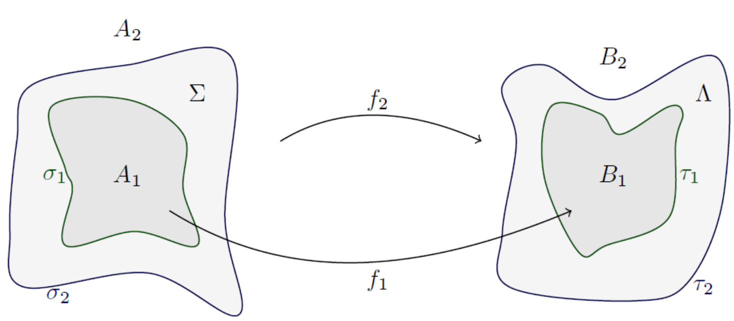

FIGURE 1. The local modification theorem

In the rest of this paragraph, we restrict ourselves to the conservative case, *id est* we state the results in $\mathcal{H}(M, \mu)$. Before stating Lax-Alpern theorem, we define the *dyadic permutations* on M.

Definition 2.9 (Dyadic permutation)**.** A dyadic permutation f of order m is a conservative automorphism which permutes the cubes of the dyadic subdivision $\mathcal{D}_m$ and is a translation when restricted to the interior of each cube, *id est* $\phi^{-1} \circ f \circ \phi$ is a real translation of from cube of I^n to another. We say that a dyadic permutation is cyclic if the orbit of any cube covers the manifold.

Of course, a dyadic permutation may not be continuous and not be defined or well-defined on the boundaries of the cubes, but this does not matter since they are of measure zero.

Theorem 2.10 (Lax-Alpern)**.** *Let $f \in \mathcal{H}(M, \mu)$ and $\varepsilon > 0$. Then, there exists an integer m and a cyclic dyadic permutation f_m of order m such that $d(f, f_m) < \varepsilon$.*

For any homeomorphism $f \in \mathcal{H}(M, \mu)$, the combination of Lax-Alpern theorem and the extension of finite maps provide, for any $\varepsilon > 0$, a $g \in \mathcal{H}(M, \mu)$ such that $d(f, g) < \varepsilon$, and g permutes cyclically the center of the cubes of a dyadic subdivision of order m, for m big enough.

These results are the key to various generic properties in $\mathcal{H}(M, \mu)$ of a topological nature. Generically, an element in $\mathcal{H}(M, \mu)$:

- is transitive (see [14], Chapter 2),
- has a dense set of periodic points (see [14], Chapter 3),
- satisfies the following dichotomy: given a period τ, either the set of periodic points of period τ is empty, or a Cantor set of Hausdorff dimension equals to 0 (see [15]).

2.3.3. *Markovian intersections.* The notion of *Markovian intersections* – and its related notion, the Smale's horseshoe – has been central in dynamical systems since its discovery, as it is one of the most striking example of a rich and complex dynamics.

Definition 2.11. We call *rectangle* a subset $R \subset M$ such that $R = f(I^n)$, where f is a homeomorphism. We call *faces* of R the image by f of the faces of I^n. We call *horizontal* the faces $f(I^{n-1} \times \{0\})$ and $f(I^{n-1} \times \{1\})$ and *vertical* the others. We say that a rectangle $R' \subset R$ is a strict horizontal (resp. vertical) subrectangle of R if the horizontal (resp. vertical) faces of R' are strictly disjoint of those those of R and the vertical (resp. horizontal) faces of R' are included in those of R.

Given $x \in \mathbb{R}^n$, we will denote by $\pi_1(x)$ its first coordinate. Following P. Zgliczynski and M. Gidea's article [39], we define Markovian intersections in the following way:

Definition 2.12. Let f be a homeomorphism of M, R_1 and R_2 two rectangles of M. We say that $f(R_1) \cap R_2$ is a *Markovian intersection* if there exists an horizontal subrectangle H of R_1 and a homeomorphism ϕ from a neighbourhood of $H \cup R_2$ to $\mathbb{R}^n$ such that:

- $\phi(R_2) = [-1, 1]^n$;
- either $\phi(f(H^+)) \subset \{x \mid \pi_1(x) > 1\}$ and $\phi(f(H^-)) \subset \{x \mid \pi_1(x) < -1\}$, or $\phi(f(H^-)) \subset \{x \mid \pi_1(x) > 1\}$ and $\phi(f(H^+)) \subset \{x \mid \pi_1(x) < -1\}$;
- $\phi(H) \subset \{x \mid \pi_1(x) < -1\} \cup [-1, 1]^n \cup \{x \mid \pi_1(x) > 1\}$.

Definition 2.13 (Smale's horseshoe)**.** For $k \in \mathbb{N}^*$, we say that a homeomorphism f contains a k-horseshoe if there exists a rectangle R such $f(R) \cap R$ contains at least k disjoint Markovian components.

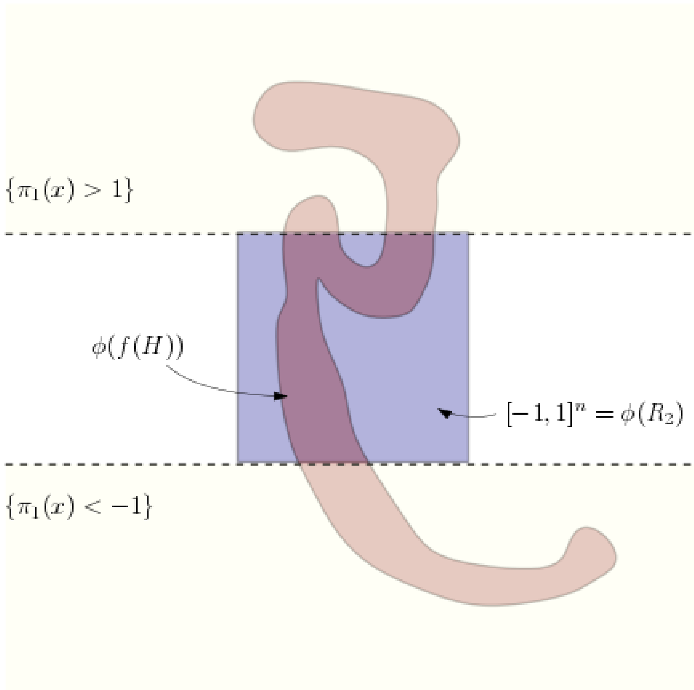

FIGURE 2. A Markovian intersection

These notions are extremely fruitful in order to study generic dynamics of homeomorphisms. For instance, one can show that a k-horseshoe a topological entropy which is greater than $\log k$. In particular, using Smale's horseshoes, one can show that generically, a conservative homeomorphism:

- has an infinite topological entropy (see [14], Chapter 3),
- is strongly topologically mixing (see [14], Chapter 3).

Our proof of the genericity of the shadowing property heavily relies on this concept.

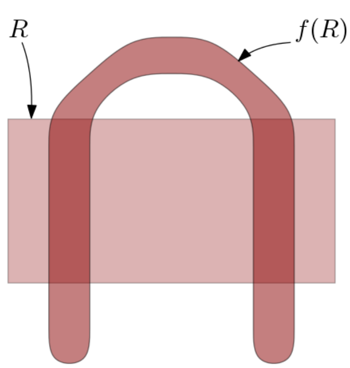

FIGURE 3. A 2-horseshoe

Part 2. On the genericity of the shadowing property for conservative homeomorphisms

This part is mainly adapted from an article written in collaboration with P.-A. Guihéneuf. We recall that $\mathcal{H}$ denotes either $\mathcal{H}(M)$ or $\mathcal{H}(M, \mu)$.

3. Introduction

3.1. The shadowing property.

3.1.1. *Main result.*

Definition 3.1 (Pseudo orbit)**.** Given $f \in \mathcal{H}$ and $\delta > 0$, a *δ-pseudo orbit* $(x_k)_{k\in\mathbb{Z}}$, is a sequence of points in M such that $\mathrm{dist}(f(x_k), x_{k+1}) < \delta$, for all $k \in \mathbb{Z}$. A *δ-periodic pseudo orbit* is a δ-pseudo orbit such that there exists an integer $N > 0$ such that $x_{k+N} = x_k$, for all $k \in \mathbb{Z}$.

As mentioned in the introductory part of this memoir, this notion is motivated by the fact that it models the roundoff error that a computer generates when trying to compute the iterations of the point x_0 under the transformation f.

Definition 3.2 (Shadowing property)**.** We say that the homeomorphism f satisfies the:

- *shadowing property*, if for every $\varepsilon > 0$, there exists $\delta > 0$, such that any δ-pseudo orbit $(x_k)_{k\in\mathbb{Z}}$ is ε-shadowed by the real orbit of a point, namely, there exists $x^* \in M$ such that $\mathrm{dist}(f^k(x^*), x_k) < \varepsilon$, for all $k \in \mathbb{Z}$,
- *periodic shadowing property*, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that any δ-periodic pseudo orbit $(x_k)_{k\in\mathbb{Z}}$ is ε-shadowed by the real orbit of a periodic point x^*, with same period as $(x_k)_{k\in\mathbb{Z}}$,
- *special shadowing property*, if it satisfies both shadowing and periodic shadowing properties.

Despite a deep study of generic properties in $\mathcal{H}(M, \mu)$, it seems that the genericity of the shadowing property is still missing in the conservative case. This paper aims to fill in this gap by showing the following result:

Theorem 3.3. *The special shadowing property is generic in $\mathcal{H}$.*

Our reasoning holds in both the dissipative and the conservative cases. Our proof mainly relies on the use of Oxtoby-Ulam's theorem, which provides an adequate subdivision of the manifold M in a very general case, namely we only require M to be a topological manifold. As a consequence, this result also extends the genericity of the special shadowing property in the dissipative case to the class of topological manifolds.

3.1.2. *A brief historical survey of the shadowing property.* The shadowing property was first introduced in the works of D. Anosov and R. Bowen, who proved independently that in a neighbourhood of a hyperbolic set, a diffeomorphism has the shadowing property. This result is known as the *shadowing lemma* (see A. Katok and B. Hasselblatt's book [19, Theorem 18.1.2], for a proof). As a consequence, examples of dynamical systems satisfying the shadowing property are provided by Anosov's diffeomorphisms. For further details on the notion of shadowing, we refer the reader to S. Y. Pilyugin's historical survey [31].

The first proof of genericity of the shadowing property was obtained by K. Yano (see [38]) in the case $M = \mathbb{S}^1$. Then, using the possibility to approximate any homeomorphism by a diffeomorphism in dimension $n \leq 3$, K. Odani obtained in [27] the genericity of the shadowing property for manifolds of dimension less than 3. S. Y. Pilyugin and O. B. Plamenevskaya were able to improve this result in [32] to any dimension in the case of smooth manifolds. In 2005, P. Koscielniak established in [20] the genericity of the shadowing property for homeomorphisms on a compact manifold which possesses a triangulation (smooth manifolds or topological manifolds of dimension ≤ 3 for example) or a handle decomposition (smooth manifolds or manifolds of dimension ≥ 6 for example). To the best of our knowledge, this is the best result obtained so far.

3.2. **The specification property.** Given two integers n, m such that $n < m$, we denote $[|n, m|] = \{n, n+1, n+2, ..., m-1, m\}$.

Definition 3.4 (Specification)**.** A specification is a finite sequence of points $x_1, ..., x_n \in M$ and strings $A_1 = [|a_1, b_1|], ..., A_n = [|a_n, b_n|]$, such that $a_1 \leq b_1 < a_2 \leq b_2 < ... < a_n \leq b_n$.

Definition 3.5 (Periodic specification property)**.** We say that a homeomorphism $f : M \to M$ has the *periodic specification property* if for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that given any specification previously defined with $a_{i+1} - b_i > N(\varepsilon)$, for every $1 \leq i < n$, there exists a periodic point x of period p, for every $p > b_n - a_1 + N(\varepsilon)$, such that

$$\text{dist}(f^j(x), f^j(x_i)) < \varepsilon, \ \forall j \in A_i, \forall\ 1 \leq i \leq n$$

The *specification property* (without *periodic*) does not require the point x to be periodic. Morally, a homeomorphism has the periodic specification property if any finite number of pieces of orbits which are sufficiently time-spaced can be shadowed by the real orbit of a periodic point. The time $N(\varepsilon)$ corresponds to the time needed to switch from one orbit to another. In the case $n = 2$, the property is sometimes referred to as the *weak specification property.*

Such a definition may look extremely strong and, therefore, not satisfied by a wide class of homeomorphisms. And yet, as an application of the genericity of the periodic shadowing property, we will actually prove that the specification property is generic in the conservative case:

Corollary 3.6. *A generic homeomorphism in $\mathcal{H}(M, \mu)$ satisfies the specification property.*

This result will simply be obtained using the genericity of topologically mixing conservative homeomorphisms and the stability of generic properties by finite intersection.

For some interesting examples of dynamical systems satisfying the specification property, we refer the reader to the paper [35] of K. Sigmund.

3.3. **Strategy of proof.** To begin with, we will apply Oxtoby-Ulam-Brown Theorem (Theorem 2.4), that will reduce the study to the case where the phase space is the unit cube. This will allow us to define dyadic subdivisions on our manifold. Then, given a generic conservative homeomorphism f, and denoting C_i the cubes of some fine enough dyadic subdivision, we will prove that:

(1) Each time $f(C_i) \cap C_j \neq \emptyset$, the set $f(C_i) \cap C_j$ has nonempty interior. Homeomorphisms satisfying this property will be called *nice* (see Definition 9.1); this will be obtained easily by a "transversality" result (Proposition 4.1).
(2) Each time $f(C_i) \cap C_j \neq \emptyset$, there exists some small cubes $c_i \subset C_i$ and $c_j \subset C_j$ such that c_i and c_j have a Markovian intersection. Homeomorphisms satisfying this property will be called *chained* (see Definition 4.2). This is the most delicate part of the proof, which will be obtained by creating transverse intersections for some foliations on the cube (see Definition 4.2).

The property of *periodic* shadowing will be deduced from the previous construction by applying a fixed point lemma (Lemma 4.5, due to [39]).

4. A toolbox of results

4.1. **Chained and sealed homeomorphisms.** We introduce in this paragraph some key notions to our proof.

Definition 4.1 (m-nice homeomorphism)**.** We say that $f \in \mathcal{H}$ is a *m-nice* homeomorphism for an integer $m > 0$ if, given the dyadic subdivision $\mathcal{D}_m = (C_m)_{1 \leq m \leq p_m}$, we have for all $i, j \in \{1, \dots, p_m\}$, either $f(C_i) \cap C_j = \emptyset$, or $f(\overset{\circ}{C_i}) \cap \overset{\circ}{C_j} \neq \emptyset$. We say that f is *nice* if it is m-nice for every $m > 0$.

