TENSOR TOMOGRAPHY FOR SURFACES

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Résumé. Ce mémoire se penche sur de récents résultats obtenus dans la théorie des problèmes inverses sur les surfaces. En particulier, nous démontrons l'injectivité de la transformée en rayon-X pour les 2-tenseurs symétriques sur les surfaces de type Anosov (surfaces sur lesquelles le flot géodésique est de type Anosov) et l'injectivité de la transformée en rayon-X pour les m-tenseurs symétriques ($m \geqslant 0$ quelconque) pour les surfaces simples à bord strictement convexe. Nous montrons comment ces résultats permettent de prouver des propriétés de rigidité spectrales ou de bord rigide.

Résumé. This memoire surveys some recent results in inverse problem theory for surfaces. More precisely, we prove the injectivity of the X-ray transform of symmetric 2-tensors for Anosov surfaces (surfaces on which the geodesic flow is Anosov) and the injectivity of the X-ray transform of symmetric m-tensors ($m \geqslant 0$) for simple surfaces with strictly convex boundary. We show how these results can be applied to prove spectral rigidity properties or boundary rigidity properties.

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1. Introduction

1.1. Historical background

1.1.1. The Radon transform

Historically, one of the first example of inverse problem was given by J. Radon in 1917, in his celebrated article *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten* (On the determination of functions from their integral values along certain manifolds). He introduced a transform — which we now refer to as the *Radon transform* — integrating functions of $\mathbb{R}^2$ along straight lines. In other words, given a function f, the Radon transform $\mathcal{R}f$ is the application :

$$\mathcal{R}f : L \mapsto \int_L f(x)dx,$$

where L denotes a line of the plane and dx the Lebesgue measure on L. By parametrizing a line by its slope p and its intercept τ, the Radon transform can be written :

$$\mathcal{R}f(p,\tau) = \int_{-\infty}^{+\infty} f(x, px+\tau)dx$$

But the usual form is that given in polar coordinates (r,θ), where θ denotes the angle of the slope and r the offset of the line. We have :

$$\mathcal{R}f(r,\theta) = \int_{\mathbb{R}^2} f(x,y)\delta_0(x\cos\theta + y\sin\theta - r)dxdy$$

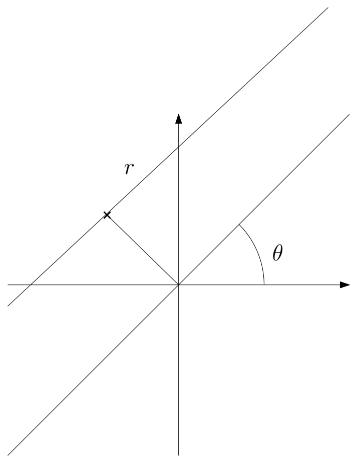

FIGURE *1.1.* The polar coordinates

This expression is rather interesting and its computation shows a characteristic sinusoid shape. It is referred to as a *sinogram*. Given a black-and-white picture, the function f that one can take is the intensity of the grey in a given point (x, y) of the picture :

FIGURE *1.2.* The initial picture ...

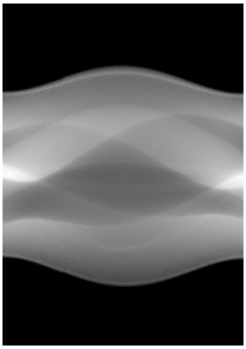

FIGURE *1.3.* ... and its sinogram.

In his article, Radon proved an *inversion formula*, that is, he pointed out the possibility to reconstruct the initial function f when given its transform, and gave the explicit formula :

$$f(x,y) = \frac{1}{(2\pi)^2} \int_0^\pi \left(\mathcal{R}f(\cdot,\theta) * h\right)(x\cos\theta + y\sin\theta)d\theta,$$

where h is the function such that $\hat{h}(\xi) = |\xi|$.

Medical imaging techniques such as CT (Computerized Tomography) actually compute Radon transforms. The idea is the following : given a certain structure (like lambs, tissues or cells) with unknown density, one sends X-ray through the shape. Given an initial beam, with known intensity, it is possible to gather its attenuated intensity when coming out of the shape — this is precisely the Radon transform. Then, using an inversion formula for the Radon transform, one recovers the unknown density.

1.1.2. Generalization

The Radon transform was just the start of a long theory of inverse problems. It gave birth to many applications in a large number of fields. Let us just quote a few of them, in order to motivate the reader's interest for it. We refer to [24] for more details.

One of the most astonishing application of the inverse problem theory arises in geophysical imaging, in order to determine the inner structure of the Earth. The seismic waves going through the Earth's crust increase their speed with depth, which curves the ray back to the surface. In some sense, they actually follow geodesics, where the metric is given by the Earth's density. The question raised is therefore : to what extent is it possible to reconstruct the Earth's density when knowing how seismic waves propagate into the Earth's crust ?

Such a problem is also studied in ultrasound tomography, when trying to detect tumors using blood flow measurements or in non-invasive industrial measurements for reconstructing the velocity of a fluid in motion. In this case, the problem involves integration formulas of 1-tensors along geodesics. The integration of 4-tensors can describe the perturbation of travel times of compressional waves propagating in anisotropic elastic media.

From a more theoretical point of view, given a Riemannian manifold (M, g), the tomography is the idea to reconstruct a tensor T, given its integral along some geodesics of the manifold. Indeed, just like in the previous case of $\mathbb{R}^2$, it is possible to define a ray transform, similar to the Radon transform. The questions which naturally rise are therefore : can one find an explicit inversion formula ? Is the ray transform an injective application ? Or what are the obstructions for the ray transform to become injective ?

The manifolds considered are usually compact with or without boundary. In the case with boundary, one of the most studied example is that of *simple* manifolds (that is, free of conjugate points) and with strictly convex boundary. As to manifolds without boundary, a very interesting class of manifolds is provided by the set of Anosov manifolds, that is, the set of manifolds for which the geodesic flow is hyperbolic (see Appendix C for a definition of Anosov flows).

1.2. Content of this memoire

All the results exposed in this memoire are related to inverse problems for surfaces. We relate recent theorems proved by G. P. Paternain, M. Salo and G. Uhlmann, mostly in [23] and [25]. Some of the theorems detailed below have been extended to greater dimensions. When it is the case, it will be mentioned, but all our proofs will be done in the two-dimensional case. Note that some of the results exposed here are still open in dimension greater than 2.

In a preliminary part, we will recall some elements of Riemannian geometry which will be used throughout the rest of the memoire. We detail the basic tools of the geometry of surfaces and their unit tangent bundle, introducing the standard moving frame $\{X, H, V\}$. Then, in Section 3, we see how these geometric properties can be related to some specific properties of the functional space $L^2(SM)$. In particular, we introduce a decomposition in Fourier elements in the circle bundles and the lower and raising operators η_+, η_-. Section 5, is devoted to the proof of the injectivity of the ray transform on negatively curved surfaces. This is based on a celebrated result of V. Guillemin and D. Kazhdan [14], published in 1979. In particular, the injectivity of I_2 will allow us to show that such a surface is *spectrally rigid*, that is an isospectral continuous deformation of the metric (a deformation preserving the spectrum of the Laplace-Beltrami operator) is always an isometry.

It is known that the geodesic flow on a negatively curved closed manifold is Anosov (see Appendix C for a definition). In Section 6, we will try to extend our previous result to *Anosov surfaces*, that are Riemannian surfaces for which the geodesic flow is Anosov (thus including negatively curved surfaces). We will show that I_2 is injective, thus proving that such a surface is also spectrally rigid, but the proof does not extend to obtain the injectivity of I_m for $m \geqslant 3$.

In a third part, we will forget boundaryless surfaces and turn to compact surfaces which are simple with strictly convex boundary, asking the same question. We will show that I_m is injective, for any $m \geqslant 0$. This will imply that the manifold is *deformation boundary rigid*.

The appendix contains a few elements of introduction to different theories referred to throughout the memoire. It is far from being exhaustive and detailed. In particular, most of the proofs are omitted but some references are given. We detail some more subtle results like the existence of local isothermal coordinates for surfaces, or the decomposition of symmetric tensors.

1.3. Acknowledgement

This memoire validates the "Master de Mathématiques Fondamentales" of University Paris VI, as well as the fourth year of the Ecole Polytechnique. It was carried out under the supervision of Colin Guillarmou. I deeply thank him for his advice and suggesting me this subject.

2. Preliminaries of Riemannian geometry

The main reference for this section is [8]. In the following, we consider (M, g) an n-dimensional smooth manifold endowed with a Riemannian metric g.

2.1. Notations

Given a local chart (U, ϕ) on M, we will denote by $(x_i)_{1 \leqslant i \leqslant n}$ the local coordinates and we write in these coordinates

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j,$$

where $g_{ij}(x) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. We denote by (g^{ij}) the coefficient of the inverse matrix. Given $x \in M$, the norm of $v \in T_xM$ is given by $|v|_x = g_x(v,v)^{1/2}$, which will sometimes be denoted $\langle v, v\rangle_x^{1/2}$. In the following, we will often drop the index x. Note that if we are given other coordinates (y_j), then one can check that in these new coordinates :

$$g = \sum_{i,j,k,l=1}^{n} g_{ij} \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_l} dy^k dy^l \tag{2.1}$$

We define the musical isomorphism at $x \in M$ by

$$\flat : \left| \begin{array}{l} T_xM \to T_x^*M \\ v \mapsto v^\flat = g(v, \cdot) \end{array} \right.$$

This is an isomorphism between the two vector bundles TM and T^*M since they are equidimensional and g is symmetric definite positive, thus non-degenerate. Given an orthonormal basis (e_i) of T_xM, we will denote by (e^i) the dual basis of T_x^*M which is, in other words, the image of the basis (e_i) by the musical isomorphism.

If E is a vector bundle over M, then the projection will be denoted $\pi : E \to M$. We denote by $\Gamma(M, E)$ the set of smooth sections of E. $\Gamma(M)$ denotes $\Gamma(M, TM)$, the set of vector fields. We will denote by $\Gamma(M, \otimes_S^m T^*M)$ the set of smooth symmetric covariant m-tensors on M. On the cotangent bundle T^*M, we denote by ω the canonical symplectic form, which we write in coordinates $\omega = \sum_{i=1}^n dp^i \wedge dx^i$. We recall that it is obtained as the differential of the canonical 1-form $\lambda \in \Omega^1(T^*M)$, defined intrinsically as

$$\lambda_{(x,p)}(\xi) = p\left(d\pi_{(x,p)}(\xi)\right),$$

for a point $(x,p) \in T^*M, \xi \in T_{(x,p)}T^*M$ and where $\pi : T^*M \to M$ denotes the projection.

Given a point $x \in M$, if (e_i) is an orthonormal basis of T_xM, we define $d\text{vol} = e^1 \wedge ... \wedge e^n$. In local coordinates, it is given by the formula :

$$d\text{vol} = \sqrt{\det(g_{ij})} dx^1 \wedge ... \wedge dx^n \tag{2.2}$$

We recall that the Laplacian in local coordinates is given by :

$$\Delta f = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j f \right) \tag{2.3}$$

We denote by ∇ the Levi-Civita connection on TM. In local coordinates, the Christoffel symbols are defined such that :

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tag{2.4}$$

They are given by the Koszul formula :

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right) \tag{2.5}$$

We denote the torsion tensor T^∇, which is defined as :

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

The curvature tensor is denoted by F^∇ and defined as :

$$F^\nabla(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.6}$$

In particular, we clearly have $F^\nabla(X,Y) = -F^\nabla(Y,X)$. We recall that the Levi-Civita connection is the unique torsion-free and g-metric connection, namely it satisfies for any $X, Y, Z \in \Gamma(M)$:

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0 \tag{2.7}$$

$$Z \cdot (g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{2.8}$$

The sectional curvature K is given at x by

$$K_x(e_1, e_2) = \langle F^\nabla(e_1, e_2)e_1, e_2 \rangle,$$

where $e_1, e_2 \in T_xM$ are orthogonal. In particular, in the case of a surface, which is what we will be mostly interested in, the sectional curvature is a real number referred to as the Gaussian curvature (or simply the curvature if the context is not ambiguous).

2.2. The geodesic flow

2.2.1. The standard point of view

Given a curve $\gamma : I \to M$, we recall that it is a geodesic if it satisfies the geodesic equation :

$$\nabla_{\dot\gamma}\ \dot\gamma = 0 \tag{2.9}$$

In local coordinates, this equation becomes :

$$\ddot\gamma_k\ + \sum_{i,j=1}^{n} \Gamma_{ij}^k \dot\gamma^i \dot\gamma^j = 0 \tag{2.10}$$

For any initial data $(\gamma(0), \dot\gamma(0)) = (x, v) \in TM$, the Cauchy theorem ensures the local existence of a solution $(\gamma, \dot\gamma)$ defined for t small. If M is complete (for instance if M is compact), then the existence is global. For a point $(x, v) \in TM$, we denote by $\varphi_t(x, v) = (\gamma(t), \dot\gamma(t))$ the geodesic flow, when it is defined. Thanks to the geodesic equation written in coordinates, we know that the infinitesimal generator X of the geodesic flow (which we will sometimes call the geodesic vector field) is given in coordinates by

$$X(x, v) = v^k \frac{\partial}{\partial x_k}\ - \Gamma_{ij}^k v^i v^j \frac{\partial}{\partial v_k}, \tag{2.11}$$

where we used the Einstein convention of summation. This will be done in the following in order to simplify the notations.

2.2.2. The dual point of view

Changing our point of view, geodesics can also be seen as critical points of a certain Lagrangian. Indeed, consider the Lagrangian density L defined on TM by $L(x, v) = \dfrac{1}{2}|v|_x^2$. Then, the critical points of the energy

$$E(\gamma) = \int_a^b L(\gamma, \dot\gamma)\ dt = \int_a^b |\dot\gamma(t)|^2_{\gamma(t)}\ dt,$$

sought among the piece-wise $\mathcal{C}^1$ curves γ joining $\gamma(a)$ to $\gamma(b)$, are the geodesics. In this perspective, one can check that the geodesic equation (2.9) is nothing less but a reformulation of the Euler-Lagrange equations :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(\gamma, \dot\gamma)\right) = \frac{\partial L}{\partial x}(\gamma, \dot\gamma)$$

In a more general frame, the Lagrange transform is the application which is given in coordinates by :

$$\mathbb{L} : \left| \begin{array}{l} TM \to T^*M \\ (x, v) \mapsto \mathbb{L}(x, v) = (x, \dfrac{\partial L}{\partial v_i}(x, v) dx^i) \end{array} \right. \tag{2.12}$$

It transports a variational problem from the Lagrangian perspective to the Hamiltonian perspective. In our particular case, one can check that the Lagrange transform is simply the musical isomorphism introduced in the previous subsection, namely $\mathbb{L}(x,v) = (x, v^\flat)$. It can be expressed in local coordinates by :

$$\mathbb{L}(x,v) = \left(x, g_{ij} v^i dx^j\right) \tag{2.13}$$

And its inverse is given by $\mathbb{L}^{-1}(x,p) = (x, p^\sharp)$, where $\sharp$ is the standard notation for the inverse of $\flat$. In coordinates, we have :

$$\mathbb{L}^{-1}(x,p) = \left(x, g^{ij} p^i \frac{\partial}{\partial x_j}\right) \tag{2.14}$$

The natural metric for covectors is given by g^{-1}, which therefore makes the musical isomorphism an isometry between TM and T^*M. For a covector ξ, we will indistinctly write $|\xi|_g^2$ or $|\xi|_{g^{-1}}^2$ but these two notations refer to the same quantity, namely $g^{ij}\xi_i\xi_j$.

We now show how to recover the geodesic flow on T^*M from the Hamiltonian point of view. We consider the Hamiltonian $H(x,p) = \dfrac{1}{2}\,|p|_x^2$ which is given in local coordinates by :

$$H(x,p) = \frac{1}{2} g^{ij} p_i p_j$$

We denote by X_H its flow on T^*M which is such that $\omega(X_H, \cdot) + dH = 0$. Let us consider an integral curve $(x(t), p(t))$ for the vector field X_H. We know that it satisfies the Hamilton equations :

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

The n first equations can be written in coordinates :

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} g^{ij} p^j,$$

that is $\dfrac{dx}{dt}^{\flat} = p(t)$. The n next equations yield to :

$$\begin{aligned}
\frac{dp_i}{dt} &= -\frac{1}{2}\sum_{k,l=1}^{n} \partial_i g^{kl} p^k p^l \\
&= \frac{1}{2}\sum_{k,l,u,v=1}^{n} g^{ku}\partial_i g_{uv} g^{vl} p^k p^l \\
&= \frac{1}{2}\sum_{u,v=1}^{n} \partial_i g_{uv} \frac{dx_q}{dt}\frac{dx_v}{dt},
\end{aligned}$$

where we used the formula giving the derivative of the coefficients of the inverse matrix according in the second last line. But since $\dfrac{dx}{dt} = p(t)^{\flat}$, we have :

$$\begin{aligned}
\frac{dp_i}{dt} &= \frac{d}{dt}\left(\sum_{k=1}^{n} g_{ik}\frac{dx_k}{dt}\right) \\
&= \sum_{k,l=1}^{n} \partial_l g_{ik}\frac{dx_l}{dt}\frac{dx_k}{dt} + \sum_{k=1}^{n} g_{ik}\frac{d^2x_k}{dt^2}
\end{aligned}$$

From the two previous formulas, we get :

$$\sum_{k=1}^{n} g_{ik}\frac{d^2x_k}{dt^2} = \frac{1}{2}\sum_{k,l=1}^{n}\left(\partial_i g_{kl} - \partial_k g_{il} - \partial_l g_{ik}\right)\frac{dx_k}{dt}\frac{dx_l}{dt}$$

By inverting this relation, it is possible to obtain the expression of dx^2/dt^2 :

$$\frac{d^2x_s}{dt^2} = -\sum_{k,l=1}^{n}\frac{1}{2}\sum_{i=1}^{n} g^{si}\left(\partial_k g_{il} + \partial_l g_{ik} - \partial_i g_{kl}\right)\frac{dx_l}{dt}\frac{dx_k}{dt}$$

Then, applying Koszul formula (2.5), we recover the Christoffel symbols, and eventually the geodesic equation (2.10) in coordinates. Given $(\gamma(0), \dot\gamma(0)) = (x_0, v_0)$ and (x_0, p_0) its image by the Lagrange transform, $(x(t), p(t))$ is nothing but the image by the Lagrange transform of the geodesic $(\gamma(t), \dot\gamma(t))$. In other words, we have proved the

Proposition 2.1. — *The Lagrange transform conjugates the geodesic flow φ_t on TM, generated by X, and the geodesic flow φ_t^* on T^*M, generated by X_H.*

Actually, this is the dual point of view : instead of looking at the tangent vector $\dot\gamma$ to the geodesic, one looks at the hyperplanes distribution which are orthogonal to $\dot\gamma$.

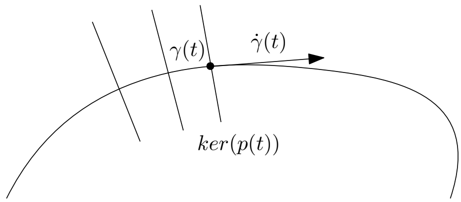

FIGURE *2.1.* The geodesic flow on TM and T^*M

2.3. Structure of SM

2.3.1. The horizontal distribution

Let us first state some very general results. We consider a vector bundle $\pi : E \to M$ of rank r with a connection ∇. Given a path $\gamma : I \to M$ such that $\gamma(0) = x$, $\dot\gamma(0) = X \in T_xM$ and an initial value $s_0 \in E_x = \pi^{-1}(\{x\})$, there exists a unique lift of γ to a path $s : I \to E$ — it is the parallel transport of s_0 along γ — such that $s(0) = s_0, \pi(s) = \gamma$ and

$$\nabla_{\dot\gamma} s = 0 \tag{2.15}$$

Now, we can associate to X the vector

$$\tilde X = \left.\frac{d}{dt}\right|_{t=0} s(t) \in T_{s_0}E \tag{2.16}$$

One can check that $\tilde X$ is well defined and only depends on the choice of $(\gamma(0), \dot\gamma(0))$. It is a linear application $T_xM \to T_{s_0}E$ which we denote by $\theta : X \mapsto \tilde X$. In local coordinates, if we write $X = X^i \dfrac{\partial}{\partial x_i}, s_0 = s^j e_j$, then the parallel transport equation (2.15) yield to :

$$\theta(X) = X^i \frac{\partial}{\partial x_i} - \Gamma_{ij}^k X^i s^j \frac{\partial}{\partial s_k} \tag{2.17}$$

We define the horizontal distribution on (E, ∇) at $s_0 \in E$ as :

$$\mathbb{H}_{s_0}^{\nabla} = \mathrm{Span}(\theta(X), X \in T_\pi(s_0)) \subset T_{s_0}E \tag{2.18}$$

In the following, we will drop the notation ∇ for $\mathbb{H}_{s_0}$, but we insist on the fact that $\mathbb{H}_{s_0}$ is entirely determined by the choice of ∇. Actually, one can check that choosing a connection ∇ is strictly equivalent to choosing a smooth distribution of vector spaces $\mathbb{H} \subset TE$. Note that we have the following theorem :

THEOREM 2.2. — *The following assertions are equivalent :*

— *The distribution $s_0 \mapsto \mathbb{H}_{s_0}$ is integrable.*

— *The connection ∇ is flat, i.e. T^∇ vanishes.*
— *The holonomy i.e. the parallel transport of a section along a closed loop is invariant by homotopy of this loop.*

Since θ is clearly injective by definition, it defines an isomorphism between $T_{\pi(s_0)} \xrightarrow{\sim} H_{s_0}$. We define the vertical distribution on (E, ∇) at s_0 as $\mathbb{V}_{s_0} = \ker d\pi_{s_0} = E_{\pi(s_0)} \subset T_{s_0}E$. Thanks to (2.15), it is easy to check that θ is exactly the inverse of the differential $d\pi_{s_0}$ restricted to the subspace H_{s_0} of $T_{s_0}E$. We therefore have the splitting :

$$T_{s_0}E = \mathbb{H}_{s_0} \oplus \mathbb{V}_{s_0} \tag{2.19}$$

Given a vector $\xi \in T_{s_0}E$, which we write in local coordinates $\xi = X^i \dfrac{\partial}{\partial x_i} + Y^k \dfrac{\partial}{\partial s_k}$, (2.17) shows that it is in $\mathbb{H}^\nabla_{s_0}$ if and only if

$$Y^k + \Gamma^k_{ij} X^i s^j = 0, \tag{2.20}$$

for all $1 \leqslant k \leqslant r$, and it is in V_{s_0} if and only if $X^i = 0$ for all $1 \leqslant i \leqslant n$.

2.3.2. The Sasaki metric

We now apply the previous formalism to the particular case when $E = TM$ and construct a canonical metric $\langle\langle \cdot, \cdot \rangle\rangle$ on TM, called the Sasaki metric. If ∇ is the Levi-Civita connection on TM, then it automatically determines a splitting like in (2.19).

In this case, there also exists a canonical application $\mathcal{K} : T_{(x,v)}TM \to T_xM$ whose kernel is exactly $\mathbb{H}_{(x,v)}$. Somehow, it can be seen as a complementary application to $d\pi$. Given $\xi \in T_{(x,v)}TM$, we can consider a local curve c in TM such that $c(0) = (x, v)$ and $\dot{c}(0) = \xi$. We can write $c(t) = (\gamma(t), Z(t))$, where Z is a vector field along the curve γ. We define :

$$\mathcal{K}_{(x,v)}(\xi) = \nabla_{\dot{\gamma}} Z(0) \tag{2.21}$$

One can easily check that the application $\mathcal{K}$ now defines an isomorphism between $\mathbb{V}_{(x,v)}$ and T_xM and its kernel is precisely $\mathbb{H}_{(x,v)}$, the horizontal distribution.

Definition 2.3. — *Given a point $(x, v) \in TM$ and $\xi, \eta \in T_{(x,v)}TM$, we set :*
— *If ξ, η are vertical, namely $\xi, \eta \in \mathbb{V}_{(x,v)} = T_xM$, then $\langle\langle \xi, \eta \rangle\rangle := \langle \mathcal{K}(\xi), \mathcal{K}(\eta) \rangle$,*
— *If ξ, η are horizontal, then $\langle\langle \xi, \eta \rangle\rangle := \langle d\pi_{(x,v)}(\xi), d\pi_{(x,v)}\eta \rangle$,*
— *$\mathbb{V}_{(x,v)}$ and $\mathbb{H}_{(x,v)}$ are orthogonal for $\langle\langle \cdot, \cdot \rangle\rangle$.*

Note that the last line induces in particular that the decomposition (2.19) is orthogonal for the Sasaki metric.