Proposition 4.1. *The set of m-nice homeomorphisms is open and dense in $\mathcal{H}$. Therefore, nice homeomorphisms are generic in $\mathcal{H}$.*

Proof of Proposition 4.1. The set of m-nice homeomorphism is open in $\mathcal{H}$: this follows from its definition. We prove its density by an argument of genericity. Consider the open set

$$H_{i,j} = \left\{ f \in \mathcal{H} \mid f(C_i) \cap C_j \neq \emptyset \Rightarrow f(\overset{\circ}{C_i}) \cap \overset{\circ}{C_j} \neq \emptyset \right\},$$

for any $1 \leq i, j \leq p_m$. Assume $f \in \mathcal{H}$ is not in $H_{i,j}$. This means that $f(\partial C_i) \cap \partial C_j \neq \emptyset$ and consider $f(x) = y$ in this intersection. Now, by the extension of finite maps (Lemma 2.6), one can find a local conservative homeomorphism φ with support in a neighbourhood of y, such that $\tilde{f} = \varphi \circ f$ is as close to f as wanted, and $\tilde{f}(x) \in \overset{\circ}{C_j}$. Therefore $\tilde{f}(\overset{\circ}{C_i}) \cap \overset{\circ}{C_j} \neq \emptyset$ (see Figure 4) and since the m-nice homeomorphisms are exactly the ones in $\cap_{i,j} H_{i,j}$, the conclusion is immediate. □

Definition 4.2 (n-chain)**.** Given f, a m-nice conservative homeomorphism, we call *n-chain* a sequence $C_0, \dots, C_n$ of cubes of the dyadic subdivision $\mathcal{D}_m$, such that for every $i \in \{0, \dots, n-1\}$, $f(C_i) \cap C_{i+1} \neq \emptyset$. We will say that a n-chain is *shadowed* if there exists a point $x \in M$ such that:

$$x \in \overset{\circ}{C}_0, f(x) \in \overset{\circ}{C}_1, \dots, f^n(x) \in \overset{\circ}{C}_n$$

We say that f is *n-chained*, if any n-chain is shadowed, and *chained* for the subdivision $\mathcal{D}_m$ if it is n-chained for every $n > 0$. We say that f is *robustly chained* if there exists a neighbourhood $\mathcal{U}$ of f in $\mathcal{H}$ such that any homeomorphism in $\mathcal{U}$ is chained (for the same subdivision $\mathcal{D}_m$).

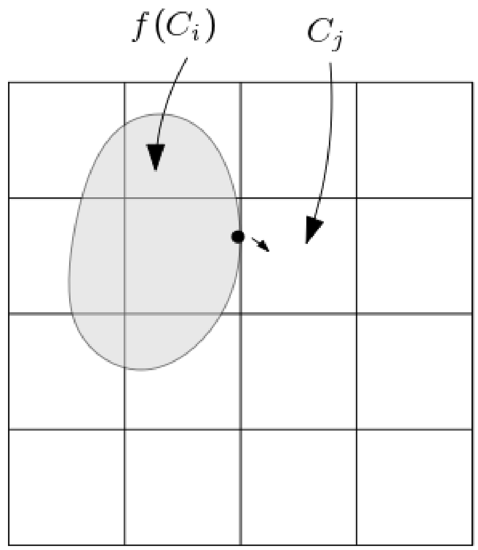

FIGURE 4. Local modification of f, seen in the cube I^2

Definition 4.3 (n-cycles)**.** Let $f \in \mathcal{H}$ be m-nice and chained. We say that f has an *n-cycle* if there exists an n-chain $C_0, C_1, \ldots, C_n \in \mathcal{D}_m$ such that $C_0 = C_n$. We will say that the cycle is *sealed* if there exists a periodic point $x \in \overset{\circ}{C}_0$ with period n, such that: $f(x) \in \overset{\circ}{C}_1, \ldots, f^{n-1}(x) \in \overset{\circ}{C}_{n-1}, f^n(x) = x$. We will say that f is *sealed* if any n-cycle is sealed, for any $n > 0$. We will say that f is *robustly sealed* if there exists a a neighbourhood $\mathcal{U}$ of f in $\mathcal{H}$ such that any homeomorphism in $\mathcal{U}$ is sealed (for the same subdivision $\mathcal{D}_m$).

4.2. **Foliations.** In this paragraph, we recall an elementary result on foliations.

Definition 4.4 (Transversal intersection)**.** Given two foliations $\mathcal{F}, \mathcal{F}'$ such that $\dim(\mathcal{F}) + \dim(\mathcal{F}') = n$, we will say that two leaves L of $\mathcal{F}$ and L' of $\mathcal{F}'$ *intersect transversally*, and denote by $L \pitchfork L'$, if either their intersection is empty, or there exists an open set ℓ of the leaf L such that $\mathrm{Card}(\ell \cap L') = 1$.

In the following, we will use this definition with $\dim(\mathcal{F}) = \mathrm{codim}(\mathcal{F}') = 1$.

Proposition 4.2. *Assume f is m-nice and consider two cubes C_i, C_j of the subdivision $\mathcal{D}_m$ such that $f(C_i) \cap C_j \neq \emptyset$. We consider a smooth foliation $\mathcal{F}$ of C_i and a smooth foliation $\mathcal{H}$ of C_j such that $dim(\mathcal{F}) + dim(\mathcal{H}) = n$. If $f(\mathcal{F})$ does not intersect transversally $\mathcal{H}$ on $\overset{\circ}{C}_j$, then there exists a $\mathcal{C}^0$ perturbation of f, as small as desired, such that this intersection is transverse.*

Note that a transverse intersection is $\mathcal{C}^0$ robust, namely it is still transverse under any small $\mathcal{C}^0$ perturbation.

Proof. We assume that there does not exist any transversal intersection of leaves in $\mathcal{U} = f(\overset{\circ}{C}_i) \cap \overset{\circ}{C}_j$. Consider a point $y \in \mathcal{U}$ and the leaf L' of $\mathcal{H}$ passing through y. We denote by $(f_1, ..., f_k)$ an orthonormal basis of $T_y L'$ and we complete it into an orthonormal basis $(f_1, ..., f_n)$ of $\mathbb{R}^n$. Now, we consider the point $x = f^{-1}(y)$, the leaf L of $\mathcal{F}$ passing through x and an orthonormal basis $(e_1, ..., e_{n-k})$ of T_xL which we complete in a basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$. We denote by $A \in O(n)$ the linear application such that $Ae_i = f_{k+i}, i \in \{1, ..., n\}$, where the index is taken modulo n. Then, considering on a neighbourhood of x the affine volume-preserving transformation taking x on y and of linear part A, and using Lemma 2.8, one gets a conservative homeomorphism as close as wanted to f, for which the intersection is transversal. □

4.3. **Markovian intersections.** The following result will provide us the $\mathcal{C}^0$ stability we seek, in order to create the open sets whose intersections will consist of the residual set of homeomorphisms satisfying the shadowing property. A proof of it can be found in [39]:

Proposition 4.3. *A Markovian intersection is $\mathcal{C}^0$ robust, namely if the intersection $f(R_1) \cap R_2$ has k Markovian components, then it is still true in a $\mathcal{C}^0$ neighbourhood of f.*

The following lemma shows how we will obtain periodic points for the periodic shadowing property. It is a simplified version of [39, Theorems 4, 16]. Their proof is based on an argument of homotopy and theory of degree, which we present below. We refer the reader to Appendx B for some basic facts on the topological theory of degree.

Lemma 4.5. *Let f be a homeomorphism and R be a rectangle such that $f(R) \cap R$ is a Markovian intersection. Then, there exists a fixed point for f in R.*

Proof. We assume $R = [-1, 1]^n \subset \mathbb{R}^n$. In the following, we denote by $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ the projection on the first coordinate and by $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection on the $(n-1)$-th last coordinates. We also define $R^+ = \{1\} \times [-1, 1]^{n-1}, R^- = \{-1\} \times [-1, 1]^{n-1}$.

By definition, there exists a strict horizontal subrectangle H of R such that $f(H)$ is a strict vertical subrectangle of R. Note that $f(H) \cap H$ is still a Markovian intersection, according to [14, Lemma 3.13]. Without loss of generality, we can assume that $f(H^+) \subset R^+$ and $f(H^-) \subset R^-$ (the other case being obtained by conjugating f to the hyperplanar symmetry whose plane of reflection is $\{\pi_1(x) = 0\}$).

We consider a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that: $\phi(H) = [-1, 1]^n$, $\phi(f(H^+)) \subset \{\pi_1(x) > 1\}$, $\phi(f(H^-)) \subset \{\pi_1(x) < -1\}$ and $\phi(f(H)) \subset \{|\pi_{n-1}(x)| < 1\}$. In the following, we will consider the application $\phi \circ f \circ \phi^{-1} : [-1, 1]^n \to \mathbb{R}^n$ and show that it has a fixed point. For the sake of simplicity, we will still denote this application f. Note that by hypothesis (see Figure 5):

(A) $f(R^-) \subset \{\pi_1(x) < -1\}$ and $f(R^+) \subset \{\pi_1(x) > 1\}$
(B) $f(R) \subset \{|\pi_{n-1}(x)| < 1\}$.

We consider a vertical segment $D_{y_0} = \{(\lambda, y_0), \lambda \in [-1, 1]\}$ for some $y_0 \in]-1, 1[^{n-1}$ and the following homotopy

$$h_t : [-1, 1]^n \to \mathbb{R}^n, x \mapsto f(\pi_1(x), ty_0 + (1-t)\pi_{n-1}(x)),$$

defined for $t \in [0, 1]$. Notice that $h_0 = f$ and $h_1 : x \mapsto f(\pi_1(x), y_0)$. By construction, $h_1(R) = f(D_{y_0})$, that is h_t retracts the image $f(R)$ onto the segment $f(D_{y_0})$ (see Figure 5).

The following homotopy

$$i_t : x \mapsto (\pi_1(h_1(x)), (1-t)\pi_{n-1}(h_1(x))),$$

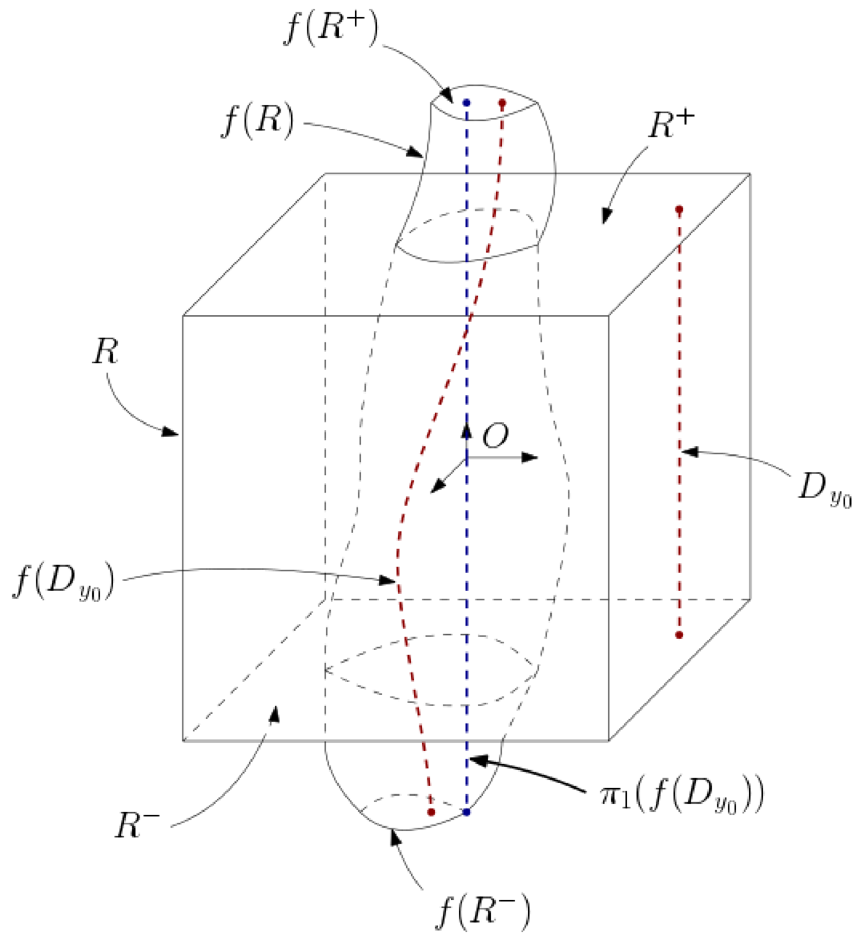

FIGURE 5. Illustration of the proof of Lemma 4.5

is such that $i_0 = h_1$ and $i_1 : x \mapsto (\pi_1(h_1(x)), 0)$. In other words, the homotopy i_t retracts the segment $f(D_{y_0})$ onto $(\pi_1(f(D_{y_0})), 0)$, its projection on the line $\{\pi_{n-1}(x) = 0\}$ (see Figure 5). Note that there exists a continuous function $\beta : [-1, 1] \to \mathbb{R}$ such that for all $x \in R$, $i_1(x) = (\beta(\pi_1(x)), 0)$. Therefore, the map $t \mapsto (i \cdot h)_t$[5] homotopes the application $(f - Id)$ to $((\beta \circ \pi_1, 0) - Id)$.

Moreover:

$$\deg((\beta, 0) - Id_{|\mathbb{R}^n}, R, 0) = (-1)^{n-1} \deg(\beta - Id_{|\mathbb{R}}, [-1, 1], 0)$$

By (A), we immediately have that $\deg(\beta - Id_{|\mathbb{R}}, [-1, 1], 0) = 1$, so $\deg((\beta, 0) - Id_{|\mathbb{R}^n}, R, 0) \neq 0$.

Now to conclude, it suffices to notice that there are no fixed points for $(i \cdot h)_t$ on the boundary of R, for any $t \in [0, 1]$. Indeed, by (B) the vertical faces are all sent strictly inside $\{-1 < \pi_{n-1}(x) < 1\}$ and by (A) the horizontal top and bottom

[5] $i \cdot h$ denotes the concatenation of the homotopies h then i.

faces are respectively sent in $\{\pi_1(x) > 1\}$ and $\{\pi_1(x) < -1\}$. Therefore, $\deg(f - Id, R, 0) \neq 0$ and f has a fixed point. □

5. Proof of the main result

5.1. **Proof.** In this section, we prove Theorem 3.3.