In the rest of this paragraph, we explain how to recover the Sasaki metric from the symplectic point of view. We recall that λ is the canonical 1-form defined on T^*M and introduced in the previous section. It can be pulled back via the musical isomorphism to get $\alpha = \flat^*\lambda \in \Omega^1(TM)$. Then

$$\begin{aligned}
\alpha_{(x,v)}(\xi) &= \lambda_{(x,v^\flat)}(d\,\flat_{(x,v)}(\xi)) \\
&= v^\flat\left(d\pi_{(x,v^\flat)} \circ d\,\flat_{(x,v)}(\xi)\right) \\
&= v^\flat\left(d\left(\pi \circ \flat\right)_{(x,v)}(\xi)\right) \\
&= \langle v, d\pi_{(x,v)}(\xi)\rangle \\
&= \langle\langle Z, \xi\rangle\rangle,
\end{aligned}$$

for some vector field Z, according to Riesz representation theorem. We claim that Z is actually the geodesic vector field X. Indeed, first observe that X is a horizontal vector field (i.e. $X(x,v) \in \mathbb{H}_{(x,v)}$), which is an immediate consequence of the expressions in local coordinates (2.11) and (2.20). In particular, in local coordinates, expression (2.17) yield to :

$$d\pi_{(x,v)}(X(x,v)) = v^i \frac{\partial}{\partial x_i} \tag{2.22}$$

In order to prove that $Z = X$, we just have to check that the expressions $\langle v, d\pi_{(x,v)}(\xi)\rangle$ and $\langle\langle X, \xi\rangle\rangle$ agree when ξ runs through a basis of $T_{(x,v)}TM$. If ξ is in $\mathbb{V}_{(x,v)}$ (that is ξ is vertical), this is immediate since $d\pi(\xi) = 0$ and X is horizontal. If $\xi = \xi_i = \dfrac{\partial}{\partial x_i} - \Gamma^k_{ij} v^j \dfrac{\partial}{\partial v_k}$, then $d\pi(\xi_i) = \dfrac{\partial}{\partial x_i}$ by (2.17) and $\langle v, d\pi_{(x,v)}(\xi)\rangle = v_i$. But by definition of the Sasaki metric, since both X and ξ_i are horizontal :

$$\langle\langle X, \xi_i\rangle\rangle = \langle d\pi(X), d\pi(\xi_i)\rangle = v_i$$

We therefore obtain the equality :

$$\alpha_{(x,v)}(\xi) = \langle\langle X, \xi\rangle\rangle \tag{2.23}$$

In other words, $\alpha = X^{\tilde{\flat}}$, where $\tilde{\flat} : TTM \to T^*TM$ is the musical isomorphism given by the Sasaki metric.

Remark 2.4. — Behind this is hidden the fact that SM (the unit tangent bundle), and therefore S^*M, both have the structure of a contact manifold, but we will not give further details about this.

2.4. Surface theory

2.4.1. Isothermal coordinates

In this paragraph, we introduce isothermal coordinates, which will be widely used is the following.

Definition 2.5. — *Let (M, g) be an n-dimensional Riemannian manifold. Isothermal coordinates are local coordinates such that the metric can be written $g = e^{2\lambda}(dx_1^2 + ... + dx_n^2)$, where λ is a smooth function.*

In dimension $n \geqslant 3$, in a neighborhood of a point, isothermal coordinates may not exist. However, in dimension $n = 2$, we have the

Theorem 2.6. — *Let (M, g) be a Riemannian surface and $x \in M$. There exists isothermal coordinates in a neighborhood of x.*

We provide a proof of this theorem in Appendix B based on the resolution of a Dirichlet problem. The existence of isothermal coordinates on a surface is closely link to the existence of a Riemann structure (or holomorphic structure) on the surface, that is a covering by charts $\{U, \varphi\}$ such that the transition maps are all holomorphic, as explained in the Appendix. A complex structure on M is an endomorphism $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}$. In particular, the data of a given conformal class together with an orientation of the manifold is equivalent to that of a complex structure.

The Koszul formula allows to compute the Christoffel symbols in the isothermal coordinates. We obtain :

Γ^1_{ij}	$j=1$	$j=2$
$i=1$	$\partial_1\lambda$	$\partial_2\lambda$
$i=2$	$\partial_2\lambda$	$-\partial_1\lambda$

Γ^2_{ij}	$j=1$	$j=2$
$i=1$	$-\partial_2\lambda$	$\partial_1\lambda$
$i=2$	$\partial_1\lambda$	$\partial_2\lambda$

Note that in this case, the curvature of M has a rather simple expression :

$$K = -e^{-2\lambda}\,\Delta\lambda \tag{2.24}$$

2.4.2. The moving frame

We now assume that M is a surface and ∇ is the Levi-Civita connection on M, associated to the Riemannian metric g. Instead of studying the tangent bundle TM, we will study the unit tangent bundle SM. Note, that since we will mostly be interested in studying properties of the geodesic flow, this will not restrict our considerations as far as geodesics are covered at constant speed (and one can always assume it is arc-length parametrized, up to a preliminary reparametrization).

SM can be seen as a subriemannian manifold of TM. In the following, we may identify the tangent space T_xM at a point with the complex plane $\mathbb{C}$ and therefore work with complex coordinates. In other words, this simply means that we see M as a Riemannian surface, endowed with a complex structure i, and i simply acts on tangent vectors by $\pi/2$-rotating them. S_x will denote the restriction of T_xM to the unit circle. Given a point $(x,v) \in SM$, the rotation $r_\theta : S_x \to S_x$ defined in coordinates by $r_\theta(x,v) = (x, e^{i\theta}v)$ is a flow on SM which provides an infinitesimal generator $V(x,v)$, spanning a direction of $T_{(x,v)}SM$. The horizontal distribution $\mathbb{H}_{(x,v)}$ is unchanged and since SM is submanifold of TM of dimension 3, one gets the orthogonal splitting :

$$T_{(x,v)}SM = \mathrm{Span}(V(x,v)) \oplus \mathbb{H}_{(x,v)}$$

Moreover, thanks to the expression (2.22), one clearly sees that the vector field X is unitary on SM. This is also the case for the vector field V. In order to obtain a orthonormal basis of $T_{(x,v)}SM$ at a point $(x,v) \in SM$, we need to provide a third vector field $H \in \mathbb{H}_{(x,v)}$, unitary and orthogonal to X. It is chosen so that $\{X, H, V\}$ is positively oriented. We call this basis the *moving frame* on SM.

We can give explicit formulas for this vector fields in isothermal coordinates. In these coordinates, in a neighborhood $\mathcal{U}$ of a point $x \in M$, we write $g = e^{2\lambda(x_1,x_2)}(dx_1^2 + dx_2^2)$. The local coordinates induced on $TM|_\mathcal{U}$ are denoted by (x_1, x_2, v_1, v_2). Thus, we can consider local coordinates on $SM|_\mathcal{U}$, given by :

$$(x_1, x_2, \theta) \mapsto (x_1, x_2, e^{-\lambda}\cos\theta, e^{-\lambda}\sin\theta)$$

Thanks to the Koszul formula (2.5), the Christoffel symbols can be computed explicitly. The expression (2.11) then yield to :

(2.25)
$$X(x,\theta) = e^{-\lambda}\left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \left(-\frac{\partial\lambda}{\partial x_1}\sin\theta + \frac{\partial\lambda}{\partial x_2}\cos\theta\right)\frac{\partial}{\partial\theta}\right)$$

$V(x,\theta)$ is simply given by :

$$V(x,\theta) = \frac{\partial}{\partial\theta} \tag{2.26}$$

Eventually, $H(x,\theta)$ can be obtained as a normal vector to the plane spanned by the vectors $\{X, V\}$, or as a positive rotation of X in the plane $\mathbb{H}_{x,v}$ of

angle $\pi/2$. We get :
(2.27)
$$H(x,\theta) = e^{-\lambda}\left(-\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_1} - \left(\frac{\partial\lambda}{\partial x_1}\sin\theta + \frac{\partial\lambda}{\partial x_2}\cos\theta\right)\frac{\partial}{\partial\theta}\right)$$
From these formulas, we can derive the Cartan structural equations. In particular, using the expression (2.24) for the curvature, one gets :

$$[X,H] = K\cdot V \tag{2.28}$$

$$[V,X] = H \tag{2.29}$$

$$[H,V] = X \tag{2.30}$$

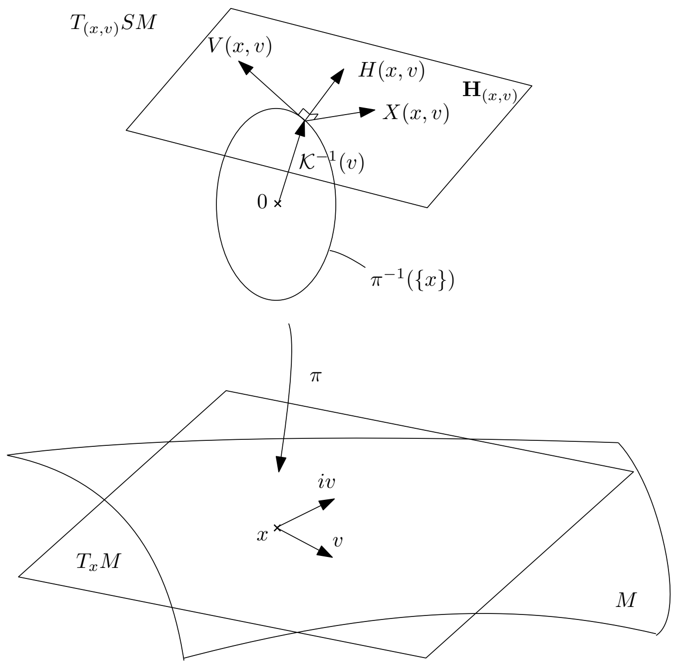

FIGURE *2.2.* The unit tangent bundle

We can study this basis from the dual point of view. Via the musical isomorphism defined thanks to the Sasaki metric, we obtain a dual basis $\{\alpha,\beta,\psi\}$ on T^*SM (note that the 1-form α has already been introduced in the previous section). Let us give a short description of this basis. Given $\xi \in T_{(x,v)}SM$, since $d\pi_{(x,v)}(X) = v, d\pi_{(x,v)}(H) = iv, \mathcal{K}_{(x,v)}(V) = iv$, we have :

— $\alpha_{(x,v)}(\xi) = \langle\langle X, \xi\rangle\rangle = \langle v, d\pi_{(x,v)}(\xi)\rangle$,
— $\beta_{(x,v)}(\xi) = \langle\langle H, \xi\rangle\rangle = \langle iv, d\pi_{(x,v)}(\xi)\rangle$,
— $\psi_{(x,v)}(\xi) = \langle\langle V, \xi\rangle\rangle = \langle iv, \mathcal{K}_{(x,v)}(\xi)\rangle$.

The kernel of ψ is precisely the horizontal distribution $\mathbb{H}$. The dual formulation [1] of the Cartan structural equations is :

$$\alpha \wedge \beta = K \cdot d\psi \tag{2.31}$$

$$\psi \wedge \alpha = d\beta \tag{2.32}$$

$$\beta \wedge \psi = d\alpha \tag{2.33}$$

Remark 2.7. — For the reader's convenience, let us just quote here the inversion formulas :

$$\frac{\partial}{\partial x_1} = e^{\lambda}\left(\cos\theta \cdot X - \sin\theta \cdot H\right) - \frac{\partial \lambda}{\partial x_2} V$$
$$\frac{\partial}{\partial x_2} = e^{\lambda}\left(\sin\theta \cdot X + \cos\theta \cdot H\right) + \frac{\partial \lambda}{\partial x_1} V$$

2.4.3. Canonical volume form on SM

As we mentioned before, the metric g induces a canonical volume form on the surface M which can be written in isothermal coordinates, thanks to formula (2.2) :

$$d\mathrm{vol} = e^{2\lambda} dx^1 \wedge dx^2 \tag{2.34}$$

We can also define a canonical volume form on SM, denoted by Θ, as the canonical volume form induced by the Sasaki metric. It is called the *Liouville measure* on SM. Since the dual basis $\{\alpha, \beta, \psi\}$ is orthonormal for the Sasaki metric, then one simply has :

$$\Theta := \alpha \wedge \beta \wedge \psi \tag{2.35}$$

From the definition (2.35), and the Cartan structural equations, one can prove that the fields X, H and V preserve the volume form Θ. Indeed, let us prove it for X. If $\mathcal{L}_X$ denotes the Lie derivative with respect to X, then one has :

$$\begin{aligned}(\mathcal{L}_X \Theta)(X, H, V) = \mathcal{L}_X\left(\Theta(X, H, V)\right) - \Theta([X, X], H, V) \\ - \Theta(X, [X, H], V) - \Theta(X, H, [X, V])\end{aligned}$$

1. Dual in the sense that we apply $d \circ \flat$ to the Cartan structural equations on the vector fields.

Since $\Theta(X, H, V) = 1$, the first term is zero and using Cartan structural equations, the three other terms are zero. So $\mathcal{L}_X\Theta$ is zero, that is, X preserves the volume-form Θ.

We also have $\Theta = \pi^*(d\text{vol}) \wedge d\theta$. Indeed, since Θ and $\pi^*(d\text{vol}) \wedge d\theta$ are both volume forms, we know that they may differ by a multiplicative function. But $\Theta(X, H, V) = 1$ and $\pi^*(d\text{vol}) \wedge d\theta(X, H, V) = d\text{vol}(d\pi(X), d\pi(H)) = 1$. Given $u \in \mathcal{C}^\infty(SM)$, this yields to the integration formula over the fibers :

$$\int_{SM} u \cdot \Theta = \int_M \left(\int_{SM_{\pi(p)}} u|_{SM_{\pi(p)}}(p) \ d\theta(p) \right) \ d\text{vol}(\pi(p)) \tag{2.36}$$

In particular :

$$vol(SM) = 2\pi \cdot vol(M) \tag{2.37}$$

Remark 2.8. — The volume form Θ can also be seen as the contact form $\alpha \wedge (d\alpha)^{n-1}$. Here, since $n = 2$, we simply obtain $\alpha \wedge d\alpha = \alpha \wedge \beta \wedge \psi = \Theta$ by the dual formulation of the Cartan structural equation (3.6).

Remark 2.9. — The construction of the Sasaki metric on the tangent bundle and the canonical volume form on the unit tangent bundle is not specific to dimension 2 and can easily be generalized to greater dimensions. Actually, the unit tangent bundle is locally a product space $U \times \mathbb{S}^{n-1}$ and the Liouville measure on SM is nothing but the product measure $d\text{vol} \times dS$ where dS denotes the euclidean volume form on $\mathbb{S}^{n-1}$.

3. Functional spaces on SM

3.1. The space $L^2(SM)$

As an introduction, we first recall the

THEOREM 3.1 (Stone). — *Let $\mathcal{H}$ be a Hilbert space and $(U_t)_{t\in\mathbb{R}}$ be a group of unitary operators defined on $\mathcal{H}$ and strongly continuous at $t = 0$. Then, there exists a self-adjoint operator A, densely defined on $\mathcal{H}$ called the infinitesimal generator of the group, such that :*

$$U_t = e^{itA}$$

The flows generated respectively by the vector fields X, H and V are volume-preserving as we saw in the previous paragraph. Therefore, they act on $L^2(SM)$ as unitary operators and by Stone's theorem, their infinitesimal generators — the operators $-iX, -iH$ and $-iV$ — are self-adjoint operators densely defined on $L^2(SM)$. In particular, their domain of definition contains $C^\infty(SM)$.

3.1.1. An orthogonal decomposition of $L^2(SM)$

We begin with a first proposition :

PROPOSITION 3.2. — *The space $L^2(SM)$ breaks up into an orthogonal sum of subspaces :*

$$L^2(SM) = \bigoplus_{n\in\mathbb{Z}} H_n,$$

where H_n is the eigenspace of the operator $-iV$ corresponding to the eigenvalue n.

Démonstration. — Since M is compact and smooth, we can always triangulate M into $(M_i)_{1\leqslant i\leqslant N}$ where each M_i is small enough so that it is contained in a local chart for which there exists isothermal coordinates. We have :

$$L^2(SM) = \bigoplus_{i=1}^{N} L^2(SM_i)$$

Now, given a M_i and $u \in L^2(SM_i)$, one may use the isothermal coordinates in order to decompose the function in a Fourier series

$$u = \sum_{n\in\mathbb{Z}} u_n,$$

where $u_n \in H_n(SM_i)$ is given in the coordinates by :

$$u_n(x,\theta) = \left(\frac{1}{2\pi}\int_0^{2\pi} u(x,t)e^{-int}\,dt\right)e^{in\theta} = \tilde{u}_n(x)e^{in\theta}$$

And :

$$H_n = \bigoplus_{i=1}^{N} H_n(SM_i)$$

$\square$

Let us insist here on the fact that the function $\tilde{u}_n$ is defined regardless of the system of isothermal coordinates that is chosen. Indeed, consider two sets of coordinates (x, θ) and (y, α) for which the metrics can be respectively written $e^{2\lambda}(dx_1^2 + dx_2^2)$ and $e^{2\mu}(dy_1^2 + dy_2^2)$ and a transition map φ. We fix x and $y = \varphi(x)$ and compare $\tilde{u}_n(x)$ with $\tilde{v}_n(y)$. Note that, since φ is conformal (see Appendix B), $d\varphi_x$ preserves the angles so it is the composite of a homothetic transformation (of ratio $e^{2(\lambda-\mu)}$) with a rotation of angle ω. Therefore, the θ coordinate in the first system is sent to $\alpha = \psi(\theta) = \theta + \omega$. Thus, replacing this in the definition with the integral, we obtain the sought result.

Remark 3.3. — In particular, one gets from this decomposition that $||u||^2_{L^2(SM)} = \sum_{k\in\mathbb{Z}} ||u_k||^2_{L^2(SM)}$. We will denote by $\pi_k : L^2(SM) \to H_k$ the orthogonal projection and by $\tilde{\pi}_k : L^2(SM) \to L^2(M)$ the map such that $\tilde{\pi}_k(u) = \tilde{u}_k$.

Definition 3.4. — *We define u_+ and u_- the respective even and odd parts of $u : SM \to \mathbb{C}$ by :*

$$u_+ := \sum_{k\in 2\mathbb{Z}} u_k, \qquad u_- := \sum_{k\in 1+2\mathbb{Z}} u_k$$

3.1.2. The Hilbert transform

Definition 3.5. — *We say that $u : SM \to \mathbb{C}$ is holomorphic if $u_k = 0$ for all $k < 0$ and antiholomorphic if $u_k = 0$ for $k > 0$.*

Definition 3.6. — *For $u_k \in H_k$, we define its Hilbert transform by $\mathcal{H}u_k := -isgn(k)u_k$.*

Observe in particular that u is holomorphic if and only if $(Id - iH)u = u_0$ and antiholomorphic if and only if $(Id + iH)u = u_0$. We have the following commutant formula :

Lemma 3.7. —

$$[\mathcal{H}, X]u = H \cdot u_0 + (H \cdot u)_0$$

We postpone the proof of this Lemma to the paragraph 3.4.4, where the proper tools will be introduced.

The previous relation can even be refined. Indeed, we can define the odd and even parts $\mathcal{H}_-$ and $\mathcal{H}_+$ of the Hilbert transform by considering for a smooth $u : SM \to \mathbb{C}$:

$$\mathcal{H}_- u = \mathcal{H} u_-, \quad \mathcal{H}_+ u = \mathcal{H} u_+$$

In particular, an even (resp. odd) function is transformed into an even (resp. odd) function by the Hilbert transform. The equality of the previous lemma can thus be written :

$$\mathcal{H}_+ X \cdot u - X \cdot \mathcal{H}_- u = (H \cdot u)_0, \quad \mathcal{H}_- X \cdot u - X \cdot \mathcal{H}_+ u = H \cdot u_0 \tag{3.1}$$

3.2. Symmetric tensors on a manifold

The references for this paragraph are [16], [28] and to a lesser extent [3]. Note that throughout this paragraph, the Einstein convention will be used in order to avoid the summation symbol on $i_1, ..., i_m$.

3.2.1. Symmetric tensors

A tensor is a section of the fiber bundle $\Gamma(M, \otimes^m T^* M)$. In local coordinates (x_i), its covariant coordinates are given by :

$$T_{i_1,...,i_m} = T\left(\frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_m}}\right)$$

By "raising" the indices, one gets the contravariant coordinates, namely :

$$T^{i_1,...,i_m} = g^{i_1 j_1} ... g^{i_m j_m} T_{j_1,...,j_m}$$

We recall that $\Gamma(M, \otimes_S^m T^* M)$ denotes the set of symmetric m-tensors on M, that are m-covariant tensors symmetric in each coordinate. Given an m-tensor $T \in \Gamma(M, \otimes^m T^* M)$, there is a natural operation of symmetrization making this tensor a symmetric one :

$$\sigma = \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} \rho_\pi,$$

where ρ_π acts on an m-tensor by :

$$\rho_\pi T = T_{i_1 ... i_m} dx^{\pi(i_1)} \otimes ... \otimes dx^{\pi(i_m)}$$

$\Gamma(M, \otimes_S^* T^* M) = \bigoplus_{m \geqslant 0} \Gamma(M, \otimes_S^m T^* M)$ is a $\mathbb{Z}$-graded algebra endowed with the symmetric product $\sigma(u \otimes v)$.

The set $\Gamma(M, \otimes_S^m T^* M)$ is naturally endowed with an L^2 scalar product q. Indeed, given $x \in M$, $T_x^* M$ is endowed with the metric g_x^{-1} as we have seen (which makes the musical isomorphism an isometry between $T_x M$ and

T_x^*M) and this extends to $\otimes_S^m T_x^*M$ by setting for two symmetric m-tensors $u = \sigma(\xi^1 \otimes ... \otimes \xi^m), v = \sigma(\eta^1 \otimes ... \otimes \eta^m)$ (with $\xi^i, \eta^j \in T_x^*M$) :

$$q_x(u,v) = \sum_{\pi \in \mathfrak{S}_m} g_x^{-1}(\xi^1, \eta^{\pi(1)})...g_x^{-1}(\xi^m, \eta^{\pi(m)})$$

Eventually, we define for $u, v \in \Gamma(M, \otimes_S^m T^*M)$:

$$q(u,v) = \int_M q_x(u,v) d\text{vol}$$

This scalar product can be extend to the algebra $\Gamma(M, \otimes_S^* T^*M)$ by declaring the sets $\Gamma(M, \otimes_S^m T^*M)$ and $\Gamma(M, \otimes_S^{m'} T^*M)$ to be orthogonal for $m \neq m'$.

We define the *inner derivative* of a symmetric tensor by :

$$dT := \sigma \nabla T,$$

where σ is the operator of symmetrization. It is a symmetric $(m+1)$-tensor. Its *divergence* is the operator defined by :

$$\delta T := -\text{tr}_{12}(\nabla T),$$

where the trace is taken over the two first coordinates. In local coordinates, one can write :

$$(\delta T)_{i_1 ... i_{m-1}} = -\frac{\partial T_{i_1 ... i_{m-1} j}}{\partial x_k} g^{jk}$$

One can prove that the formal adjoint of d is δ for the L^2-product introduced previously.

We can now state the main theorem of this paragraph :

Theorem 3.8. — *Let (M,g) be a compact Riemannian manifold with or without boundary (in this case, we will use the convention $\partial M = \emptyset$). Let T be a smooth symmetric m-tensor field. Then there exists a unique smooth symmetric m-tensor field f^s and a unique smooth $(m-1)$-tensor field v such that :*

$$f = f^s + dv, \qquad \delta f^s = 0, v|_{\partial M} = 0$$

The last condition is empty if M is boundaryless. We call f^s the *solenoidal part* of the tensor f and v its *potential part*. We provide a proof of this theorem in the Appendix D, where this result is even proved in the Sobolev regularity H^k.

3.2.2. Functions on SM and symmetric tensors

There exists a canonical map

$$\Phi^m : \Gamma(M, \otimes_S^m T^*M) \to \mathcal{C}^\infty(TM),$$

defined by $\Phi^m(T)(x,v) = T_x(v,...,v)$. If T is written in coordinates

$$T = T_{i_1...i_m} dx^{i_1} \otimes ... \otimes dx^{i_m}$$

then $\Phi^m(T)$ the m-homogenous polynomial in v given in the same coordinates by :

$$\Phi^m(T)(x,v) = T_{i_1...i_m}(x) v^{i_1}...v^{i_m},$$

where v^{i_k} denotes the i_k-th coordinate of the vector v. Most of the time, we will only consider the restriction of this function to the unit tangent bundle SM. Note in particular that Φ^0 is just the pull-back of a function $f \in \mathcal{C}^\infty(M)$ via the projection $\pi : SM \to M$. In the isothermal coordinates, for $(x,v) = (x, e^{-\lambda}e^{i\theta}) \in SM$, we have

$$\Phi^m(T)(x,v) = T_{i_1...i_m}(x) e^{-m\lambda} (\cos\theta)^p (\sin\theta)^{m-p},$$

where $p = \#\{k, i_k = 1\}$ which, by using Euler formulas, yields to

$$\Phi^m(T)(x,v) = T_{i_1...i_m}(x) \frac{e^{-m\lambda}}{2^m i^{m-p}} \sum_{0 \leqslant k \leqslant p, 0 \leqslant l \leqslant m-p} \binom{p}{k}\binom{m-p}{l}(-1)^l e^{i\theta(m+2(k-p-l))}$$

We therefore obtain a decomposition $T = \sum_{k=-m}^m T_k$, where each $T_k \in H_k$. Note in particular, according to the previous formula, that if m is even, then $T = T_+$ and if m is odd, then $T = T_-$. Moreover, if T is real, then $T_k = \bar{T}_{-k}$.