Proof of Theorem 3.3. Let us define:

$$A_{\varepsilon,n} = \left\{ f \in \mathcal{H} \;\middle|\; \begin{array}{l} f \text{ is robustly chained, robustly sealed} \\ \text{and } m\text{-nice for some } \mathcal{D}_m \text{ with } \chi(\mathcal{D}_m) < \varepsilon \end{array} \right\}$$

The proof of the result announced immediately follows from these two lemmas:

Lemma 5.1. *A_ε is open and dense in $\mathcal{H}$.*

Lemma 5.2. *If $f \in \cap_{p\in\mathbb{N}} A_{1/p}$, then f satisfies the special shadowing property.*

Proof of Lemma 5.1. We fix $\varepsilon > 0$. Let $f \in \mathcal{H}$ and $\kappa > 0$. We want to show that there exists $g \in A_\varepsilon$ such that $d(f,g) < \kappa$. We consider a dyadic subdivision $\mathcal{D}_m$ such that $\chi(\mathcal{D}_m) < \min(\varepsilon, \kappa, \omega(\kappa))$, where $\omega(\kappa)$ denotes the modulus of uniform continuity of f. Note that up to a small perturbation of f, by Proposition 4.1, we can always assume that f is m-nice. The proof is divided into 3 parts. First, we create Markovian intersections between each pair of cubes C_i and C_j such that $f(C_i) \cap C_j \neq \emptyset$ (1), and then we check that f is robustly chained (2) and robustly sealed (3).

(1) On each cube $C_i \in \mathcal{D}_m$, we consider the foliation $\mathcal{F}$ by vertical lines and the foliation $\mathcal{H}$ by horizontal hyperplanes. For each intersection $f(C_i) \cap C_j \neq \emptyset$, we look at both foliations $f(\mathcal{F} \cap C_i)$ and $\mathcal{H} \cap C_j$. By Proposition 4.2, we can always assume that, up to a small perturbation of f, this intersection is transversal. We denote by L_i^j and L'^j_i the leaves of $\mathcal{F} \cap C_i$ and $\mathcal{H} \cap C_j$, such that $f(L_i^j)$ intersects transversally L'^j_i.

Now, on each cube C_i, we consider a smooth path γ_i such that for every transverse intersection $L_i^j \pitchfork f^{-1}(L'^j_i) \neq \emptyset$, the path γ_i coincides with the leaf L_i^j on a small neighbourhood of $L_i^j \pitchfork f^{-1}(L'^j_i)$. Looking at the cubes C_k such that $f(C_k) \cap C_i \neq \emptyset$, we also consider a smooth codimension 1 submanifold σ_i such that for every transverse intersection $f(L_k^i) \pitchfork L'^i_k \neq \emptyset$, σ_i coincides with the leaf L'^i_k on a small neighbourhood of the transverse intersection $f(L_k^i) \pitchfork L'^i_k$. Remark that by construction, for every i and j, we have the transverse intersection $f(\gamma_i) \pitchfork \sigma_j$.

Then, we consider a δ tubular neighbourhood Γ_i of the path γ_i and a δ' tubular neighbourhood Σ_i of the submanifold σ_i, as well as two conservative homeomorphisms ϕ_i and ϕ'_i of the cube C_i, such that (see Figures 6 and 7):

- Γ_i and Σ_i have the same volume,
- ϕ_i and ϕ'_i have support in C_i,
- if we denote c_i the cube with same centre as C_i and same volume as Γ_i, we have $\phi_i(c_i) = \Gamma_i$ and $\phi'_i(c_i) = \Sigma_i$;
- the image of the vertical (resp. horizontal) faces of the cube c_i by ϕ_i (resp. ϕ'_i) is contained in a small neighbourhood of the boundary of γ_i (resp. σ_i).

As γ_i and σ_i are smooth, for δ (and thus δ') small enough, for any i,j such that $f(C_i) \cap C_j \neq \emptyset$, the rectangles $f(\phi_i(c_i))$ and $\phi'_j(c_j)$ have a Markovian intersection.

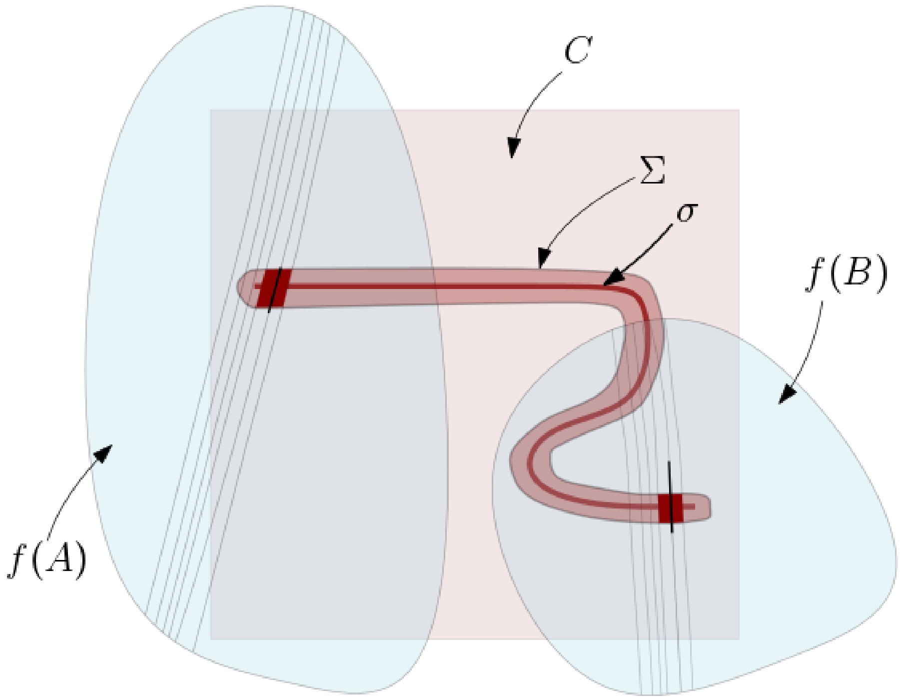

FIGURE 6. Cubes A and B have their image intersecting cube C. The vertical lines represent the image by f of the vertical foliation of cubes A and B. Σ is transversal to the image of the foliation by f in the dark red areas.

Then, we set:

$$g = \big(\prod_j (\phi'_j)^{-1}\big) f \big(\prod_i \phi_i\big),$$

By the inequality $\chi(\mathcal{D}_m) < \min(\varepsilon, \kappa, \omega(\kappa))$, and since the ϕ_i and ϕ'_i have their support[6] included in a single cube of the subdivision, one gets a homeomorphism which is κ-close to f and which has the property: for every i, there exists a sub-cube c_i of C_i, such that for every cube C_i and C_j such that $f(C_i) \cap C_j \neq \emptyset$, there is a Markovian intersection with a single component between $f(c_i)$ and c_j.

(2) The following Lemma shows that the conclusions of (1) implies that f is chained. By Proposition 4.3, this property remains true on a whole $\mathcal{C}^0$-neighbourhood of f, so f is robustly chained.

Lemma 5.3. *Assume that f is m-nice and that for any C_i, C_j such that $f(C_i) \cap C_j \neq$, there exists subcubes $c_i \subset C_i$ and $c_j \subset C_j$ such that $f(c_i)$ and c_j have a Markovian intersection. Then f is chained.*

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[6]The support of a homeomorphism ψ is defined as the closure of the largest set K such that $\psi|_K \neq Id$.

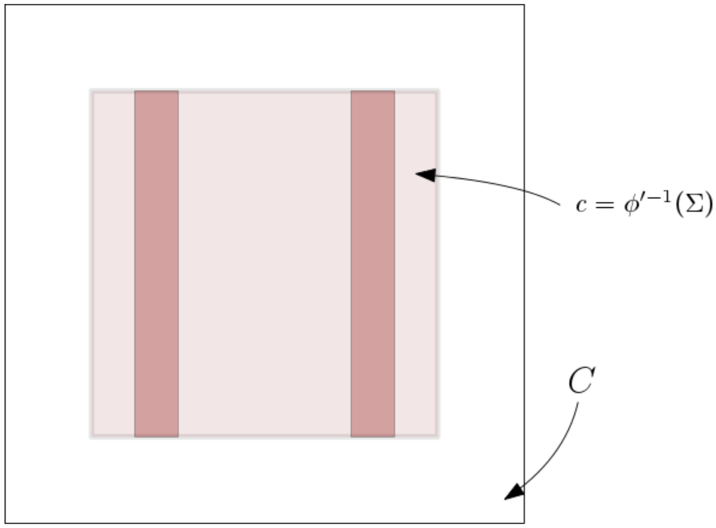

FIGURE 7. The application ϕ' unfolds the cube c in the manifold Σ. The previous dark red areas in Figure 6 are the image by ϕ' of the dark red areas above.

Proof. We show by recurrence on n that f is n-chained for any $n \geq 0$. The exact property that we will show is the following : given any n-chain $C_0 \to \cdots \to C_n$, there exists a strict horizontal subrectangle $\mathcal{R} \subset c_0$ such that $f^i(R) \subset \overset{\circ}{C}_i$, for any $0 \leq i \leq n$ and $f^n(R)$ is a strict vertical subrectangle of c_n.

The initialization for $n = 0$ is trivial. Assume the property is true at rank $n \geq 0$ and consider any $n+1$-chain $C_0 \to \cdots \to C_n \to C_{n+1}$. By hypothesis, there exists a strict horizontal subrectangle $R \subset c_0$ such that $f(R) \subset \overset{\circ}{C}_1, \ldots, f^n(R) \subset \overset{\circ}{C}_n$ and $f^n(R)$ is a strict vertical sub-rectangle of c_n. By construction, since $f(C_n) \cap C_{n+1} \neq \emptyset$, there exists a strict horizontal subrectangle H of c_n such that $f(H)$ is a strict vertical subrectangle of c_{n+1}. Since $f^n(R)$ is a strict vertical subrectangle of c_n, we know that $f^n(R) \cap H$ contains at least one connected component $\tilde{R}$ which is a strict vertical subrectangle of H and a strict horizontal subrectangle of $f^n(R)$. But therefore, $f(\tilde{R})$ is a strict vertical subrectangle of c_{n+1} and $\tilde{R}$ is a strict horizontal subrectangle of R, and subsequently of c_0. Moreover, since $\tilde{R} \subset R$, it also shadows the chain.

□

(3) Given an n-chain $C_0 \to \cdots \to C_n = C_0$, we know that there exists a rectangle $R \subset c_0$ such R shadows the chain and $f^n(c_0) \cap R$ has a connected component which is a strict vertical subrectangle of R. Therefore, By Lemma 4.5 applied with f^n, we know that there exists a fixed point for f^n in R, namely a periodic point of period n for f.

□

Proof of Lemma 5.2. Let us consider $f \in A_\varepsilon$. This gives us a subdivision $\mathcal{D}_m$ such that f is chained and m-nice for $\mathcal{D}_m$ and $\chi(\mathcal{D}_m) < \varepsilon$. We set:

$$\delta < \min_{f(C_i)\cap C_j=\emptyset} \text{dist } (f(C_i), C_j)$$

Note that the set of index on which the minimum is taken may be empty: in that case, any $\delta > 0$ works.

Consider any δ-pseudo orbit $(y_i)_{i\in\mathbb{Z}}$. We enumerate the cubes so that $y_i \in C_i$. Therefore, $f(C_i) \cap C_{i+1} \neq \emptyset$, otherwise we would have $d(f(C_i), C_{i+1}) > \delta$, which is impossible because we have a δ-pseudo orbit. Consider the finite δ-pseudo orbits $(y_i)_{-n\leq i\leq n}$. By construction of A_ε, there exists a point $x_n \in \overset{\circ}{C}_0$, such that $f^{-n}(x_n) \in \overset{\circ}{C}_{-n}, \dots, x_n \in C_0, \dots, f^n(x_n) \in \overset{\circ}{C}_n$. By construction, we have for all $-n \leq k \leq n$:

$$d(f^k(x_n), y_k) < \chi(\mathcal{D}_m) = \max_{1\leq i\leq p_m} \text{diam } C_i < \varepsilon$$

Therefore, x_n ε-traces the finite δ-pseudo orbit $(y_k)_{-n\leq k\leq n}$. Since M is a compact manifold, we can assume that $(x_n)_{n>0}$ converges towards x, which ε-traces $(y_k)_{k\in\mathbb{Z}}$.

In the case of a periodic δ-pseudo orbit, the shadowing by the real orbit of a periodic point follows immediately from the fact that f is chained and sealed.

Eventually, this shows that the G_δ set $\cap_{p\in\mathbb{N}} A_{1/p}$ is contained in the set of conservative homeomorphisms satisfying the special shadowing property.

□

This completes the proof of Theorem 3.3 (SP).

□

5.2. **Remarks.** As a conclusion to this section, we formulate a two remarks on the proof:

- The non-shadowing property also holds on a dense set in $\mathcal{H}$. Indeed, in [14] (pages 33-34), the density of the maps $f \in \mathcal{H}$ which have an iterate equal to the identity on an open set ($f^p = Id$ on an open set $V \subset M$, for some $p > 0$) is proved. This property contradicts immediately the shadowing property.
- Given a $\varepsilon > 0$, the proof also provides an upper-bound for the $\delta > 0$ that can be chosen. If $\mathcal{D}_m$ is a subdivision of diameter less than ε, then on can take:

$$\delta < \min_{f(C_i)\cap C_j=\emptyset} \text{dist } (f(C_i), C_j)$$

6. Genericity of the specification property

6.1. **Proof.** Recall that a homeomorphism f is *topologically mixing* if for any non-empty open set $U, V \subset M$, there exists a $N \in \mathbb{N}$ such that for any $n \geq N$, $f^n(U) \cap V \neq \emptyset$. In order to prove Theorem 1.2, we will need the following result:

Theorem 6.1 (Guihéneuf)**.** *Topologically mixing homeomorphisms are generic in $\mathcal{H}(M, \mu)$.*

For a proof of this result, we refer the reader to [14]. Therefore, thanks to Theorem 1.1 and 4.1, we know that generically a conservative homeomorphism is topologically mixing and has the periodic shadowing property. We are going to show that this implies the specification property.

Corollary 3.6. Let f be topologically transitive and have the periodic shadowing property. We fix $\varepsilon > 0$. Therefore, there exists a $\delta > 0$ such that every δ-periodic pseudo orbit is $\varepsilon/2$-shadowed and we can always assume $\delta < \varepsilon/2$. By uniform continuity of f, there exists a $\eta > 0$ such that: $\text{dist}(x, y) < \eta$ implies $\text{dist}(f(x), f(y)) < \delta$. We consider a dyadic subdivision $\mathcal{D}_m = (C_i)_{1 \leq i \leq p_m}$ of maximum diameter less than $\max(\eta, \delta)$. Since f is topologically mixing, given any pair of cubes C_i, C_j, we know that there exists a $N(i,j) \geq 0$ such that for all $n \geq N(i,j)$, $f^n(C_i) \cap C_j \neq \emptyset$. Denote $N(\varepsilon) = \max_{i,j} N(i,j)$.

Let us consider any sequence of points $x_1, ..., x_n$ and any sequence of strings $A_1, ..., A_n$ like in the definition of the specification property. Note that we can always assume $a_1 = 0$. Take a $p > b_n + N(\varepsilon)$.

We consider the δ-periodic pseudo orbit $(y_n)_{0 \leq n \leq p}$ defined in the following way. First, for all $a_1 = 0 \leq j \leq b_1$, $y_j = f^j(x_1)$. Then consider a cube C containing $f^{b_1}(x_1)$ and a cube $\tilde{C}$ containing $f^{a_2}(x_2)$. Since $a_2 - b_1 > N(\varepsilon)$ and f is topologically mixing, we know that there exists a $z \in C$ such that $f^{a_2 - b_1}(z) \in \tilde{C}$. Take for the δ-pseudo orbit, $y_j = f^{j-b_1}(z), b_1 + 1 \leq j \leq a_2 - 1$, and $y_{a_2} = f^{a_2}(x_2)$. Since $\text{dist}(z, f^{b_1}(x_1)) < \text{diam } C < \eta$, we know that $\text{dist}(y_{b_1+1}, f(y_{b_1})) < \delta$ and we also have $\text{dist}(y_{a_2}, f(y_{a_2-1})) < \text{diam } \tilde{C} < \delta$. Then, one simply has to iterate the same process with the different $x_i, i \geq 2$. In the end, the point $f^{b_n}(x_n)$ is brought back in a neighbourhood of x_1 in time $p - b_n > N(\varepsilon)$, and the δ-pseudo orbit is made periodic.

By this construction, we therefore have a δ-periodic pseudo orbit $(y_n)_{0 \leq n \leq p}$ which $\varepsilon/2$-shadows (it is actually δ-shadowed but $\delta < \varepsilon/2$) each piece of orbit $\Lambda_i = \left\{f^j(x_i), j \in A_i\right\}, 1 \leq i \leq n$. In turn, this δ-periodic pseudo orbit is $\varepsilon/2$-shadowed by the real orbit of a periodic point y^* of period p, by the periodic shadowing property. By triangular inequality, the real orbit of the periodic point y^* ε-shadows each piece of orbit Λ_i. □

Remark 6.2. It appeared in a discussion with A. Arbieto that the implication "topologically mixing & shadowing property ⇒ specification property" is actually a well-known fact in topological dynamics since [13]. Since this proof was redacted before this discussion, we left it here intentionally.

To conclude, we want to remark that the specification property does not hold generically in the dissipative case. Indeed, it is possible to show in this case, that there exists an open dense set of homeomorphisms who possess at least two different periodic attractive orbits, whose basin of attraction has non-empty interior. One can check that this contradicts the specification property.

6.2. **About measures.** The specification property is a very strong property which substantially impacts the ergodic properties of a homeomorphism f. It allows to investigate the space $\mathcal{M}_f$ of f-invariant measures in greater details. For the proofs of the various results stated in this paragraph, we refer the reader to [13, Chapter 21].

If f satisfies the specification property, then it can be easily shown that f has a positive topological entropy (see [13, Proposition 21.6]). This shows, in particular, that generic homeomorphisms in $\mathcal{H}(M, \mu)$ have positive topological entropy. However, this result does not have a great interest since it has been shown, as mentionned in the introduction to this memoir, that generic homeomorphisms in $\mathcal{H}(M, \mu)$ actually have infinite topological entropy (see [14] for instance).

In general, given any dynamical system f (not necessarily satisfying the specification property), the set of f-invariant measures with full support is either empty or a dense G_δ in $\mathcal{M}_f$. Since we only consider $f \in \mathcal{H}(M,\mu)$, we have obviously $\mu \in \mathcal{M}_f$ and therefore the set of f-invariant measures with full support is not empty, so it is a dense G_δ.

More specifically, when f satisfies the specification property, then one can show (see [13, Propositions 21.9, 21.10]) that the set of ergodic and nonatomic measures is a dense G_δ in $\mathcal{M}_f$.

Remark 6.3. This situation is quite peculiar, and impossible to picture. Recall that if $T : M \to M$ is a continuous application, then $\mathcal{M}_T(X)$ is a compact convex space (see Proposition C.3). The extremal points of this set consist of the T-ergodic measures ... but these measures are dense and even uncountable (see Proposition A.1) in $\mathcal{M}_T(X)$!

We tried to sum up the situation Figure 6.2, where we suggest a cartography of the set $\mathcal{M} = \cup_{f\in\mathcal{H}(M,\mu)} \{f\} \times \mathcal{M}_f$. $\mathcal{R} \subset \mathcal{H}(M,\mu)$ denotes the residual set of μ-conservative homeomorphisms satisfying the specification property. Given, $f \in \mathcal{R}$, $\mathcal{R}_f \subset \mathcal{M}_f$ denotes the residual set of f-invariant, fully supported, non-atomic and ergodic measures. For $f \in \mathcal{H}(M,\mu)$, the sets $\mathcal{M}_f$ intersect at least in the point μ which is invariant by all $f \in \mathcal{H}(M,\mu)$.

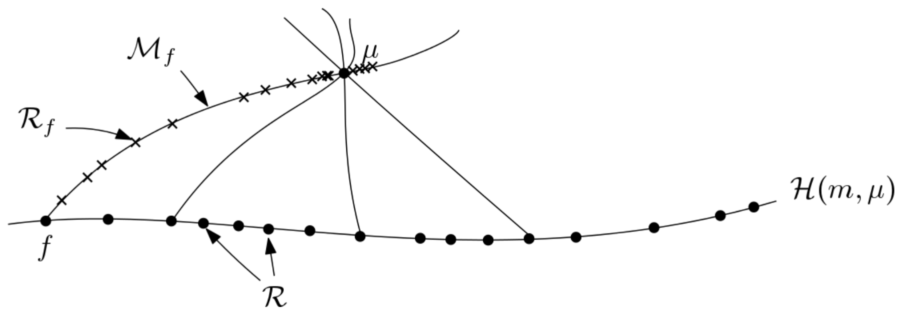

FIGURE 8. A cartography of $\mathcal{M}$

In 1941, J. Oxtoby and S. Ulam proved one of the first significant theorems in the theory of conservative homeomorphisms, namely that a generic conservative homeomorphism is ergodic. This result actually initiated the theory of generic dynamics. As an interesting remark, we thought that this cartography was surprisingly enough close to the Oxtoby-Ulam theorem in the sense that there exists a residual set of homeomorphisms f, such that μ can be approximated by non-atomic, fully supported, f-invariant and ergodic measures. In the case of the Oxtoby-Ulam theorem, the point μ would be contained in the sets $\mathcal{R}_f$, for $f \in \mathcal{R}$, a residual subset of $\mathcal{H}(M,\mu)$.

7. ON OTHER SHADOWING PROPERTIES

There exists quite a few alternate definitions of shadowing and their meaning is usually slightly changing according to the authors: inverse shadowing, weak shadowing, limit shadowing, two-sided limit shadowing, average shadowing, asymptotic average shadowing ... In this paragraph, we discuss on the last four of them.

7.1. **Average shadowing property.** The side notion of *average pseudo orbit* arises from the fact that the behaviour of the roundoff error of a computation can often be described by the realization of independant Gaussian random perturbations with zero mean. In other words, this notion allows large deviations in the distance between $f(x_k)$ and x_{k+1}, as long as they are rare enough and balanced by a number of very small deviations.

Definition 7.1 (Average δ-pseudo orbit)**.** Given a $\delta > 0$, we say that a sequence $(x_n)_{n\in\mathbb{N}}$ is an *average δ-pseudo orbit* for the homeomorphism f if there exists an integer $N > 0$ such that for all $n \geq N$ and $k \geq 0$, we have:

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathrm{dist}(f(x_{i+k}), x_{i+k+1}) < \delta$$

Therefore, some authors naturally introduced the notion of *average shadowing.*

Definition 7.2 (ε-shadowing in average)**.** Given $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ is a *ε-shadowed in average* by the point x^* if:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{dist}(f^i(x^*), x_i) < \varepsilon$$

Definition 7.3 (Average shadowing property)**.** We say that a homeomorphism f satisfies the *average shadowing property* if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that any average δ-pseudo orbit is ε-shadowed in average by the real orbit of a point.

In a recent work, M. Kluczycki, D. Kwientniak and P. Oprocha proved, among some other results, that a homeomorphism satisfying the shadowing property and topologically mixing was also satisfying the average shadowing property and the asymptotic average shadowing property (see [21, Theorem 3.8]). We therefore obtain the following corollary:

Corollary 7.4. *A generic element in $\mathcal{H}(M, \mu)$ satisfies the average shadowing property and the asymptotic average shaodiwng property.*

7.2. **Limit shadowing property.** This other notion of shadowing is interested in the behaviour of the pseudo orbit and its shadowing as time goes to infinity.

Definition 7.5 (Limit δ-pseudo orbit)**.** We say that $(x_n)_{n\in\mathbb{Z}}$ is a limit δ-pseudo orbit if it is a δ-pseudo orbit such that $\lim_{n\to+\infty} \mathrm{dist}(f(x_n), x_{n+1}) = 0$. It is a two-sided limit δ-pseudo orbit if $\lim_{|n|\to+\infty} \mathrm{dist}(f(x_n), x_{n+1}) = 0$.

Definition 7.6 (Limit shadowing property)**.** We say that f satisfies the limit shadowing property if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that any limit δ-pseudo orbit is ε-shadowed by the real orbit of a point x^* such that $\lim_{n\to+\infty} \mathrm{dist}(f^n(x^*), x_n) = 0$. Moreover, if we have $\lim_{n\to-\infty} \mathrm{dist}(f^n(x^*), x_n) = 0$, then wa say that f satisfies the two-sided limit shadowing property.

In their article [11], B. Carvalho and D. Kwietniak proved that the two-sided limit shadowing is among the strongest notion of shadowing that can be defined. In particular, they prove (see [11, Theorem B]) that two-sided limit shadowing implies topological mixing, specification, average shadowing and asymptotic average shadowing. In a recent article [22], M. Mazur and P. Oprocha proved that limit shadowing was $\mathcal{C}^0$ dense in the set of homeomorphisms and suggested that it may not be generic. As a conclusion of this part, we raise here two interesting open questions:

Question 7.1. Is the limit shadowing property generic for (conservative) homeomorphisms?

Question 7.2. Can we say something about density or genericity of the two-sided limit shadowing property?

Part 3. Ergodicity in the dissipative case

This part initially aimed to provide an answer to a question raised by C. Bonatti and E. R. Pujals, and stated in the article [1] of F. Abdenur and M. Andersson, namely: on a compact manifold, is the dynamics of a homeomorphism $\mathcal{C}^0$-densely *finitely wonderful*[7] ? We were able to provide a partial answer to this question, that is to show that $\mathcal{C}^0$-densely a homeomorphism is *finitely δ-almost wonderful*. We discuss this result in the last section.

The terminology used throughout this article mostly refers to the definitions introduced by F. Abdenur and M. Andersson in their paper [1]. We refer the reader to their article for further details. λ will denote the[8] Lebesgue measure on M.

8. Introduction

8.1. **A historical lack.** Since the work of J. Oxtoby and S. Ulam in the 1940's, it is a well-known fact that generically a conservative homeomorphism on a compact manifold (for a good Borel probability measure μ) is ergodic. In the $\mathcal{C}^0$ topology, on which we will focus in the rest of this part, ergodic properties of generic volume-preserving homeomorphisms are now quite well understood (see S. Alpern and V. S. Prasad's classical book [4] for elementary techniques of this theory, and [14] for a general survey on the subject), as well as topological properties of generic dissipative homeomorphisms (see [3]). However, the study of ergodic properties of generic dissipative homeomorphisms has long been ignored and no equivalent of Oxtoby-Ulam's theorem had been found in the dissipative case, until very recently. Indeed, in 2012, F. Abdenur and M. Andersson proved in [1] a result which put an end to this long-standing question, and explained the generic ergodic behaviour of a dissipative homeomorphism. Before stating their theorem, we recall a few definitions which will be essential throughout the rest of this part.

8.2. **Elementary definitions.** We refer the reader to the Appendix D.1 for a summary of the properties of the space of Borel probability measures of a compact metric space.

Definition 8.1 (Birkhoff average)**.** Given a continuous dynamical system $f : M \to M$, and a point $x \in M$, the Birkhoff limit of the point x is, under the condition it exists, the probability measure

$$\mu_x = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)},$$

where δ_y denotes the Dirac probability measure supported on y, and the limit is understood in the weak-* topology.

Of course, when this limit exists, it is, in particular, an f-invariant probability measure. Moreover, it is characterized by the following condition: given any continuous function $\varphi : M \to M$, the average $\lim_{n\to\infty} \dfrac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x))$ coincides with the integral $\int_M \varphi \, d\mu_x$.

---

[7]A proper definition of this notion will be given in the following section.

[8]In the sense of the paragraph 2.1.1.

Definition 8.2 (Totally singular maps)**.** We say that a map f is totally singular with respect to the Lebesgue measure λ, if there exists a Borel measurable set Λ such that $\lambda(\Lambda) = 1$ and $\lambda(f^{-1}(\Lambda)) = 0$.

In other words, f maps a set of measure 0 to a set of measure 1.

Definition 8.3 (Basin of attraction)**.** Given any probability measure μ on M, we define its basin to be the set

$$B(\mu) = \{x \in M, \mu_x \text{ is well-defined and coincides with } \mu\}$$

We say that the measure μ is physical if $\lambda(B(\mu)) > 0$.

One can formulate a few remarks about this definition :

(1) Since μ_x, if it exists, is f-invariant, only invariant measures can have a non-empty basin. In particular, a physical measure is necessarily f-invariant. But physical measures are not necessarily ergodic.

(2) If x is a periodic point of period p (for f), then its orbit supports a unique f-invariant probability measure $\mu_x = \dfrac{1}{p}\sum_{k=0}^{p-1} \delta_{f^k(x)}$ (which coincides with Definition 8.1).

(3) If x is an *asymptotically stable* (AS) periodic point of period p, namely there exists a neighbourhood V of x in M such that any $y \in V$ satisfies $f^{pn}(y) \to x$, as $n \to \infty$, then the basin of attraction of $\mu_x = \dfrac{1}{p}\sum_{k=0}^{p-1} \delta_{f^k(x)}$ contains the union $\cup_{k=0}^{p-1} f^k(V)$, and is therefore physical since m is strictly positive on any open set.

8.3. **Ergodic behaviour.** In their paper, F. Abdenur and M. Andersson introduce a new terminology to characterize some remarkable (either pathological or well-behaved) behaviours of ergodic dynamics. We do not reproduce here the five "w-definitions" they suggest, but only focus on the two most important (in our perspective) ones. We also introduce a new "almost"-type definition.