The orthogonal projection u_k of a function $u \in L^2(SM)$ on H_k can be identified with a section of the k-th tensor power of the canonical line bundle κ i.e. $\kappa^{\otimes k}$ (see Appendix E for further details). Namely, if $k \geqslant 0$, then we consider $u_k \mapsto \tilde{u}_k e^{k\lambda}(dz)^{\otimes k}$ (if $k \leqslant 0$, we consider $u_k \mapsto \tilde{u}_k e^{k\lambda}(d\bar{z})^{\otimes(-k)}$). Then in coordinates if $(x,v) = (x, e^{-\lambda}(\cos\theta + i\sin\theta)) \in SM$, we get :

$$\tilde{u}_k e^{k\lambda}(dz)^{\otimes k}\left((x, e^{-\lambda}(\cos\theta + i\sin\theta))\right) = \tilde{u}_k e^{k\lambda} e^{-k\lambda} e^{ik\theta} = u_k(x,\theta)$$

Now, we show how to do "the way back", that is how to recover a m-symmetric tensor, given a certain class of smooth function on SM. Let us denote by $\mathcal{R}_m$ the set of smooth, real-valued functions u on SM such that $u_k = 0$ for $|k| \geqslant m+1$ and $u = u_+$ (resp. $u = u_-$) if m is even (resp. odd). Consider $u \in \mathcal{R}_m$ and assume m is even (the same arguments apply for m odd). In the isothermal coordinates (x,θ), we have $u_k = \tilde{u}_k e^{ik\theta}$ and we define

$$U_k = 2 \cdot \mathrm{Re}\left(\tilde{u}_k e^{k\lambda}(dz)^{\otimes k}\right)$$

It is a k-symmetric tensor such that if $(x,v) = (x, e^{-\lambda}e^{i\theta}) \in SM$, then :

$$U_k(x,v,...,v) = f_{-k}(x,v) + f_k(x,v)$$

We can raise the degree of this tensor by two, by tensoring with the metric g and symmetrizing, namely we define $\alpha U_k = \sigma\left(U_k \otimes g\right)$. Moreover, the restriction of the metric to SM is the constant function $\mathbf{1}_{SM}$ defined on SM, namely $\Phi^2(g) = \mathbf{1}_{SM}$. Thus, the restriction of αU_k to SM is still $u_k + u_{-k}$. Now we set

$$U := \sum_{k=0}^{m/2} \alpha^k U_{m-2k},$$

and we have by construction $\Phi^m(U) = u$. In other words, we have proved that :

Proposition 3.9. — *For each $m \geqslant 0$, $\Phi^m : \Gamma(M, \otimes_S^m T^*M) \to \mathcal{R}_m$ is bijective. In particular, $\mathcal{R}_0$ can be identified with the set of smooth functions $\mathcal{C}^\infty(M)$.*

3.2.3. The inner derivation

The main idea which will be used throughout the rest of this memoire is to transfer the analysis of tensor fields on functions defined on the unit tangent bundle via the previous identification. Thus, we need to understand from this functional point of view what is the inner derivative of a tensor. We have the

Proposition 3.10. —

$$\Phi^{k+1}(dT) = X \cdot \Phi^k(T) \tag{3.2}$$

Démonstration. — Let T be a smooth symmetric m-tensor. In local coordinates $(x_1, ..., x_n)$, we can express the coordinates of ∇T :

$$\begin{aligned}
(\nabla T)_{i_0 \ldots i_m} &= \nabla T \left(\frac{\partial}{\partial x_{i_0}}, ..., \frac{\partial}{\partial x_{i_m}} \right) \\
&= \left(\nabla_{\frac{\partial}{\partial x_{i_0}}} T \right) \left(\frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_m}} \right) \\
&= \frac{\partial}{\partial x_{i_0}} \cdot \left(T \left(\frac{\partial}{\partial x_{i_1}}, ..., \frac{\partial}{\partial x_{i_m}} \right) \right) \\
&\qquad - \sum_{j=1}^{m} T \left(\frac{\partial}{\partial x_{i_1}}, ..., \nabla_{\frac{\partial}{\partial x_{i_0}}} \frac{\partial}{\partial x_{i_j}}, ..., \frac{\partial}{\partial x_{i_m}} \right) \\
&= \frac{T_{i_1 \ldots i_m}}{\partial x_{i_0}} - \sum_{j=1}^{m} \sum_{k=1}^{n} \Gamma^k_{i_0 i_j} T_{i_1 \ldots i_{j-1} k i_{j+1} \ldots i_m}
\end{aligned}$$

Thus, ∇T can be written in coordinates :

$$\nabla T = (\nabla T)_{i_0 \ldots i_m} dx_{i^0} \otimes ... \otimes dx_{i^m},$$

where the $(\nabla T)_{i_0 \ldots i_m}$ have just been given and the sum is taken over $1 \leqslant i_0, ..., i_m \leqslant n$. As a consequence, $\sigma \nabla T$, that is the symmetrization of ∇T can be seen via the application Φ^{m+1} as the function (the sum is still implicit) :

$$\frac{\partial T_{i_1 \ldots i_m}}{\partial x_{i_0}} v^{i_0} v^{i_1} ... v^{i_m} - \sum_{j=1}^{m} \sum_{k=1}^{n} \Gamma^k_{i_0 i_j} T_{i_1 \ldots i_{j-1} k i_{j+1} \ldots i_m} v^{i_0} ... v^{i_m}$$

Now, the first term can easily be factored as

$$\left(v^{i_0} \frac{\partial}{\partial x_{i_0}} \right) \cdot \left(T_{i_1 \ldots i_m} v^{i_1} ... v^{i_m} \right),$$

while the second term involves some more efforts in the computation (one has to rearrange the summation) and can also be factored as

$$\left(\Gamma^{i_0}_{ij} v^i v^j \frac{\partial}{\partial v_{i_0}} \right) \cdot \left(T_{i_1 \ldots i_m} v^{i_1} ... v^{i_m} \right)$$

We recognize thanks to the expression (2.25) the vector field X acting on $T_{i_1 \ldots i_m} v^{i_1} ... v^{i_m}$. □

3.2.4. Actions on 1-forms

Eventually, let us end this paragraph with two short computations which will be used in the sequel. Let A be a smooth real-valued 1-form on M. According to the previous paragraph, we can identify A with the function $\Phi^1(A) \in \Omega_{-1} \oplus \Omega_1$. Let us denote $a = \Phi^1(A)$. Then we have the :

Lemma 3.11. —
$$\Phi^0(\delta A) = \eta_+ a_{-1} + \eta_- a_1$$

In particular, a 1-form is solenoidal if and only if $\eta_+ a_{-1} + \eta_- a_1 = 0$.

Démonstration. — In local coordinates, using the Christoffel symbols for the isothermal coordinates, one can see that (the other terms vanish) :
$$\delta A = -\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2}\right)$$
Applying the expressions given in Remark 2.7 linking the $\partial/\partial x_i$ to the moving frame (and thus to $\eta_\pm$) and the fact that
$$a_1 = e^{-2\lambda}\frac{A_1 - iA_2}{2}$$
$$a_{-1} = e^{-2\lambda}\ \frac{A_1 + iA_2}{2},$$
we obtain the sought result. □

Lemma 3.12. —
$$\Phi^1(\star db) = H \cdot \Phi^0(b)$$

Démonstration. — A computation in local coordinates using expression (2.27) for H provides the result. □

3.3. The Riccati equation

The reference for this paragraph is [22].

3.3.1. The Jacobi vector fields

Let $\gamma : [a, b] \to M$ be a geodesic joining $\gamma(a)$ and $\gamma(b)$. The Jacobi vector fields are the vector fields along γ for which the hessian of the energy functional is degenerate. The normal Jacobi vector fields are those among the Jacobi vector fields which are orthogonal to $\dot\gamma$. They are the vector fields along γ for which the hessian of the length functional is degenerate. Standard computations allow to prove that Jacobi vector fields satisfy the Jacobi equation along γ
$$Y'' + R(Y, \dot\gamma)Y = 0,$$
where R denotes the curvature tensor and with initial conditions $Y(a) = v, Y'(a) = w$. The normal Jacobi vector fields are obtained for the same equations but with initial values $v, w \perp \dot\gamma(a)$. The vector space of Jacobi (resp. normal Jacobi) vector fields along γ is a $2n$-dimensional (resp. $2(n-$

1)-dimensional) vector space : this comes from the Cauchy theorem, and the fact that a Jacobi field is completely determined by its initial values v and w. Note that the most interesting Jacobi fields are those for which $v = 0$ (which geometrically means that the ends in the variation formula of the hessian of the energy functional are fixed).

The Jacobi vector fields can be used in order to compute the differential of the geodesic flow φ_t. Indeed, given $(x, v) \in TM$, if $\xi \in T_{(x,v)}TM$, and Y denotes the Jacobi vector field with initial conditions $J_\xi(0) = d\pi_{(x,v)}(\xi), \dot{J}_\xi(0) = \mathcal{K}_{(x,v)}(\xi)$, then :

$$d(\varphi_t)_{(x,v)}(\xi) = (d\pi_{\varphi_t(x,v)})^{-1}(Y(t)) + (\mathcal{K}_{\varphi_t(x,v)})^{-1}(\dot{Y}(t)) \tag{3.3}$$

In the two-dimensional case, the normal Jacobi equation reduces to a scalar equation, namely :

$$f'' + Kf = 0,$$

with $f(a) = \lambda, f'(a) = \mu$. We say that a point $q = \gamma(t_0)$ is conjugate to $p = \gamma(a)$ if there exists a normal Jacobi field along γ such that $f(t_0) = 0$ and $f(a) = 0$. In particular, if $(x_1, v_1) = \varphi_t(x_0, v_0)$ are conjugate, taking $\xi = V(x_0, v_0)$ in the previous equation (3.3) (which satisfies $Y(0) = d\pi_{(x,v)}(V) = 0$), we obtain

$$d(\varphi_t)_{(x_0,v_0)}(V) = (\mathcal{K}_{\varphi_t(x,v)})^{-1}(\dot{Y}(t)) \in \mathbb{V}_{(x_1,v_1)} = \mathrm{Span}(V(x_1, v_1))$$

3.3.2. The Riccati equation

Setting $u = f'/f$, we obtain the equivalent equation along γ,

$$u' + u^2 + K = 0$$

which can be rewritten as the Riccati equation on SM :

$$X \cdot u + u^2 + K = 0 \tag{3.4}$$

We will admit the following theorems. They will be used in the next paragraph in order to provide an *α-control.*

THEOREM 3.13 (Hopf). — *If (M, g) is free of conjugate points, then there exists a bounded measurable function $u : SM \to \mathbb{R}$, differentiable in the direction X, solution to the Riccati equation (3.4).*

For a proof of Hopf's theorem, we refer to his short original article [17]. It is possible to better Hopf's theorem for Anosov manifolds :

THEOREM 3.14. — *If (M, g) is Anosov, then there exists two bounded solutions r^+, r^- of class $\mathcal{C}^1$ to the Ricatti equation (3.4), which are everywhere distinct ($r^+ - r^- > 0$ actually). Moreover, $-H(x, v) + r^+(x, v)V(x, v)$ (resp. $-H + r^-V$) spans the bundle $E^s(x, v)$ (resp. $E^u(x, v)$).*

Remark 3.15. — Actually, the regularity can be bettered up to $\mathcal{C}^{2-\varepsilon}$ for any $\varepsilon > 0$, as established by Katok-Hurder in [18] because we are in dimension 2.

We will also omit the proof of this theorem. We refer to [20], where it is proved for an even wider class of flows, that of Anosov λ-geodesic flows. There are various implications of this theorem. Let us just quote two of them. The first one is that there cannot exist conjugate points on an Anosov manifold because there exists bounded (measureble and $\mathcal{C}^1(SM)$) solutions to the Riccati equation (and therefore non-vanishing solutions to the initial Jacobi equations). The second implication is rather surprising. This last theorem actually allows to prove that the sphere and the torus cannot carry an Anosov flow, in an elegant fashion. Indeed, assume this is the case, then we have the existence of such functions $r^{\pm}$. Take $r = r^+$ or r^- and integrate the Riccati equation over the unit tangent bundle SM :

$$\int_{SM} X \cdot r + \int_{SM} r^2 + \int_{SM} K = 0$$

The first term vanish since the Liouville form on SM is preserved by the geodesic flow X. Since K is constant over the fibers, the formula of integration over the fibers (C.1) gives :

$$\int_{SM} K = 2\pi \int_M K = 4\pi^2 \chi(M),$$

by the Gauss-Bonnet formula. Thus :

$$\int_{SM} r^2 = -4\pi^2 \chi(M)$$

Since $\chi(\mathbb{S}^2) = 2$, we obtain a contradiction. For the torus, $\chi(\mathbb{T}^2) = 0$, so we obtain $r = 0$, that is $r^+ = r^- = 0$, which is absurd too.