Definition 8.4. A continuous dynamical system $f : M \to M$ is said to be:

- **weird** if m-a.e. point $x \in M$ has a well-defined Birkhoff limit μ_x, but f is totally singular and admits no physical measure,
- **wonderful** if there exists a finite or countable family of physical measure μ_n such that $\lambda(\cup_n B(\mu_n)) = 1$,
- **δ-almost wonderful**, for some $0 < \delta \ll 1$, if there exists a finite or countable family of physical measure μ_n such that $\lambda(\cup_n B(\mu_n)) > 1 - \delta$.

In particular, one can check that a homeomorphism cannot be, at the same time, weird and wonderful. But there also exists homeomorphisms which are neither weird nor wonderful (*wacky* homeomorphisms for exemple, see [1]). In the general case, for a compact manifold of dimension $n \geq 2$, F. Abdenur and M. Andersson succeeded to prove the following result:

Theorem 8.5 (Abdenur-Andersson)**.** *A generic dissipative* $f \in \text{Homeo}(M)$ *is weird.*

In other words, the Birkhoff average of almost every point $x \in M$ well-behaves: at least, it converges towards a limit-measure whose support is, by definition, contained in the orbit of x. The inexistence of physical measure is actually due to the fact that given two points x and y, Lebesgue-almost surely their Birkhoff average will exist, but will not coincide.

The article also points out an interesting question, first raised by C. Bonatti and E. Pujals:

Question 8.1. Is the dynamics of a dissipative homeomorphism $\mathcal{C}^0$-densely finitely wonderful ?

For the $\mathcal{C}^0$ topology, this question can be seen as an analogue to the Palis' conjecture stated in the introductory part of this memoir, even though the $\mathcal{C}^0$ topology presents *a priori* less difficulties than the $\mathcal{C}^1$ topology.

Of course, since a homeomorphism cannot be weird and wonderful at the same time, and by genericity of the weird homeomorphisms, a generic homeomorphism cannot be wonderful. However, there are many examples of generic properties which do not hold on a dense set and a positive answer to this question would therefore not contradict Abdenur-Andersson's theorem.

A partial answer to this question is already known. Indeed, it has been shown by R. Bowen in [8] that $\mathcal{C}^\infty$ diffeomorphisms are densely finitely wonderful. Since the work of J. Munkres in the 1960's, it is known that in dimension $n \leq 3$ any homeomorphism can be uniformly approximated by $\mathcal{C}^\infty$ diffeomorphisms, thus providing a complete positive answer to Question 8.1 in dimension ≤ 3. Since the short survey [25] of S. Müller compiling and extending already-existing results of approximation, we know that in dimension $n \geq 5$, a homeomorphism can be uniformly approximated by diffeomorphisms if and only if it is isotopic to a diffeomorphism (see Part 4, Section 12 for further details).

As a consequence, the question stated above is still open in dimension $n \geq 4$. The aim of this part is to provide a partial answer to it. We will prove the following result:

Proposition 8.1. *δ-almost wonderful homeomorphisms are dense in* $\mathrm{Homeo}(M)$*, for any* $0 < \delta \ll 1$.

F. Abdenur and M. Andersson's proof of Theorem 1 is mostly based on their Shredding Lemma (see [1, Section 5]) which gives them very easily the genericity of weird homeomorphisms. In his thesis (see [14, Chapter 4]), P.-A. Guihéneuf suggests a much shorter proof of the Shredding Lemma, based on an argument *à la* Oxtoby-Ulam. We will follow his ideas in our proof of Theorem 8.1.

9. Main result

9.1. **Starting tools.** By Oxtoby-Ulam's theorem, we know that given a subdyadic division $\mathcal{D}_m$ of M, it will behave under any homeomorphism on M like the collection of cubes on I^n endowed with the Lebesgue measure. Therefore, in the sequel, we can always assume that M is the unit cube $[-1,1]^n$ endowed with the Lebesgue measure λ.

Definition 9.1 (m-centre-nice homeomorphism)**.** We say that $f \in \mathrm{Homeo}(M)$ is a *m-centre-nice* homeomorphism for an integer $m > 0$ if, given the dyadic subdivision $\mathcal{D}_m = (C_m)_{1 \leq m \leq p_m}$, we have for all $i, j \in \{1, ..., p_m\}$, either $f(e_i) \notin C_j$, or $f(e_i) \in \mathring{C}_j$. We say that f is *centre-nice* if it is m-centre-nice for every $m > 0$.

Proposition 9.1. *The set of m-centre-nice homeomorphisms is open and dense in* Homeo(M)*. Therefore, the set of centre-nice homeomorphisms is generic in* Homeo(M).

Proof. It is similar to that of Proposition 4.1. □

Given two compacts $K, K' \subset M$, we will denote by $K \subset\subset K'$ the fact that there exists an open set O such that $K \subset O \subset K'$ and we will say that K is strictly included in K'. Given a cube $C \subset I^n$ centred in x, the cube $C(\delta)$ (for some $\delta > 0$) will refer to the cube centred in x whose lengths have been diminished by 2δ:

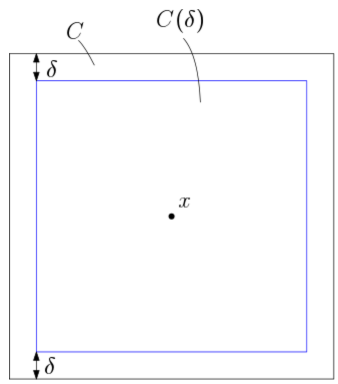

FIGURE 9. Definition of the set $C(\delta)$.

9.2. **Proof.** We first state the following lemma:

Lemma 9.2. *Let $f \in$ Homeo(I^n) and x be a stable periodic point of period p for f, in the sense that there exists a closed cube C containing x such that $f^p(C) \subset\subset C$ (S). Then, there exists a $\tilde{f} \in$ Homeo(I^n) such that $d(f, \tilde{f}) <$ diam (C) and p is asymptotically stable in the sense (AS).*

Proof. Up to a shrinking and a small perturbation of f, we can always assume that the iterates $C, f(C), ..., f^{p-1}(C)$ are disjoint and x is the center of C. We denote by τ_α the homeomorphism which is the contraction of factor $\alpha > 1$ on C and the identity on $I^n - C$. If C were the whole unit cube, this contraction would be expressed by the formula:

$$\phi_\alpha(x) = |x|_\infty^\alpha x$$

In our case, one could also easily compute an analytic expression of τ_α, but it will not be useful to the rest of our proof. Consider a homeomorphism h defined as:

$$h = \begin{cases} Id, & \text{on } M - C \\ \tau_\alpha \circ \sigma \circ f^{-p}, & \text{on } f^p(C) \end{cases}$$

Here σ is a possible change of the orientation according to the fact that f reverses, or not, the orientation. Note that there exists at least one such h by the local modification lemma. We set $\tilde{f} = h \circ f$. Since $h = Id$ on $M - C$, we have immediately that $d(f, \tilde{f}) <$ diam (C). Moreover, one can check that for any $y \in \mathring{C}$, we have dist$(\tilde{f}^{pn}(y), x) \to 0$, as $n \to \infty$. Therefore, x is an asymptotically stable periodic point for $\tilde{f}$ with basin $\mathring{C}$. □

This result shows that, up to a small perturbation, a periodic point can be made attractive. According to (3) of the remarks formulated in the introduction, this technique actually creates a basin of attraction with a strictly positive Lebesgue measure. We now prove Proposition 8.1.

Proof. Let $f \in \mathrm{Homeo}(I^n)$ and $\varepsilon > 0$. By uniform continuity of f, there exists a $\eta > 0$ such that: dist $(x, y) < \eta$ implies $\mathrm{dist}(f(x), f(y)) < \varepsilon/3$. By Proposition 2, we can always assume f is centre-nice. Let $\mathcal{D}_m = (C_i)_{1\le i\le p_m}$ be a subdivision of diameter less than $\min(\eta, \varepsilon/3)$. We denote by e_i the centres of the cubes C_i. We define the map $\sigma : (e_i)_{1\le i\le p_m} \to (e_i)_{1\le i\le p_m}$ such that $\sigma(e_i) \in C_{\sigma(e_i)}$. It is well-defined since f is centre-nice (and therefore m-centre-nice). For each centre e_i, we consider a small cube E_i, centred in e_i, such that $f(E_i) \in \mathring{C}_{\sigma(e_i)}$. Now, we consider a $\delta > 0$ (uniform) small enough, so that for each cube C_i, the cube $C_i(\delta)$ strictly contains each $f(E_j)$, if $\sigma(e_j) = e_i$.

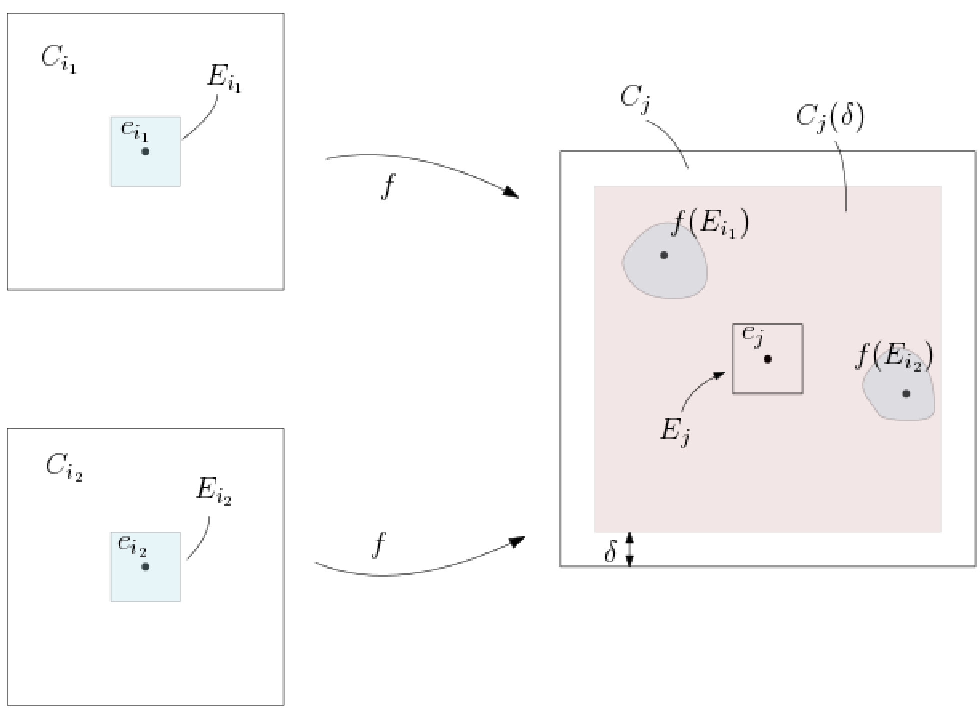

FIGURE 10. Definition of the sets E_i and $C_i(\delta)$. In this case, $\sigma(e_{i_1}) = \sigma(e_{i_2}) = e_j$.

Now, let us consider the homeomorphism τ_α defined as the contraction of factor α on each cube of the subdivision. We take a α big enough so that for each C_i, we have $\tau_\alpha(C_i(\delta)) \subset\subset E_i$. Now, we consider the map $f_1 = f \circ \tau_\alpha$. By construction, we have $d(f, f_1) < \chi(\mathcal{D}_m) < \varepsilon/3$. Given a point $x \in M$, we are interested in the sequence of cubes that the iterates of the point x (by f_1) are going to visit. By our construction, we know that if $x \in C_i(\delta)$, then the positive iterates of x will visit the same cubes as the centre e_i.

Consider any centre e of some cube C of the subdivision. We re-enumerate the cubes so that $f_1^i(e) \in C_i$. Since there are only a finite number of cubes, we will necessarily have $f_1^i(e) \in C_i = \tilde{C} = C_{i'} \ni f_1^{i'}(e)$ for some $i' > i$. We consider the first two such integers, denote $p = i' - i$ and $\tilde{e}$ the centre of $\tilde{C}$. Note that by our construction, we actually have $f_1^i(e) \in C_i(\delta)$, for any $i \geq 0$. Therefore, the points $f_1^i(e)$ and $f_1^{i'}(e)$ have the same sequence of cubes as the point $\tilde{e}$, and since $f_1^p(f_1^i(e)) = f_1^{i'}(e)$, we know that this sequence is p-periodic. As a consequence, any x in some $C(\delta)$ has a sequence of cubes which eventually ends up being periodic. In the following, we consider again the standard enumeration of the dyadic subdivision.

We now consider the centres of cubes $(e_i)_{i \in I}$ which have a periodic sequence of cubes, namely if $i \in I$, then there exists a period $p > 0$ such that $f_1^p(e_i) \in C_i(\delta)$. Up to a small perturbation θ_i with support strictly contained in $C_i(\delta)$, we can assume that each e_i is periodic of period p_i for $f_2 = (\theta_i)_{i \in I} \circ f_1$. By triangular inequality, we have $d(f_2, f) < \dfrac{2}{3}\varepsilon$.

In particular, each e_i is a periodic point for f_2 and $f_2(C_i(\delta)) \subset\subset C_i(\delta)$. By Lemma 1, we can modify locally f_2 on each $C_i(\delta)$ to obtain a f_3 such that $d(f_3, f_2) < \varepsilon/3$ and $C_i(\delta)$ is asymptotically stable for f_3. Of course, since any x in some $C(\delta)$ eventually falls in a $C_i(\delta)$ with $i \in I$, by our remark (*), it will be attracted by the orbit of e_i. If we denote by μ_i the Birkhoff average of the periodic point $e_i, i \in I$, then we have:

$$\lambda\left(\bigcup_{i \in I} B(\mu_i)\right) > \lambda\left(\bigcup_{i \in I} C_i(\delta)\right) = 1 - O_{\delta \to 0}(\delta)$$

Note that the cardinal of I is independant of the δ that we choose (however it depends on ε). And moreover, by triangular inequality, we have $d(f, f_3) < \varepsilon$.

□

10. Discussion

The following paragraphs are quite informal and tend to motivate hopes for a better result, rather than providing an accurate mathematical material.

10.1. Obstruction to a complete proof.

We set:

$$\mathcal{C} = \bigcup_{i=1}^{p_m} C_i - \overset{\circ}{C}_i(\delta)$$

According to the previous proof, we are able to control the dynamics of f, except on the set

$$\Lambda = \bigcap_{n \geq 0} f^{-n}(\mathcal{C}),$$

which consists of the points $x \in I^n$ which never fall in a $C_i(\delta)$ and are therefore never attracted by one of the $e_i, i \in I$. Let us call the behaviour of these points the *bad behaviour*. By definition, Λ is f-invariant, namely $f(\Lambda) \subset \Lambda$, and one can actually check by an elementary argument that $f(\Lambda) = \Lambda$. Of course, if $\lambda(\Lambda) = 0$, the proof would be over, but it is clearly not the case in general.

Therefore, one could try to iterate the previous technique in a small open neighbourhood of the set Λ, in order to decrease its measure. It is very likely that by this technique, one may obtain the $\mathcal{C}^0$-density of wonderful maps in the end. However, it is also obvious that such a technique would lead to an explosion of the number of basins, which we want to avoid.

Remark 10.1. After some discussions at IMPA, it appears that A. Avila would have completely solved the question by using a tricky key-lemma so that, in the end, $\lambda(\Lambda) = 0$. This result is not yet published.

10.2. **Towards a better result?** However, it is very likely that this result could be improved by showing that not only, $\mathcal{C}^0$-densely, the set $M - \cup_n B(\mu_n)$ is of Lebesgue-measure 0, but that it is of Hausdorff dimension 0 or even of topological dimension 0. We refer the reader to Appendix D for a definition of these notions. We introduce here the notion of expansive homeomorphisms and show how this could be related to our problem.

Definition 10.2. A homeomorphism f is said to be *expansive* if there exists a $c > 0$, such that for any $x, y \in M$, there exists a $n \in \mathbb{Z}$, such that $\mathrm{dist}(f^n(x), f^n(y)) > c$. It is *positively expansive* if n can be chosen in $\mathbb{N}$.

Expansive homeomorphisms are a prototype of chaos since, by definition, they are extremely sensitive to initial conditions. The theory of expansive homeomorphisms has enriched since the 1980's with the work of R. Mãné and P. Walters, but some important questions are still open. For the reader's curiosity, we state one of them. We recall that an Anosov diffeomorphism satisfies the shadowing property (see [19, Theorem 18.2] for a reference), and one can show that it is actually expansive.

Definition 10.3. We define a topological Anosov (TA) homeomorphism as an expansive homeomorphism satisfying the shadowing property.

One of the first result about TA homeomorphism is due to P. Walters who showed that a TA homeomorphism is *topologically structurally stable*: there exists a $\mathcal{C}^0$ neighbourhood of a TA homeomorphism f such that any g in this neighbourhood is semi-conjugated to f. On the other hand, it is known that topological structural stability implies the shadowing property. But it is still a question to determine whether a topologically structurally stable is expansive on its non-wandering set $\Omega(f)$ (which is equal to the whole manifold in the conservative case) or not. This result was proved by R. Mãné in the $\mathcal{C}^1$ case.

Back to our initial problem, R. Mãné proved the following theorem:

Theorem 10.4 (Mãné)**.** *If $f : X \to X$ is an expansive homeomorphism of a compact metric space, then every minimal set for f is topologically zero-dimensional.*

Therefore, one attempt could be to modify locally f on $\mathcal{C}$, so that, in the end, the application f exhibits an expansive behaviour on the set Λ of points which never fall in a $C_i(\delta)$. However, this appears to be rather tricky: for instance, it is very likely that the use of a local modification theorem in an open neighbourhood of Λ would not work. Indeed, this result acts as a "black box" since it allows no control of the application on the set Σ (in the notations of Lemma 2.8), which could therefore exhibit wild behaviours on this set.

Part 4. Genericity for conservative flows

11. Introduction

11.1. Motivations for the problem. Surprisingly enough, it seems that the study of generic $\mathcal{C}^0$ flows has been left unturned in the field of dynamical systems. Of course, these flows do not proceed *a priori* from a physical equation since they are not obtained by integration of a vector field. In particular, they may exhibit some very exotic behaviours. But one could argue that some important physical transformations exist, which do not guarantee the good definition of a speed vector at any time, and thus, of a vector field generating the flow. For instance, the Brownian motion of a particle is a very classical illustration of a trajectory which is everywhere continuous but almost-everywhere non-derivable. Following this idea, one can actually construct a continuous dynamical system whose orbits are almost everywhere non-derivable.

11.2. Obstructions to a $\mathcal{C}^0$ theory. In the case of continuous flows, the absence of vector fields leads to important difficulties when one tries to assess the behaviour of such a transformation:

- One of the most simple and fundamental notion of the $\mathcal{C}^1$ theory, that is the existence of transverse sections to the flow, misses with this lack of regularity, which makes it harder to understand the local action of the flow.
- More crucial is the absence of any obvious means of perturbation of such a flow, like Lemma 2.8. Indeed, "perturbing" a continuous flow $(\phi^t)_{t\in\mathbb{R}}$ can mostly be obtained by conjugating it with a homeomorphism – thus not affecting its topological properties – but it is hard to use any other means, mostly because of commutation issues. Since we wish to show the genericity of dynamical properties (properties invariant by a conjugacy), this will not be of great interest.

Note that evaluating if another continuous flow $(\psi^s)_{s\in\mathbb{R}}$ commutes with $(\phi^t)_{t\in\mathbb{R}}$ is much harder than in the regular case, because the theory of Lie brackets does not hold at this level of regularity.

11.3. Approximating a dynamical system. However, instead of looking at the continuous flow $(\phi^t)_{t\in\mathbb{R}}$, one could be interested in looking at a regular flow which approximates it. Then, by perturbing this smoothed flow, that is by perturbing the vector field generating it – which is much easier! –, one would obtain a perturbation of the initial system, thus providing a similar technique to Lemma 2.8 for the study of generic properties.

This could be made possible if a theorem of uniform approximation of continuous flows by $\mathcal{C}^\infty$ flows existed, but this is not the case. In a section below, we conjecture this result, but we are unable to prove it. To the best of our knowledge, this is the first time that this conjecture is stated in literature.

Before detailing it, we provide a brief survey on approximation results in the case of homeomorphisms and discuss how this can be related to the approximation of flows. In particular, we complete an argument due to J.-C. Sikorav on the uniform approximation of volume-preserving homeomorphisms by volume-preserving diffeomorphisms.

Remark 11.1. As mentioned briefly in the introductory part, we set ourselves in the natural frame for diffeomorphisms, that is volume forms and not good measures anymore. In the following, we will even assume that M is a smooth connected compact Riemannian manifold of dimension $n \geq 2$. It is very likely that some of the results presented below still hold with less regularity.

12. Approximation results for homeomorphisms

12.1. **Dissipative case.** As defined in the introductory part, we recall that $d(\cdot,\cdot)$ denote the $\mathcal{C}^0$ distance between applications. Given f, a homeomorphism on the manifold M, a natural question one can ask is the following:

Question 12.1. Given $\varepsilon > 0$, is there a diffeomorphism g such that $d(f,g) < \varepsilon$, that is, g is ε-close to f in the $\mathcal{C}^0$ topology?

If the answer is positive, then we will say that f can be uniformly approximated by diffeomorphisms. Similar results for other classes of applications are well-known facts in differential topology (see [16, Chapter 2], for instance):

- In the case of $\mathcal{C}^0$ applications between manifolds (not requiring any property of bijection), it is known that given any continuous application $f : M \to N$ between two smooth manifolds M and N and any $\varepsilon > 0$, there exists a $\mathcal{C}^\infty$ application $g : M \to N$ such that $d(f,g) < \varepsilon$.
- Given $f : M \to N$, a $\mathcal{C}^1$ diffeomorphism between two smooth manifolds, there exists $g : M \to N$, a $\mathcal{C}^\infty$ diffeomorphism such that $d(f,g) < \varepsilon$. Therefore, it is not even necessary to make a distinction between a $\mathcal{C}^1$- or a $\mathcal{C}^\infty$-approximation.

Question 12.1 is almost solved and the only case which is still open is the case of dimension 4[9]. Indeed, in his paper [25], S. Müller shows the following result, using the fundamental work of J. Munkres in the 1960's and recent advances:

Theorem 12.1 (Munkres-Connell-Bing-Müller)**.** *If $n \leq 3$, then any homeomorphism can be uniformly approximated by diffeomorphisms. If $n \geq 5$, then a homeomorphism can be uniformly approximated by diffeomorphisms if and only if it is isotopic to a diffeomorphism.*

12.2. **Conservative case.** Assume the answer to Question 12.1 is positive and f preserves a certain structure such as a volume form. Then one can ask whether these approximating diffeomorphisms can be chosen such that they preserve this structure or not. The following result of uniform approximation of a conservative homeomorphism by volume-preserving diffeomorphisms is a "well-known" fact:

Theorem 12.2. *If f is a conservative homeomorphism, with regard to a volume form ω, which can be uniformly approximated by diffeomorphisms, then the diffeomorphisms can be chosen conservative.*

Actually, this result has been a folklore for a while. The only first real proof was given in 2006 by Y.-G. Oh in his paper [29] but his proof is very sophisticated and goes along with other estimate results in Moser's theorem. In 2007, J.-C. Sikorav suggested a new proof for this "elementary" result which was completely independent and much easier. A slight argument was missing to complete the proof, which we provide here:

[9] As we mentionned before, this was also the last open case for the *annulus* theorem.

Proof. Let φ be a homeomorphism preserving the smooth volume element $\omega \in \Omega^n(M)$, and which can be uniformly approximated by diffeomorphisms. We fix $\varepsilon > 0$. Let K be a smooth triangulation, such that diam $(\varphi(\sigma)) < \varepsilon$, for each n-simplex of $K^{(n)}$. For any $\eta > 0$, there exists $\psi^\eta \in \mathrm{Diff}^\infty(M)$ such that $d(\varphi, \psi^\eta) < \eta$. In particular :

$$\max_{\sigma \in K^{(n)}} |\mathrm{vol}(\psi^\eta(\sigma)) - \mathrm{vol}(\sigma)| = O(\eta)$$

There exists a smooth function $f^\eta : M \to \mathbb{R}^+$ such that for any $\sigma \in K^{(n)}$:

$$\int_{\psi^\eta(\sigma)} f^\eta \omega = \int_\sigma \omega$$

Note that f^η can be chosen uniformly close to the constant function $\mathbf{1} : M \to \mathbb{R}$, as $\eta \to 0$. Moreover, since $\psi^\eta(K)$ is another smooth triangulation of the manifold, we have :

$$\int_M f^\eta \omega = \sum_{\sigma \in K^{(n)}} \int_{\psi^\eta(\sigma)} f^\eta \omega = \sum_{\sigma \in K^{(n)}} \int_\sigma \omega = \int_M \omega$$

Therefore, $f^\eta \omega$ and ω are smooth volume forms with same total volume. By Moser's celebrated result (Theorem 2.3), we know that there exists a $\phi^\eta \in \mathrm{Diff}^\infty(M)$ such that $(\phi^\eta)_* (\omega) = f^\eta \omega$. Since f^η can be chosen uniformly close to $\mathbf{1}$ as $\eta \to 0$, ϕ^η can be chosen uniformly close to the identity as $\eta \to 0$. For some η small enough, if we set $\Psi^1 = \phi^\eta \circ \psi^\eta$, we have :

$$d(\varphi, \Psi_1) + \max_{\sigma \in K^{(n)}} \mathrm{diam}\ \Psi^1(\sigma) < \varepsilon \tag{12.1}$$

And for all $\sigma \in K^{(n)}$:

$$\mathrm{vol}(\Psi^1(\sigma)) = \int_{\phi^\eta \circ \psi^\eta(\sigma)} \omega = \int_{\psi^\eta(\sigma)} \phi^\eta_*(\omega) = \int_{\psi^\eta(\sigma)} f \omega = \int_\sigma \omega = \mathrm{vol}(\sigma)$$

The rest of the proof is identical to [36]. We copy it here to provide a coherent proof of the result to the reader. The idea is that if a diffeomorphism preserves the volume of each simplex of the triangulation, then it can be locally modified on the interior of each simplex so that it becomes volume-preserving. By construction, the n-form $\alpha = {\Psi^1}_* \omega - \omega$ satisfies $\int_\sigma \alpha = 0$ for any n-simplex $\sigma \in K^{(n)}$. By [37, Lemma 2, p. 148], it has a primitive β_1 such that $\int_\tau \beta_1 = 0$, for every $\tau \in K^{(n-1)}$. Thus β_1 is exact on $K^{(n-1)}$ and there exists $\gamma \in \Omega^{n-2}(M)$ such that $\beta_1 = d\gamma$ on $K^{(n-1)}$. We consider $\beta = \beta_1 - d\gamma$. It is a primitive of α that vanishes on $K^{(n-1)}$.

Moser's method applied to the path $\omega_t = \omega + t d\beta$ provides a time-dependant vector field X_t such that $\omega_t(X_t, \cdot) = -\beta$, whose integration gives an isotopy θ_t with $\theta_0 = Id$, $(\theta_t)_* \omega_t = \omega$, so that $\Psi = \Psi^1 \circ \theta_1$ preserves ω. Moreover, since $\beta|_{K^{(n-1)}} = 0$, we have $X_t = 0$ on $K^{(n-1)}$ and $\theta_t = Id$ on $K^{(n-1)}$. Therefore, $\Psi|_{K^{n-1}} = \Psi^1$ and by 12.1, we have $d(\Psi, \varphi) < \varepsilon$ which concludes the proof.

□

The case of higher regularity for diffeomorphisms has already been investigated. Actually, it is known that smooth diffeomorphisms are $\mathcal{C}^1$-dense among $\mathcal{C}^1$-diffeomorphisms (see [16, Theorem 2.7], for a reference). In the conservative case, this result is only very recent and was proved in 2010 by A. Avila in [6].

12.3. **Towards the flows.** Earlier, in 1979, an approximation result had been proved for conservative vector fields in a paper by C. Zuppa (see [41]) : divergence-free smooth vector fields are $\mathcal{C}^k$-dense in divergence-free $\mathcal{C}^k$ vector fields for any integer $k \geq 1$. The proof is quite elementary and mainly relies on the use of convolution for vector fields. But it seems that a question has not been adressed so far in literature.

Question 12.2. Given a (conservative) $\mathcal{C}^0$-flow, can it be uniformly approximated by a (conservative) $\mathcal{C}^\infty$-flow?

From the previous paragraphs, we can already formulate a few remarks. Indeed, one can observe that if a homeomorphism is isotopic to the identity, then it can be uniformly approximated by diffeomorphisms according to Theorem 12.1. As a consequence, any element ϕ^t, at a fixed $t \in \mathbb{R}$, of a continuous flow can be individually uniformly approximated by diffeomorphisms. But then, the question is whether these approximations embed in a flow or not. Since a work of Palis [28], we know that only few diffeomorphisms embed in a flow: they actually form a meagre set (in the Baire sense) in the set of diffeomorphisms. It is still an open question to characterize the diffeomorphisms that can be embeded. On this question, some progress have been made recently with the work of X. Zhang [40], but it does not seem that the question stated above can be tackled from this point of view.

13. Approximation of continuous flows

We are unable to provide a complete and positive (or negative!) answer to Question 12.2. Therefore, we state it as a conjecture and present a few elements which could initiate a proof, at least in the conservative case.