3.3.3. α-controlled surfaces

Let us conclude this section by a less common result on surfaces. It will be used in Section 6. Given $\alpha \in [0,1]$, we will say that a surface is α-controlled if for any $\psi \in \mathcal{C}^\infty(SM)$ such that $\psi|_{\partial(SM)} = 0$, one has :

$$||X \cdot \psi||^2 - (K\psi, \psi) \geqslant \alpha ||X \cdot \psi||^2$$

THEOREM 3.16. — *A surface (M, g) which is free of conjugate points is 0-controlled. In other words, for any $\psi \in \mathcal{C}^\infty(SM)$, such that $\psi|_{\partial(SM)} = 0$,*

$$||X \cdot \psi||^2 - (K\psi, \psi) \geqslant 0 \tag{3.5}$$

Démonstration. — Consider a solution $u : SM \to \mathbb{R}$ to the Riccati equation given by Hopf's Theorem 3.13. For $\psi \in \mathcal{C}^\infty(SM)$, we obtain :

$$|X\cdot\psi - u\psi|^2 = |X\cdot\psi|^2 - 2\mathrm{Re}(u\bar{\psi}X\cdot\psi) + u^2|\psi|^2 = |X\cdot\psi|^2 + |\psi|^2(X\cdot u + u^2) - X\cdot(u|\psi|^2)$$

Using the invariance of the Liouville volume form Θ on SM by the geodesic flow X and the fact that ψ vanishes on $\partial(SM) = 0$, we obtain :

$$||X\cdot\psi - u\psi||^2 = ||X\cdot\psi||^2 + (X\cdot u + u^2, |\psi|^2) = ||X\cdot\psi||^2 - (K\psi,\psi),$$

where we used the Riccati equation in the last equality. □

Remark 3.17. — Actually, one can prove that the equality in (3.5) holds if and only if $\psi = 0$. Indeed, in this case, using the previous equalities, we obtain $X\cdot\psi = u\psi$ for $u = r^\pm$, so $(r^+ - r^-)\psi = 0$ and $\psi = 0$.

For an Anosov surface, one can even prove a better result, namely :

Theorem 3.18. — *Assume (M, g) is Anosov. Then, there exists $\alpha > 0$, such that for all $\psi \in \mathcal{C}^\infty(SM)$,*

$$||X\cdot\psi||^2 - (K\psi,\psi) \geqslant \alpha\left(||X\cdot\psi||^2 + ||\psi||^2\right) \tag{3.6}$$

Démonstration. — Consider the solutions $r^\pm$ to the Riccati equation (3.4). We know that they are bounded and $\mathcal{C}^1$, $r^+ - r^- > 0$ and the manifold is compact, so there exists constants $C, D > 0$, such that $C \leqslant r^+ - r^- \leqslant D$. We define $A = X\cdot\psi - r^-\psi, B = X\cdot\psi - r^+\psi$. Thanks to our previous computations, we know that $||A||^2 = ||B||^2 = ||X\cdot\psi||^2 - (K\psi,\psi)$. Moreover, we have :

$$\psi = \frac{1}{r^+ - r^-}(A - B)$$

$$X\cdot\psi = \lambda A + (1-\lambda)B,$$

with $\lambda = r^+/(r^+ - r^-)$. From these equations, we obtain the constant α such that :

$$2\alpha||\psi||^2 \leqslant ||A||^2$$

$$2\alpha||X\cdot\psi||^2 \leqslant ||A||^2,$$

which gives the sought result. □

3.4. A technical toolbox

The following results can be found in [14] and [25].

3.4.1. The raising and lowering operators η^+ and η^-

DEFINITION 3.19. — *We define the following operators on $C^\infty(SM)$:*

$$\eta^+ = \frac{1}{2}(X - iH) \tag{3.7}$$

$$\eta^- = \frac{1}{2}(X + iH) \tag{3.8}$$

An immediate computation shows that the Cartan structural equations can be rewritten as :

$$[-iV, \eta^+] = \eta^+ \tag{3.9}$$

$$[-iV, \eta^-] = -\eta^- \tag{3.10}$$

$$[\eta^+, \eta^-] = \frac{iK}{2}V \tag{3.11}$$

We set $\Omega_k = H_k \cap C^\infty(SM)$. In the rest of this paragraph, we detail some useful results concerning these operators.

PROPOSITION 3.20. — *$\eta^\pm : \Omega_k \to \Omega_{k\pm1}$ and the formal adjoint of η^+ is $-\eta^-$.*

Démonstration. — Let $u \in \Omega_k$. Then, we have by (3.9) :

$$(-iV)\eta^+u = [-iV, \eta^+]u + \eta^+(-iV)u = \eta^+u + k\eta^+u = (k+1)\eta^+u$$

The same computation can be applied to η^-, using (3.10). The computation of the adjoint comes from the fact that $-iX$ and $-iH$ are self-adjoint operators on a dense set of $L^2(SM)$ containing $\mathcal{C}^\infty(SM)$. □

We now assume that the surface (M, g) is topologically equivalent to a torus with $g \geqslant 2$ holes. In particular, this is the case if (M, g) has negative curvature, or if (M, g) is Anosov, that is the geodesic flow is Anosov (see Theorem C.4 in the Appendix). Then we have the following result :

PROPOSITION 3.21. —
- *For $k \leqslant -2$, η_+ is surjective and* $\dim$ *ker* $(\eta_+) = (-2k-1)(g-1)$.
- *For $k = -1$,* $\dim$ *ker* $(\eta_+) = g$.
- *For $k = 0$,* $\dim$ *ker* $(\eta_+) = 1$.
- *For $k \geqslant 1$, $\eta_+ : \Omega_k \to \Omega_{k+1}$ is injective.*

PROPOSITION 3.22. —
- *For $k \leqslant -1$, $\eta_- : \Omega_k \to \Omega_{k-1}$ is injective.*
- *For $k = 0$,* $\dim$ *ker* $(\eta_-) = 1$.
- *For $k = 1$,* $\dim$ *ker* $(\eta_-) = g$.
- *For $k \geqslant 2$, η_- is surjective and* $\dim$ *ker* $(\eta_-) = (2k-1)(g-1)$.

Démonstration. — Thanks to the explicit expression of the vector fields X and H in isothermal coordinates (x, θ), it is possible to compute explicitly $\eta_\pm u$ for $u \in \Omega_k$. Indeed, if $u(x, y, \theta) = h(x, y)e^{ik\theta}$ in local isothermal coordinates, then one has

$$\eta_-(u) = e^{-(k+1)\lambda}\bar{\partial}(he^{k\lambda})e^{i(k-1)\theta}, \tag{3.12}$$

$$\eta_+(u) = e^{(k-1)\lambda}\partial(he^{-k\lambda})e^{i(k+1)\theta}, \tag{3.13}$$

where $\partial = \dfrac{1}{2}(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y})$ and $\bar{\partial} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y})$.

Therefore, the operators $\eta_\pm$ are almost the operators ∂ and $\bar{\partial}$, which have been well-studied in literature so far. For instance, as far as η_- is concerned, its surjectivity and/or injectivity is equivalent to that of $\bar{\partial}$. This stems from the Riemann-Roch theorem (see Appendix E or [29] (Chapter 10) for instance). ☐

Proposition 3.23. — *$\eta^+ : \Omega_k \to \Omega_{k+1}$ and $\eta^- : \Omega_k \to \Omega_{k-1}$ are first order elliptic operators.*

Démonstration. — Let us use the expression (3.12). Ω_k can be seen as a line bundle over M and we can write :

$$\eta_-(he^{ik\theta}) = (e^{-\lambda}\bar{\partial}h + ah)e^{i(k-1)\theta},$$

for some smooth functions a. Now, the symbol of $\bar{\partial}$ is given by $-i(\xi_x + i\xi_y)$ so the principal symbol of η_- is $e^{-\lambda}(-i\xi_x + \xi_y) \neq 0$, as long as $\xi \neq 0$. ☐

Corollary 3.24. — *$\Omega_k = \eta_+(\Omega_{k-1}) \oplus \ker(\eta_-)$, and the decomposition is orthogonal.*

Corollary 3.25. — *If f is smooth and $\eta_+ u = f$ (resp. $\eta_- u = f$), then u is smooth.*

3.4.2. Max Noether's theorem

Let us end this section by a result of Max Noether which appears in the proof of the injectivity of the ray transform on Anosov manifold. We will not give a proof of this theorem and refer to [6] for further details. If (M, g) is a smooth oriented surface, we know that it admits an underlying holomorphic structure, making it a Riemann surface (see Appendix B for instance). We say that M is *hyperelliptic* if there exists a holomorphic map $f : M \to \mathbb{S}^2$ of degree two.

Theorem 3.26 (Noether). — *On a non-hyperelliptic surface of genus $g \geqslant 2$, the kernel of $\bar{\partial} : \kappa^{\otimes m} \to \kappa^{\otimes m} \otimes \bar{\kappa}$ is spanned by the m-fold products of the abelian differentials of the first kind.*

In the case M is hyperelliptic, one can use the fact the a closed Riemann surface of genus $g \geqslant 2$ admits Galois covers of arbitrary degree. Given an integer $n \geqslant 1$, there exists a Galois cover $\pi : N \to M$ of degree n, where N is a Riemann surface with genus $n(g-1)+1$. If M is hyperelliptic, then one can show that N will be hyperelliptic only if $n = 2$ or 4 (see [19]), so by taking $n \geqslant 5$, we can ensure that N is not hyperelliptic. This trick is used once in the memoire.

A consequence of Max Noether's theorem is the following. Assume that M is a non-hyperelliptic surface. Then, according to Theorem 3.26, we know that the k-fold products of abelian differentials of the first kind span the space of holomorphic k-differentials. In other words, writing this in terms of smooth functions on SM, if $\eta_- u = 0$ and $u \in \Omega_k$, then there exists $a_i^{(j)} \in \Omega_1$ such that $\eta_- a_i^{(j)} = 0$ and :

$$u = \sum_{i=1}^{N} a_i^{(1)} ... a_i^{(k)}$$

This fact will be crucial in Section 6.

3.4.3. The Pestov identity

In this paragraph, the scalar product considered is the canonical scalar product in $L^2(SM)$.

Proposition 3.27. — *Let $u \in \mathcal{C}^\infty(SM)$. Then :*

$$||XVu||^2 - (KVu, Vu) + ||Xu||^2 - ||VXu||^2 = 0 \tag{3.14}$$

Démonstration. — We denote by P the partial differential operator on SM defined as :

$$P := VX$$

The formal adjoint P^* of P is XV. We can compute the bracket $[P^*, P]$ thanks to the Cartan structural equations :

$$\begin{aligned}
[P^*, P] &= XVVX - VXXV \\
&= ([X,V] - VX)\, VX - VX\, ([X,V] - VX) \\
&= VXVX + HVX - VXVX - VXH \\
&= V[H,X] - X^2 \\
&= -X^2 + VKV
\end{aligned}$$

Now, we have for $u \in \mathcal{C}^\infty(SM)$

$$||Pu||^2 = (Pu, Pu) = (u, P^*Pu) = (u, [P^*, P]u) + ||P^*u||^2,$$

which is the identity sought when replacing the operator P by the operators V and X. □

Remark 3.28. — In particular, if $u \in H_k$, then :

$$(k^2 + 1)||Xu||^2 - k^2(Ku, u) - ||VXu||^2 = 0$$

Using $X = \eta^+ + \eta^-$, we obtain :

$$-\frac{k}{2}(Ku, u) + ||\eta^- u||^2 = ||\eta^+ u||^2 \tag{3.15}$$

3.4.4. Proof of Lemma 3.7

Démonstration. — Note that it is sufficient to check the formula for $u \in \Omega_k$. If $|k| > 1$, then the right member clearly vanishes and it is easy to check that X and $\mathcal{H}$ commute. If $k = 1$ (the case $k = -1$ being analogous), $H \cdot u_0$ vanishes, $X \cdot u \in \Omega_0 \oplus \Omega_2$ and $\mathcal{H}X \cdot u = -i(X \cdot u - (X \cdot u)_0)$. On the other hand, $X\mathcal{H}u = -iX \cdot u$. Thus $[\mathcal{H}, X]u = i(X \cdot u)_0$, which coincides with $(H \cdot u)_0$ since $X = \eta_+ + \eta_-, H = -i(\eta_+ - \eta_-)$. The case $k = 0$ is also immediate. □

4. The X-ray transform

4.1. Definition of the X-ray transform

4.1.1. A first definition

Let us first give the general idea behind the X-ray transform, before detailing it according to the nature of the manifold involved. Let (M, g) be a n-dimensional Riemannian manifold with or without boundary. We denote by $\mathcal{G}$ the set of its *closed* unit-speed geodesics. In the boundaryless case, this simply means periodic geodesic while in the case with boundary, this means geodesics which *exit* the manifold in finite time. Let T be a smooth symmetric m-tensor. In the following, we will still denote by T the smooth function $\Phi^m(T) \in \mathcal{C}^\infty(SM)$, canonically associated to T, as explained in Section 3.2.2. We define the ray transform I_m for $\gamma \in \mathcal{G}$ by :

$$I_m T(\gamma) = \int_0^T T_{\gamma(t)}(\dot{\gamma}(t))dt,$$

where L denotes the length of the geodesic. Of course, this definition can be extended to the case of infinite time geodesics like *trapped* geodesic in the case with boundary (geodesics which do not exit the manifold) as long as the tensor integrated satisfies some good integration properties, but we will not go that much into this issue.