Conjecture 13.1. *Let Φ be a $\mathcal{C}^0$ flow, $a, b \in \mathbb{R}$ such that $a < b$ and $\varepsilon > 0$. Then, there exists a $\mathcal{C}^\infty$ flow Ψ such that:*

$$\sup_{t \in [a,b]} d(\Phi^t, \Psi^t) < \varepsilon$$

To the best of our knowledge, this conjecture has never been made (nor proved!) in literature so far. In the following, we assume that the flow $(\phi_t)_{t \in \mathbb{R}}$ is conservative regarding a good measure μ, that is, each homeomorphism ϕ^t, for $t \in \mathbb{R}$ preserves μ. We try to approach the flow ϕ by a $\mathcal{C}^\infty$ flow ψ. We will not care whether ψ preserves the measure μ or not.

Ideas. Without loss of generality, we can always assume that $[a, b] = [0, T]$, for some $T > 0$. We take T small enough such that, for any $x \in M$, the orbit $x \xrightarrow{\Phi} \Phi^T(x)$ is contained in a single local chart. This is always possible by uniform continuity of the application $(t, x) \mapsto \phi^t(x)$ on any compact $[-a, a] \times M$, $0 < a < \infty$.

We fix $\varepsilon > 0$. Let $\eta > 0$ be such that, by uniform continuity of Φ, we have: dist $(x, y) < \eta$ and $|t - t'| < \eta$ implies dist $(\Phi^t(x), \Phi^{t'}(y)) < \varepsilon/3$. We may also assume that $\eta < \varepsilon/3$.

By Poincaré's recurrence theorem, we know that μ-almost every point in M is recurrent, which means, since μ is a good measure, that the set of recurrent point is at least dense in M. Consider a recurrent point $z \in M$ and t_z such that $d(z, \Phi^{t_z}(z)) < \eta/10$. We claim, by uniform continuity of Φ, that there exists a $\mathcal{C}^\infty$

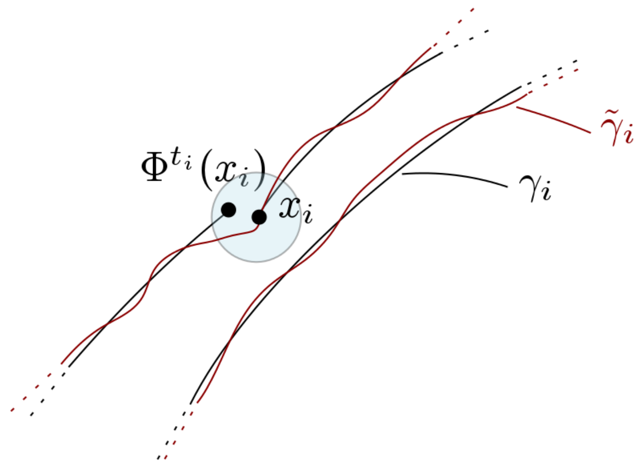

FIGURE 11. Closing the orbits

closed injective curve $\tilde{\gamma}$ and a flow $\tilde{\Phi}$ defined on it such that :

$$\forall x \in \tilde{\gamma}, \forall t \in [0,T], \mathrm{dist}(\Phi^t(x), \tilde{\Phi}^t(x)) < \eta/5 < \varepsilon/3$$

Now, using this technique, we consider a finite cover of M by the sets $B(\gamma_i, \eta/2)_{1 \leq i \leq N}$, defined in the following way. γ_i is defined as the orbit $x_i \xrightarrow{\Phi} \Phi^{t_i}(x_i)$ of a recurrent point x_i, where t_i is chosen to be the first (positive) time when the orbit of x_i comes back to the ball $\overline{B}(x_i, \eta/10)$. $B(\gamma_i, \eta/2)$ is defined as the set $\{y \in M, \exists x \in \gamma_i, \mathrm{dist}\ (x,y) < \eta/2\}$. Furthermore, we request that the orbits γ_i are all disjoint. Such a cover exists, thanks to the density of recurrent points. We also consider smooth approximating curves $\tilde{\gamma}_i$ and smooth approximating flows $\tilde{\Phi}_i$ defined on them just like in the previous paragraph.

The dynamics of the points $x \in \tilde{\gamma}_i$ under Φ_i is controlled and approximates well the dynamics of the flow Φ. Note that each flow $\tilde{\Phi}_i$ gives a smooth vector field X_i defined on the curve $\tilde{\gamma}_i$. By Tietze's extension lemma, there exists at least one smooth vector field Y defined on M, such that $Y|_{\tilde{\gamma}_i} = X_i$, since the $\tilde{\gamma}_i$ are all closed and disjoint. We define

$$\mathcal{F} = \{X, \mathcal{C}^\infty \text{ vector field s.t. } X|_{\tilde{\gamma}_i} = X_i\} = Y + \{X, \mathcal{C}^\infty \text{ vector field s.t. } X|_{\tilde{\gamma}_i} = 0\}$$

The problem is now to control the dynamics of a point $x \in M$ which does not belong to any $\tilde{\gamma}_i$. Take any $X \in \mathcal{F}$ and denote by Ψ its flow. If x_i denotes the projection of x on the closest curve $\tilde{\gamma}_i$, we have by triangular inequality:

$$\mathrm{dist}(\Psi^t(x), \Phi^t(x)) \leq \mathrm{dist}\ (\Psi^t(x), \Psi^t(x_i)) + \mathrm{dist}\ (\Psi^t(x_i), \Phi^t(x_i)) + \mathrm{dist}\ (\Phi^t(x_i), \Phi^t(x))$$

By construction, the last two terms are controlled by $\varepsilon/3$. One idea could be to construct a X such that the first term is controlled by $\varepsilon/3$. But how to find such a X ?

14. Perturbation techniques for flows

For a class of dynamical systems $\mathcal{S}$ endowed with a certain topology, proving that a property $\mathcal{P}$ (which we identify to the set of elements of $\mathcal{S}$ satisfying this property) is generic mainly consists in two steps:

- identifying the open sets $O_n \subset \mathcal{S}$ such that $\bigcap_n O_n \subset \mathcal{P} \subset S$ and proving that they are open (though this is not usually the hardest part!),
- proving that the sets O_n are dense.

Therefore, in the case of continuous flows, if Conjecture 13.1 happened to be true, this would mean that proofs of density for the sets O_n could be carried out assuming the flow considered is $\mathcal{C}^\infty$ and not just $\mathcal{C}^0$, thus enabling the use of perturbation techniques of smooth flows (and vector fields). In this section, we present some techniques of perturbations which may be used in that case on a $\mathcal{C}^\infty$ flow, especially in the conservative case.

Note that we will now assume that $n = \dim M \geq 3$. We will also restrict ourselves to volume forms. Therefore, by conservative, we mean that the flow induced by a vector field X preserves this volume form, or in an equivalent way that $\operatorname{div}(X) = 0$.

14.1. **Flow-box theorem for a volume-preserving vector field.** In this section, we present a conservative version of the celebrated flow-box theorem, stating that a $\mathcal{C}^k, k \geq 1$ vector field which does not vanish in p is $\mathcal{C}^k$-conjugated in a neighbourhood of p to a constant vector field.

Theorem 14.1 (Flow-box theorem)**.** *Let X be a $\mathcal{C}^k$ vector field, $k \geq 1$, $p \in M$ such that $X(p) \neq 0$ and Σ be a transverse section to X containing p. There exists a neighbourhood Ω of p and $\varphi : \Omega \to \varphi(\Omega) \subset \mathbb{R}^n$ such that $\varphi(p) = 0$ and:*

(1) $\varphi_* X = (1, 0, ..., 0)$
(2) $\varphi(\Omega \cap \Sigma) \subset \{x_1 = 0\}$

In other words, this means that the flow generated by X is locally conjugated to a translation in a neighbourhood of p. Note that C. Calcaterra and A. Boldt proved in [10] an extension of this theorem to the case of Lipschitz vector fields. The homeomorphism φ they obtain, which gives the conjugacy, is a lipeomorphism.

We now investigate the case when X is a conservative vector field. *A priori*, the diffeomorphism φ which gives the conjugacy is not volume-preserving. The following proposition states that it is actually possible to find such a φ which is conservative:

Theorem 14.2 (Flow-box theorem (conservative case))**.** *Let X be a $\mathcal{C}^k$ vector field, $k \geq 1$, $p \in M$ such that $X(p) \neq 0$ and Σ be a transverse section to X containing p. There exists a neighbourhood Ω of p and $\varphi : \Omega \to \varphi(\Omega) \subset \mathbb{R}^n$ such that $\varphi(p) = 0$ and:*

(1) $\varphi_* X = (1, 0, ..., 0)$
(2) $\varphi_* \omega = dx_1 \wedge ... \wedge dx_n$
(3) $\varphi(\Omega \cap \Sigma) \subset \{x_1 = 0\}$

To the best of our knowledge, there are mainly two proofs of this result. One is given in V. Araù and M. J. Pacifico's book [5] on three dimensional vector fields. Their proof, although it can be extended to any dimension, is quite sophisticated insofar as it relies on Moser and Dacorogna's theorem on an equation with determinant (see [24]) and requires a regularity at least $\mathcal{C}^2$ for the vector field. In their paper [12], F. Castro and F. Oliveira provide a new elementary proof for this result for $\mathcal{C}^\infty$ vector fields in any dimension. However, their argument can immediately be adapted to $\mathcal{C}^k$ vector fields, ≥ 1, and we present it below. Before that, we thought that the following question may be of interest:

Question 14.1. Is there an equivalent of C. Calcaterra and A. Boldt's result (see [10]) on Lipschitz vector fields in the conservative case ?

Proof. We first straighten out X in a neighbourhood U_1 of p with the $\mathcal{C}^k$-diffeomorphism $\varphi : U_1 \subset M \to V_1 \subset \mathbb{R}^n$ satisfying 1. and 2. of Theorem 14.1. We define $\varphi_*\omega = \eta = \psi(z)dz_1 \wedge ... \wedge dz_n$, where $\psi : V_1 \to \mathbb{R}$ is a $\mathcal{C}^k$ function, which we can always assume to be positive (up to a change of the orientation), and $Z = (Z_1, ..., Z_n) = (1, 0, ..., 0) = \varphi_*X$. We know that (see [2, Proposition 2.5.23], for instance):

$$\operatorname{div}_\eta(Z) = \operatorname{div}_{dz_1\wedge...\wedge dz_n}(Z) + \frac{L_Z\psi}{\psi} = \sum_{i=1}^n \frac{\partial Z_i}{\partial z_i}(z) + \frac{1}{\psi(z)}\sum_{i=1}^n \frac{\partial \psi}{\partial z_i}(z)Z_i(z) = \frac{\partial \psi}{\partial x_1}(z),$$

where L_Z denotes the Lie derivative. Moreover, since X preserves ω, Z preserves η, so for all $z \in V_1$:

$$\operatorname{div}_\eta(Z) = \frac{\partial \psi}{\partial x_1}(z) = 0$$

Therefore, we can write $\psi(z_1, ..., z_n) = f(z_2, ..., z_n)$. Let us define $\xi : V_1 \to \mathbb{R}^n$ by:

$$\xi(z_1, ..., z_n) = (z_1, \int_0^{z_2} f(t, z_3, ..., z_n)\ dt, z_3, ..., z_n)$$

If z' denotes $(z_2, ..., z_n)$, its Jacobian is of the form:

$$\det D\xi(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & f(z') & * & * & \cdots & * \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

Therefore $\xi_*Z = Z$. Moreover, $\xi(0) = 0$ and $\det D\xi(0) = f(0) = \psi(z_1, 0, ..., 0) > 0$. As a consequence, there exists a $V_2 \subset V_1$, such that $\xi : V_2 \to V_3$ is a $\mathcal{C}^{k+1}$-diffeomorphism. Now, setting $(y_1, ..., y_n) = \xi(z_1, ..., z_n)$, we have:

$$dy_1 \wedge ... \wedge dy_n = \det D\xi\ dz_1 \wedge ... \wedge dz_n = \psi\ dz_1 \wedge ... \wedge dz_n = \eta$$

This means exactly that $\xi_*(\varphi_*\omega) = \xi_*(\psi\ dz_1 \wedge ... \wedge dz_n) = dy_1 \wedge ... \wedge dy_n$. Setting $\Omega = \varphi^{-1}(V_2)$ and $\beta : \Omega \to V_3,\ x \mapsto \xi \circ \varphi(x)$, we obtain the result.

□

14.2. **Example of perturbation.** As suggested by the proof of Lemma 2.8 (see [14], for instance), perturbations in the conservative case are usually harder to produce than in the dissipative case. The same phenomenon occurs here for flows. In this paragraph, we show how the previous conservative version of the flow-box theorem can help generating a conservative perturbation. Recall that in our strategy, if an approximation of conservative continuous flows by conservative smooth flows existed, this step would consist in the perturbation of the smoothed flow. The following result is a slight adaptation of [12, Lemma 4.1]. We state it in the case where M is three-dimensional because it is visually significant, but it can naturally be extended to any greater dimension.

Lemma 14.3 (Castro-Oliveira)**.** *Let X be a smooth conservative vector field and $x \in M$ a regular point. Let $\mathcal{U} \ni x$ be an open set given by the conservative flow-box theorem, on which the flow can be straightened out by the diffeomorphism φ, and $y \in \mathcal{U}$ which does not belong to the same transverse section as x. Then, there exists a smooth conservative vector field Y such that $X = Y$, outside $\mathcal{U}$ and the positive orbit of x contains y. Moreover, we can choose $||\varphi_* X - \varphi_* Y|| < h/t$ (see Figure 14.2).*

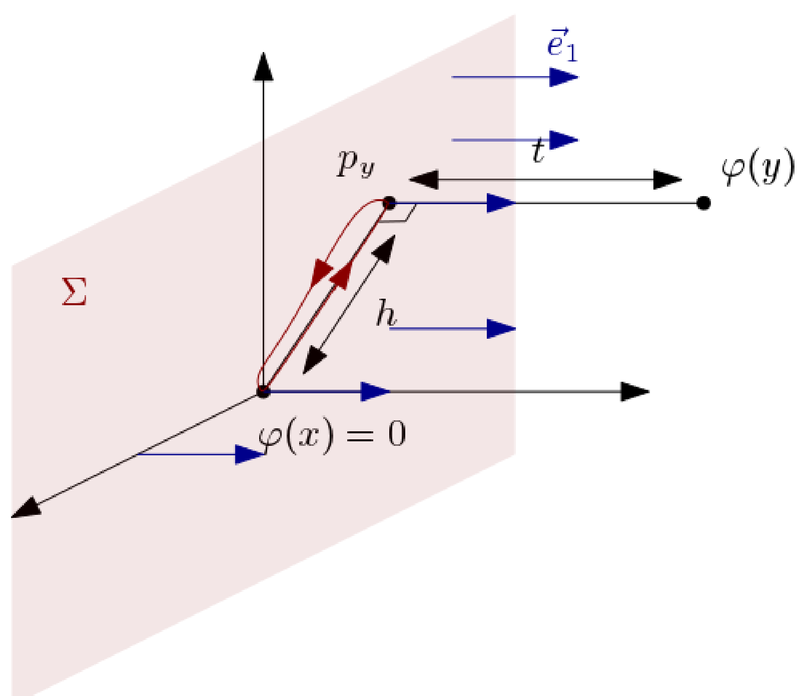

FIGURE 12. X is straightened out in the blue constant vector field. γ is the path in dark red, contained in Σ.

Ideas of proof: We explain the main ideas of the perturbation technique. A detailed demonstration can be found in [12].

We consider a closed smooth injective loop γ contained in the plane Σ (in dark red in Figure 14.2) such that the segment $[0, p_y]$ is included in γ. Consider the vector field Z, tangent to γ, of constant norm h/t. Since γ is smooth, this vector field can be thickened in a neighbourhood of γ by the following trick. Consider $x \in \gamma$ and the plane Σ_x passing through x and perpendicular to $Z(x)$. We also consider a smooth bump function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 1$ and $\psi \equiv 0$ outside $[-\delta, \delta]$ for a δ small enough. For $y \in \Sigma_x \cap B(x, \delta)$, we define $Z(y) = \psi(||y - x||)Z(x)$, and we set $Z \equiv 0$ for all other y. It is easy to compute that the flow Z is conservative. Informally, this holds because no point accumulates mass: mass either rotates around γ, or stays still.

Now, considering the flow $\tilde{X} = \varphi_*^{-1}(\varphi_* X + Z)$, one gets a conservative flow such that y is in the positive orbit of x and the inequality announced holds. □

Remark 14.4. Note that by Gronwall's lemma, we can link the perturbation on the vector field to the perturbation on the flow it generates. If $||X - Y||_\infty < \varepsilon$, then we have $d_{\mathcal{F}}(\phi_X, \phi_Y) \lesssim \varepsilon \cdot e^{||dX||_\infty}$

Part 5. Conclusion

As a conclusion to this memoir, we sum up the open questions that we have encountered throughout this work:

- Generic dynamics:
  - Is the limit shadowing property generic for (conservative) homeomorphisms?
  - Is the two-sided limit shadowing property generic for (conservative) homeomorphisms?
- Ergodic and chaotic behaviour:
  - Can we prove that $\mathcal{C}^0$-densely the *bad behaviour* of a wonderful homeomorphism is of Hausdorff or topological dimension 0?
  - Is a TA homeomorphism expansive on its non-wandering set?
- Continuous flows:
  - Is it possible to approximate uniformly a continuous flow by a smooth flow on a compact manifold?
  - If a continuous conservative flow can be uniformly approximated by smooth flows, can we choose them conservative?
  - Is there a conservative version of the flow-box theorem for Lipschitz vector fields?

Appendix: miscellaneous elements of topology

Appendix A. Baire spaces

In this paragraph, we recall the elementary definitions at the root of the theory of generic dynamics. In particular, we reproduce the proof of Baire's theorem.

Definition A.1 (G_δ, F_σ sets)**.** Let X be a topological space. We call G_δ (resp. F_σ) a set of X which is a countable intersection (resp. union) of open (resp. closed) sets. We call *residual* (resp. meagre) a (resp. nowhere dense F_σ) dense G_δ set. We say that a property is *generic* in X, if it is satisfied on a residual set.

Definition A.2 (Baire space)**.** A Baire space is a topological space in which a countable intersection of dense open sets is dense.

Definition A.3 (Polish space)**.** A Polish space is a separable completely metrizable topological space.

In particular, a Polish space is a Baire space. Indeed, this is due to the celebrated Baire's theorem :

Theorem A.4 (Baire)**.** *Every complete metric space is a Baire space.*

Proof. We denote by dist the distance on the complete metric space X. Let $V_1, V_2, ...$ be open dense set of X and W an open sets of X. We have to show that $\cap_n V_n \cap W \neq \emptyset$. We denote by $B(x,r)$ the ball centred in $x \in X$ of radius r.

Since V_1 is dense, $W \cap V_1$ is an non-empty open set so we can find x_1 and r_1 such that $\overline{B}(x_1, r_1) \subset W \cap V_1$ and $0 < r_1 < 1$. If $n \geq 2$ and x_{n-1} and r_{n-1} are chosen, the density of V_n shows that $V_n \cap B(x_{n-1}, r_{n-1})$ is non-empty and we can find x_n and r_n such that $\overline{B}(x_n, r_n) \subset V_n \cap B(x_{n-1}, r_{n-1})$ and $0 < r_n < 1/n$.

By recurrence, we construct a sequence $(x_n)_{n \geq 1}$ in X. Moreover, if $i, j > n$, x_i and x_j are both in $B(x_n, r_n)$ so that $\text{dist}(x_i, x_j) < 2/n$. The sequence has the Cauchy property and X is complete so it converges to a point x^* belonging to each V_n and to W by construction. This completes the proof. □

Some other examples of Baire spaces are given by locally compact Hausdorff spaces, or by open sets of a Baire space. The following result shows that in a Baire space without isolated points, a dense G_δ is a relatively "big" space:

Proposition A.1. *If X is a Baire space without isolated points, no countable dense set is a G_δ.*

Proof. Assume that there exists a countable dense set $E = \{x_k\}$ which is a G_δ. Therefore, $E = \cap_n V_n$, where the V_n are open and dense. If we set

$$W_n = V_n - \bigcup_{k=1}^{n} \{x_k\},$$

then the W_n are still open and dense (since X does not have isolated points). But $\cap_n W_n = \emptyset$, which is absurd. □

Appendix B. The topological degree

This paragraph details the tools used in the proof of Lemma 4.5.

B.1. **Local Brouwer degree.** The reference for this paragraph is [18].

We consider a continuous map $f : \Omega \to \mathbb{R}^n$ and an open domain $\mathcal{U} \subset \Omega$. We assume that f is d-compact, that is $f^{-1}(0) \cap \mathcal{U}$ is compact. If $\overline{\mathcal{U}} \subset \Omega$ and $\overline{\mathcal{U}}$ is compact, this is equivalent to $0 \notin f(\partial D)$.

Definition B.1. Given f defined as above, we define the following properties:

- Localization: Let $i : \mathcal{V} \to \mathcal{U}$ be the inclusion of an open subset satisfying $f^{-1}(0) \subset \mathcal{V}$. Then $\deg(f_{|\mathcal{V}}) = \deg(f)$.
- Units: Let $i : \mathcal{U} \to \mathbb{R}^n$ be the inclusion. Then:

$$\deg(i) = \begin{cases} 1, & \text{if } 0 \in \mathcal{U} \\ 0, & \text{if } 0 \notin \mathcal{U} \end{cases}$$

- Additivity: If $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$, $f_{|\mathcal{U}_1}, f_{|\mathcal{U}_2}$ are d-compact and $\mathcal{U}_1 \cap \mathcal{U}_2$ is disjoint from $f^{-1}(0)$, then:

$$\deg(f) = \deg(f_{|\mathcal{U}_1}) + \deg(f_{|\mathcal{U}_2})$$

- Homotopy invariance: Let $F : \mathcal{U} \times [0,1] \to \mathbb{R}^n$ be a d-compact homotopy, that is each $f_t = F(\cdot, t)$ is d-compact. Then:

$$\deg(f_0) = \deg(f_1)$$

- Multiplicativity: Let $f : \mathcal{U} \to \mathbb{R}^n, f' : \mathcal{U}' \to \mathbb{R}^p$ be two d-compact maps. We consider $F : \mathcal{U} \times \mathcal{U}' \to \mathbb{R}^n \times \mathbb{R}^p$. Then:

$$\deg(F) = \deg(f) \times \deg(f')$$

We are now able to define the topological degree of a continuous map. For a proof of the following result, we refer the reader to [18].

Theorem B.2. *There exists a unique application* $\deg : (f, \mathcal{U}) \mapsto \deg(f) \in \mathbb{Z}$ *defined for d-compact maps and satisfying the five previous properties: localization, units, additivity, homotopy invariance and multiplicativity.*

We now present two useful properties of the degree.

Proposition B.1. *Let* $f : \mathcal{U} \to \mathbb{R}^n$ *be a d-compact map. If* $\deg(f) \neq 0$*, then there exists a point* $x \in \mathcal{U}$ *such that* $f(x) = 0$.

Proposition B.2. *Assume* $f : \mathcal{U} \to \mathbb{R}^n$ *is a* $\mathcal{C}^1$ *map and* 0 *is a regular value for* f*. Then:*

$$\deg(f) = \sum_{x \in f^{-1}(0)} \operatorname{sgn}(\det(d_x f))$$

The theory of degree can easily been extended to the case of continuous applications between manifolds. The previous proposition is often used as a definition of the degree for differentiable applications. In order to extend it to the continuous applications, one has to use the fact that any continuous map is homotopic to a differentiable map, and that almost every point is regular for a differentiable application (Sard's theorem).

B.2. **Degree of a map $S^n \to S^n$.** We consider a continuous application $f : S^n \to S^n$, where S^n denotes the n-sphere. Recall that the n-th group of homology of the sphere $H^n(S^n)$ is isomorphic to $\mathbb{R}$, via the canonical map $u \mapsto \int_{S^n} u$. Therefore, the homomorphism $f^* : H^n(S^n) \to H^n(S^n)$ is of the form $f^*(u) = \alpha \cdot u$ for $\alpha \in \mathbb{R}$, for any $u \in H^n(S^n)$. The following proposition is a classical exercise in differentiable topology:

Proposition B.3. *For a continuous map $f : S^n \to S^n$, the degree* $\deg(f)$ *defined in the previous paragraph coincides with the number α.*

The two following propositions are used in Part 2, in order to prove Lemma 4.5. Here B^n denotes the closed unit ball of $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$.

Proposition B.4. *Let $f : B^n \to \mathbb{R}^n$ be a continuous map such that $0 \notin f(S^{n-1})$. We define*

$$s_f : \begin{cases} S^{n-1} \to S^{n-1} \\ x \mapsto \dfrac{f(x)}{||f(x)||} \end{cases}$$

Then:

$$\deg(f) = \deg(s_f)$$

In particular, for $n = 0$ and $S^0 = \{-1, 1\}$, the degree of $f : S^0 \to S^0$ is:

$$\deg(f) = \begin{cases} 1, & \text{if } f(1) = 1, f(-1) = -1, \\ -1, & \text{if } f(1) = -1, f(-1) = 1, \\ 0, & \text{otherwise} \end{cases}$$

Appendix C. The space $\mathcal{M}(X)$

Let X be a topological space. We denote by $\mathcal{M}(X)$ the set of Borel probability measure on X. In this paragraph, we introduce the topology on $\mathcal{M}(X)$.

Definition C.1 (Weak-* topology)**.** We say that a sequence $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}(X)^{\mathbb{N}}$ converges to μ for the weak-* topology if and only if $\mu_n(\phi) \to_{n\to\infty} \mu(\phi)$, for any $\phi \in \mathcal{C}(X)$. We denote this by $\mu_n \to^* \mu$.

Note that this actually defines the closed sets of the topology.

Proposition C.1. *If X is compact, then $\mathcal{M}(X)$ is compact. In other words, given any sequence $(\mu_n)_{n\in\mathbb{N}}$, one can find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a measure μ such that $\mu_{n_k} \to^* \mu$.*

We assume now that X is a compact metric space. This implies, in particular, that the space $\mathcal{C}(X)$ of continuous functions endowed with the uniform topology is separable, that is, it admits a dense family $(f_n)_{n\in\mathbb{N}}$.

Definition C.2. We define the following metric on $\mathcal{M}(X)$:

$$\forall \mu, \nu \in \mathcal{M}(X),\ d(\mu, \nu) = \sum_{n\in\mathbb{N}} 2^{-n} \cdot \min(1, |\mu(f_n) - \nu(f_n)|$$

Proposition C.2. *The metric d is compatible with the weak-* topology.*

Given a continuous application $T : X \to T$, we denote by $\mathcal{M}_T(X)$ the set of Borel T-invariant probability measures.

Proposition C.3. *The sets $\mathcal{M}(X)$ and $\mathcal{M}_T(X)$ are convex. The extremal points of $\mathcal{M}(X)$ are the Dirac measures. The extremal points of $\mathcal{M}_T(X)$ are the T-ergodic measures.*

Appendix D. On dimensions

D.1. **Hausdorff dimension.** Let X be a compact metric space X of distance dist. In this paragraph, we recall the definition of the Hausdorff dimension. The reference is the note [34] by J. Shah.

Let X be a metric space whose distance is denoted by d. Given $A \subset X$, we denote by $|A|$ the diameter of A. Let E be a subset of X. We define:

$$\mathcal{H}_\alpha^\delta(E) = \inf_{E \subset \cup_i F_i, |F_i| < \delta} \sum_{i=1}^{\infty} |F_i|^\alpha$$

Definition D.1 (α-Hausdorff content)**.** The α-Hausdorff content of the set E is the number:

$$m_\alpha(E) = \sup_{\delta \geq 0} \mathcal{H}_\alpha^\delta(E) = \lim_{\delta \to 0} \mathcal{H}_\alpha^\delta(E)$$

Given a set E, one can show that the function $\alpha \mapsto m_\alpha(E)$ has the following graph:

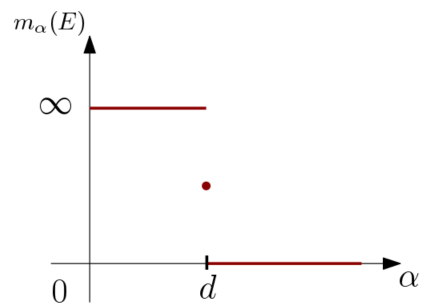

Figure 13. The function $\alpha \mapsto m_\alpha(E)$

This behaviour allows to define the Hausdorff dimension:

Definition D.2 (Hausdorff dimension)**.** The Hausdorff dimension $d(E)$ of the set E is defined as:

$$d(E) = \sup_{m_\alpha(E)=E} \alpha = \inf_{m_\alpha(E)=0} \alpha$$

If $0 < m_{d(E)}(E) < \infty$, we say that E has strict Hausdorff dimension $d(E)$.

D.2. **Topological dimension.** We define here the topological dimension of a topological space X, which is a topological invariant. We relate it to the Hausdorff dimension.

Definition D.3. We say that X has a topological dimension less or equal to n if for any finite cover $\cup_i U_i$ of X, there exists a refinement $\cup_j V_j$ of $\cup_i U_i$, such that any $x \in X$ belongs to at most $(n+1)$ different V_j. We say that X is topologically n-dimensional if it has topological dimension less or equal to n but not less or equal to $n-1$.

We have the following description of zero-dimensional spaces.

Proposition D.1. *Let $F \subset X$. F is topologically zero-dimensional if and only if F is totally disconnected.*

We now assume that X is a metrizable topological space. Then:

Proposition D.2. *The topological dimension of X is the infimum of the Hausdorff dimensions of X, taken on all the metrics coinciding with the topology on X.*

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