4.1.2. The notion of s-injectivity

The fundamental question we want to answer is : what properties on f (or T) can we recover from the knowledge of its ray transform If (or $I_m T$) ? For instance, if $I_m T = 0$, that is $I_m T(\gamma) = 0$ for any $\gamma \in \mathcal{G}$, can we prove that $T = 0$? The answer to this question is usually negative. If T is a symmetric $(m-1)$-tensor, we know, according to (3.2), that its inner derivative $dT = \sigma \nabla T$ is given, in terms of functions on SM by $X \cdot T$. Moreover, on a manifold with boundary, if we assume that $T|_{\partial M} = 0$, then it becomes clear that $I_m(dT) = 0$ by the fundamental theorem of Analysis.

Recall that, according to Theorem 3.8, one can decompose any smooth symmetric m-tensor T in $T = T^s + dh$, where T^s is a symmetric m-tensor with zero divergence and h is an $(m-1)$-tensor which is zero on ∂M (in the case M admits a boundary). We refer to Appendix D for a proof of this result. The part T^s is called the *solenoidal* part of the tensor T, while the part dh is called the *potential* part. We have just seen that we cannot expect to recover the potential part of a tensor from the knowledge of its ray transform.

DEFINITION 4.1. — *We will say that the ray transform I_m is s-injective, if it is injective on the set of solenoidal m-tensors. In other words, it it s-injective if for any symmetric m-tensor T, $I_mT = 0$ implies $T = dh$, for some $(m-1)$-tensor h such that $h|_{\partial M} = 0$. In the case of functions, that is $m = 0$, the s-injectivity reduces to the injectivity, that is for any $f \in \mathcal{C}^\infty(M)$, $I_0 f = 0$ implies $f = 0$.*

Remark 4.2. — In the rest of this memoire, I will denote the ray transform acting on smooth functions in $\mathcal{C}^\infty(SM)$ (or, when explicitly detailed in $L^2(SM)$). We will use the index I_m to insist on the fact that we refer to the ray transform acting on symmetric m-tensors or, equivalently, on smooth functions in the space $\mathcal{R}_m$.

4.1.3. The Livcic property

Given $x \in \partial M$, we define the second fundamental form :

$$S_x : \left| \begin{array}{l} T_x(\partial M) \times T_x(\partial M) \to \mathbb{R} \\ (v, w) \mapsto g(\nabla_v \nu, w) \end{array} \right.$$

We say that the manifold is *strictly convex* if S_x is definite positive for any $x \in \partial M$. A manifold is said to be *simple* if it is simply connected, does not possess any conjugate points and is strictly convex. One can prove that such a manifold is diffeomorphic to a ball of $\mathbb{R}^n$ and the exponential map is a diffeomorphism in each point of the manifold. Indeed, the absence of conjugate points imply that the exponential map is a local diffeomorphism (this can be seen using equation (3.3) where the differential of the exponential is computed explicitly in terms of the Jacobi vector fields). Since M is complete (by the Hopf-Rinow theorem, because it is compact), the exponential map is surjective and becomes in each point a covering map and since M is simply connected, it is a diffeomorphism. Thus, given $x \in \mathring{M}$, there exists a closed compact set $K_x \subset T_xM$ such that $\exp_x : K_x \to M$ is a diffeomorphism. Now K_x is diffeomorphic to a ball because it is star-shaped in 0 (the geodesics in K_x are sent on "rays" or straight lines) and ∂K_x is a smooth graph over a sphere εS_x for some $\varepsilon > 0$ (where S_x denotes the unit tangent sphere in x) because of the strictly convex boundary condition (see Section 4.4.1). In particular, one obtains that such a manifold is non-trapping because any geodesic starting from x will hit the boundary in finite time. Note that the converse is also true : if the surface is non-trapping, then the $\pi_1(M)$ has to be trivial, otherwise there would exist a periodic geodesic (see [8], Section 2.98 for instance).

The two cases that we shall consider are :

— compact closed Anosov manifolds (manifolds for which the geodesic flow is Anosov on the unit tangent bundle),
— compact and simple manifolds.

The interest of these two sets of manifolds lies in the fact that they both satisfy the *Livcic property* : if the integral of a smooth function $f \in \mathcal{C}^\infty(SM)$ is zero along every closed integral curve of the geodesic field X, then then there exists a function $u \in \mathcal{C}^\infty(SM)$ such that $f = X \cdot u$. In the case of a compact and simple manifold, this means that the integral of f is zero along every geodesic (in particular, all the geodesics have their extremal points on the boundary of M) and the function u has to satisfy $u|_{\partial M} = 0$ by the fundamental theorem of Analysis. In the case of a compact closed Anosov manifold, this means that the integral of f is zero along every periodic geodesic.

Let us end this paragraph with an important remark. We assume that the manifold (M, g) satisfies the smooth Livcic property. Thus, we know that if $I_m T = 0$, there exists a smooth function h such that $T = X \cdot h$. Now, assume we can prove that $u \in \mathcal{R}_{m-1}$, then going back to the tensors via the application Φ, this exactly means that there exists a smooth tensor h such that $T = dh$, and I_m is s-injective. As a consequence, on a manifold satisfying the Livcic property, the injectivity of I_m actually "reduces" to proving that if $I_m T = 0$, then there exists a smooth function $u \in \mathcal{R}_{m-1}$ such that $T = X \cdot u$.

4.2. Definition on a compact manifold with boundary

4.2.1. Another definition of the X-ray transform

Let (M, g) be a n-dimensional manifold with boundary. We denote by ν the unit outer normal to ∂M. The unit tangent bundle SM of M is a $(2n-1)$-dimensional manifold with boundary. We denote by $\pi : SM \to M$ the projection. $\partial(SM)$ is therefore a $(2n-2)$-dimensional manifold which we can write $\partial(SM) = \partial_+(SM) \cup \partial_-(SM)$ where :

$$\partial_\pm(SM) = \{(x, v) \in \partial(SM), \mp g(\nu(x), v) \geqslant 0\}$$

We also define :

$$\partial_0(SM) = \{(x, v) \in \partial(SM), g(\nu(x), v) = 0\}$$

For $(x, v) \in SM$, we denote by $\varphi_t(x, v)$ the geodesic flow starting from the point x in the direction v.

It is always possible to endow the manifold (M,g) in a boundaryless manifold (N,h) such that if i denotes the inclusion, then $i^*h = g$. We define the travel time $\tau : SM \to [0,\infty]$ by :

$$\tau(x,v) = \inf\left\{t > 0, \pi(\phi^t(x,v)) \in N \setminus M\right\}$$

Of course, this definition does not depend on the embedding chosen. Clearly, this time can be infinite. We say that the manifold M is *non-trapping* if $\tau(x,v) < \infty$ for any point $(x,v) \in SM$. In the following, we will assume M to be non-trapping.

Given a smooth function f defined on M, we define its ray transform by :

$$I_0 : \left| \begin{array}{l} \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\partial_+(SM)) \\ f \mapsto \left((x,v) \mapsto I_0 f(x,v) = \int_0^{\tau(x,v)} f(\pi(\varphi_t(x,v)))\, dt\right) \end{array}\right. \tag{4.1}$$

It is the integral of f taken along the geodesic curves. Just like in the previous paragraph, it can be easily generalized to any symmetric m-tensor T on M. Indeed, let us still denote by T the smooth function induced by the tensor on SM (it is $\Phi^m(T)$ in our previous notations). Then we define the ray transform I_m of the tensor by :

$$\forall (x,v) \in \partial_+(SM), \quad I_m T(x,v) = \int_0^{\tau(x,v)} T(\varphi_t(x,v))\, dt$$

4.2.2. The map I

From now on, we assume that the manifold is simple. We will discuss these assumptions in a next paragraph. In particular, we insist on the fact that such a manifold is necessarily non-trapping.

The natural measure $\tilde{\Theta}$ on $\partial(SM)$ is the restriction of the Liouville measure Θ on SM to the boundary. On $\partial_+(SM)$, we consider the measure $d\mu(x,v) = \langle \nu(x), v\rangle \tilde{\Theta}_{(x,v)}$. The ray transform can naturally be extended to a bounded operator $I : L^2(SM) \to L^2(\partial_+(SM), \mu)$.

Proposition 4.3. — *$I : L^2(SM) \to L^2(\partial_+(SM), \mu)$ is bounded*

The proof of this proposition relies on Santaló's formula, which traduces, in terms of measure, a disintegration of the Liouville measure Θ on SM along the flow lines of the geodesic vector field X :

Lemma 4.4. — *Let (M,g) be a manifold with boundary such that the set of trapped points has zero measure and $h \in L^1(SM)$. Then :*

$$\int_{SM} h \cdot \Theta = \int_{\partial_+(SM)} \left(\int_0^{\tau(x,v)} h(\varphi_t(x,v))dt\right) d\mu$$

Proof of Lemma 4.4. — It is sufficient to prove the result for $h \in \mathcal{C}_c^\infty(\overset{\circ}{SM})$ and a density argument allows to conclude. Given $(x,v) \in SM$, we have

$$\int_0^{\tau(\varphi_t(x,v))} h(\varphi_s(\varphi_t(x,v)))ds = \int_0^{\tau(x,v)-t} h(\varphi_{s+t}(x,v))ds = \int_t^{\tau}(x,v)h(\varphi_s(x,v))ds,$$

so we obtain $X \cdot \int_0^{\tau(x,v)} h(\varphi_s(x,v))ds = -h(\varphi_t(x,v))$. Thus, applying Green's formula, using the fact that X preserves the Liouville measure :

$$\begin{aligned}\int_{SM} h \cdot \Theta &= -\int_{SM} X \cdot \left(\int_0^{\tau(x,v)} h(\varphi_s(x,v))ds\right) \cdot \Theta \\ &= \int_{\partial SM} \left(\int_0^{\tau(x,v)} h(\varphi_s(x,v))ds\right) \langle X, \nu \rangle \cdot \tilde{\Theta}\end{aligned}$$

It is immediate that the integral vanishes on $\{(x,v) \in \partial(SM), \langle v, \nu(x) \rangle > 0\}$, which gives the sought result.

$\square$

Proof of Proposition 4.3. — Since M is strictly convex, we know (see Paragraph 4.4.1) that the travel time $\tau : \partial_+(SM) \to [0,\infty)$ is continuous and the manifold is non-trapping, so there exists a maximum travel time $T = \sup \tau$.

$$\begin{aligned}||If||^2_{L^2(\partial_+(SM),\mu)} &= \int_{\partial_+(SM)} \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v))dt\right)^2 d\mu \\ &\leqslant \int_{\partial_+(SM)} T \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v))^2 dt\right) d\mu \\ &= T \int_{SM} f^2 \cdot \Theta,\end{aligned}$$

where we used Santalo's formula in the last equality. Note in particular that I is bounded by the square root of the maximum travel time. $\square$

4.2.3. The adjoint map I_0^*

The natural adjoint of $I_0 : L^2(M) \to L^2(\partial_+(SM),\mu)$ is an operator $I_0^* : L^2(\partial_+(SM),\mu) \to L^2(M)$. For $h \in \mathcal{C}^\infty(\partial_+(SM))$, we define $h_\psi(x,v) = h(\varphi_{-\tau(x,-v)}(x,v))$.

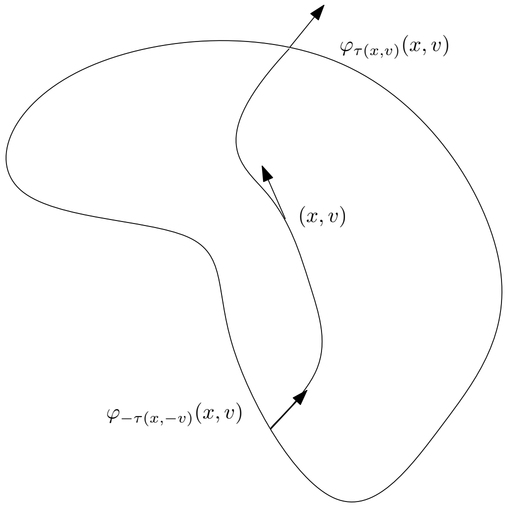

FIGURE *4.1.* The travel time

PROPOSITION 4.5. — *The adjoint of I_0 for the natural L^2 inner products is the operator*

$$I_0^* : \left| \begin{array}{l} L^2(\partial_+(SM),\mu) \to L^2(M) \\ h \mapsto \left(x \mapsto I_0^* h(x) = \int_{S_x} h_\psi(x,v)dv\right), \end{array} \right.$$

where S_x is the sphere bundle in x and dv is the volume form on S_x, i.e. the restriction of the Liouville form on S_x (which is also the euclidean volume form on the sphere).

Démonstration. — Consider $f \in \mathcal{C}^\infty(M), w \in \mathcal{C}^\infty(\partial_+(SM))$. Note that for $(x,v) \in \partial_+(SM)$, we have $w(x,v) = w_\psi(\varphi_t(x,v))$, for any $t \in [0, \tau(x,v)]$. Then, we have :

$$\begin{aligned} (I_0 f, w)_{L^2(\partial_+(SM),\mu)} &= \int_{\partial_+(SM)} \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v))dt \right) w(x,v) d\mu \\ &= \int_{\partial_+(SM)} \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v)) w_\psi(\varphi_t(x,v))dt \right) d\mu \\ &= \int_{SM} f w_\psi \cdot \Theta \\ &= \int_M f(x) \left(\int_{S_x} w_\psi(x,v) dv \right) \mathrm{dvol}^g, \end{aligned}$$

where we applied Santalo's formula in the second last equality and the formula of integration over the fibers (C.1) in the last one. □

Remark 4.6. — Note that for $h \in \mathcal{C}^\infty(SM)$, $h_0(x) = \dfrac{1}{2\pi}\int_{S_x} h(x,v)dv$. Here, we obtain :

$$I_0^* h(x) = 2\pi (h_\psi)_0(x)$$

We are mostly interested in smooth functions on the unit sphere bundle (which will be seen for us as the associated function to a smooth tensor). A problem may come from the fact that it is not clear that $I_0 : \mathcal{C}^\infty(SM) \to \mathcal{C}^\infty(\partial_+(SM))$, because of the non-smoothness of τ (see below). Moreover, it is also rather uncertain that $I_0^* : \mathcal{C}^\infty(\partial_+(SM)) \to \mathcal{C}^\infty(SM)$, all the more so since this is actually wrong. Indeed, for $w \in \mathcal{C}^\infty(\partial_+(SM))$, the smoothness of w_ψ is only guaranteed on $SM \setminus \partial_0(SM)$ and thus the smoothness of

$$I_0^* w(x) = \int_{S_x} w_\psi(x) dv$$

is not guaranteed on ∂M. In order to avoid problems of regularity, we consider $\mathcal{C}^\infty_\alpha(\partial_+(SM))$, the set of functions $h \in \mathcal{C}^\infty(\partial_+(SM))$ such that h_ψ is smooth on SM. We will only study the restriction of $I_0^* : \mathcal{C}^\infty_\alpha(\partial_+(SM)) \to \mathcal{C}^\infty(M)$. In the following, we will see that some proofs of injectivity for the X-ray transform rely on the surjectivity of this operator.

4.2.4. The operator $I_0^* I_0$

The reference for this paragraph is [26], where it was proved for the first time. We also refer to Appendix A for a short review on some elementary facts about pseudodifferential operators. Note that this paragraph is not specific to dimension 2, and we consider a dimension $n \geqslant 2$, even though it will only be applied in the two-dimensional case in the following. In this paragraph, we are going to prove the

Theorem 4.7. — *The operator $I_0^* I_0$ is an elliptic pseudodifferential operator of order -1. Its principal symbol is given by $\sigma(I_0^* I_0) = c|\xi|^{-1}_{g^{-1}}$.*

It will be used in order to obtain the surjectivity of I_0^* in a further section. Let us discuss a bit about this operator. We have :

$$\begin{aligned}(I_0^* I_0 f)(x) &= \int_{S_x} \left(\int_{-\tau(x,-v)}^{\tau(x,v)} f(\varphi_t(x,v)) dt \right) dv \\ &= 2 \int_{S_x} \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v)) dt \right) dv\end{aligned}$$

Note that for a geodesic, we have $\varphi_t(x,v) = \varphi_1(x,tv) = \exp_x(tv)$. Since the manifold is simple, $\exp_x$ is a diffeomorphism onto M, i.e. there exists a closed ball B_x such that $\exp_x : B_x \to M$ is a diffeomorphism. We change variables in the previous formula. Setting $\Phi : (t,v) \mapsto \exp_x(tv) = y$, we have $|\det d\Phi_{(t,v)}| = t^{n-1}|\det d(\exp_x)_{tv}|$. Note that $t = d(x,y)$ (where the distance is computed with the Riemannian metric). The change of variable in the integral gives :

$$(I_0^* I_0 f)(x) = \int_M K(x,y) f(y) d\mathrm{vol}^g(y),$$

with the kernel :

$$K(x,y) = 2\frac{|\det d(exp_x^{-1})_y|}{d^{n-1}(x,y)}$$

This is clearly a smooth function outside the diagonal. Since $d(\exp_x)_0 = id$, we have $|\det d(exp_x^{-1})_0| = 1$ so the singularity of the kernel around the diagonal is contained in the term $1/d^{n-1}(x,y)$.

We recall that the manifold is smooth, as well as the metric. We have the following

Lemma 4.8. — *In coordinates, there exists smooth functions $G_{ij}(x,y)$ such that $G_{ij}(x,x) = g_{ij}(x)$ and :*

$$d^2(x,y) = G_{ij}(x,y)(x-y)^i(x-y)^j$$

Démonstration. — We fix x and define $f(y) = d^2(x,y)$. We can always assume that the neighborhood considered is smaller than the radius of injectivity of the manifold. Given y, we can thus write $y = \exp_x(v(y))$, where $v(y) = (\exp_x)^{-1}(y) \in T_xM$ and $f(y) = ||v(y)||_x^2$. We denote by φ_t^i the flow generated by the $\partial/\partial x_i$. Thus :

$$\frac{\partial f}{\partial x_i}(x) = \frac{d}{dt}\left(f(\varphi_t^i(x))\right)\Big|_{t=0} = 2\langle dv_0(\partial/\partial x_i), v(x)\rangle_x = 0,$$

since $v(x) = 0$. So x is a critical point and we can compute the hessian, using that $dv_0 = id$:

$$\mathrm{Hess}_x f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = 2||\partial/\partial x_i||_x^2 = 2g_{ii}(x)$$

By polarizing the previous identity and developing f up to the second order, we obtain :

$$f(y) = g_{ij}(x)(x-y)^i(x-y)^j + O(||x-y||^3) = g_{ij}(x)(1+O(||x-y||))(x-y)^i(x-y)^j$$

This provides the sought result with $G_{ij}(x,y) = g_{ij}(x)(1+O(||x-y||))$. $\square$

This will allow to compute the principal symbol of $I_0^* I_0$:

Démonstration. — Thanks to the previous lemma, we can therefore write in coordinates, in a vicinity of a point x :

$$K(x,y) = 2\frac{|\det d(exp_x^{-1})_y|}{(G_{ij}(x,y)(x-y)^i(x-y)^j)^{(n-1)/2}}$$

Consider a covering of M by a finite number of charts $\{(U_\alpha, \varphi_\alpha)\}$ and a partition of unity $\sum \chi_\alpha = 1$ related to this covering. Each χ_α has compact support in U_α. We take χ'_α with compact support in U_α such that $\chi'_\alpha = 1$ on $\mathrm{supp}(\chi_\alpha)$. Consider a smooth function u on M. We write everything in coordinates in $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ (and we drop the index α) :

$$\begin{aligned}
I_0^* I_0(\chi u)(x) &= \int K(x,y)\chi'(y)\chi(y)u(y)d\mathrm{vol}^g(y) \\
&= \int_U K(x,y)\chi'(y)\left(\int_{\mathbb{R}^n} e^{iy\cdot\xi}\widehat{u\chi}(\xi)d\xi\right)\sqrt{\det g(y)}dy \\
&= \int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\int_U e^{i(y-x)\cdot\xi}K(x,y)\chi'(y)\sqrt{\det g(y)}dy\right)\widehat{u\chi}(\xi)d\xi \\
&= \int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\widehat{u\chi}(\xi)d\xi,
\end{aligned}$$

where $\cdot$ denotes the usual scalar product on $\mathbb{R}^n$, with

$$p(x,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi}K(x,x-z)\chi'(x-z)\sqrt{\det g(x-z)}dz = \int_{\mathbb{R}^n} e^{-iz\cdot\xi}F(x,z)dz,$$

which can also be written

$$F(x,z) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} p(x,\xi)e^{iz\cdot\xi}d\xi$$

Now, F is a smooth (outside the diagonal) function and $F(x,\cdot) \in L^1(\mathbb{R}^n)$ since it is compactly supported and we have established that $F(x,z) \sim_{z\to 0} 2|z|^{-(n-1)}$. Let us prove that $F \in S^{-(n-1)}(\mathbb{R}^n_x \times \mathbb{R}^n_z)$ in the sense that (here, we are interested in the behavior as $z \sim 0$)

$$\forall \alpha,\beta, \forall z \neq 0, \quad |\partial_x^\beta \partial_z^\gamma F(x,z)| \leqslant C_{\beta\gamma}|z|^{-n+1-|\gamma|}, \tag{4.2}$$

and then by [29] (Proposition 2.7, page 9), we will be able to conclude that $p = \mathcal{F}_2(F) \in S^{-1}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ (where the Fourier transform is taken over the second variable) [2].

2. Let us just recall that this stems from the fact that if $\mathcal{H}^\sharp_k(\mathbb{R}^n)$ denotes the set of smooth homogeneous functions on $\mathbb{R}^n \setminus \{0\}$, it is well known that the Fourier transform acts as $\mathcal{F} : \mathcal{H}^\sharp_m(\mathbb{R}^n) \to \mathcal{H}^\sharp_{-m-n}(\mathbb{R}^n)$. This result can actually be extended somehow to symbols.

We can write :

$$
\begin{aligned}
F(x,z) &= \frac{2|\det d(exp_x^{-1})_{x-z}|\chi'(x-z)\sqrt{\det g(x-z)}}{(G_{ij}(x,x-z)z^iz^j)^{(n-1)/2}} \\
&= \frac{2|\det d(exp_x^{-1})_{x-z}|\chi(x-z)^{(n+1)/2}\sqrt{\det g(x-z)}}{(\chi'(x-z)G_{ij}(x,x-z)z^iz^j)^{(n-1)/2}} \\
&= \lambda(x,z)/\theta(x,z)
\end{aligned}
$$

Since λ is smooth and compactly supported, all its derivatives are bounded so the problem reduces to proving (4.2) for $1/\theta$ instead of F, that is :

$$\forall\alpha,\beta,\forall z\neq 0,\qquad \left|\partial_x^\beta\partial_z^\gamma\left(\frac{1}{\theta(x,z)}\right)\right| \leqslant C_{\beta\gamma}|z|^{-n+1-|\gamma|}$$

One way of proving this estimate is to use the Leibnitz formula in an induction. Actually, the whole proof starts from the observation that θ satisfies the estimates

$$\forall\alpha,\beta,\forall z\neq 0,\qquad \left|\partial_x^\beta\partial_z^\gamma\left(\theta(x,z)\right)\right| \leqslant C_{\beta\gamma}|z|^{n-1-|\gamma|}$$

Thus, using $\partial_x^\beta\partial_z^\gamma(\theta/\theta)=0$:

$$\theta\cdot\partial_x^\beta\partial_z^\gamma\left(1/\theta\right) = \sum_{\substack{\beta'+\beta''=\beta,\gamma'+\gamma''=\gamma,\\ |\beta'|+|\gamma'|<|\beta|+|\gamma|}} c(\beta',\gamma')\partial_x^{\beta'}\partial_z^{\gamma'}\left(1/\theta\right)\partial_x^{\beta''}\partial_z^{\gamma''}\theta$$

And :

$$\left|\theta\cdot\partial_x^\beta\partial_z^\gamma\left(1/\theta\right)\right| \leqslant C\sum_{\beta'+\beta''=\beta}|z|^{-n+1-|\beta'|}\cdot|z|^{n-1-|\beta''|}\leqslant C|z|^{-|\beta|}$$

Since θ is compactly supported and satisfies $\theta(x,z)\sim_{z\to 0}|z|^{n-1}$, we can write $|\theta(x,z)| \geqslant \frac{1}{2}|z|^{n-1}$ around zero, and thus :

$$\left|\partial_x^\beta\partial_z^\gamma\left(1/\theta\right)\right| \leqslant C|z|^{-n+1-|\beta|}$$

This proves the estimate (4.2).

Let us end the computation. Since $g(x)$ (the matrix of the bilinear form g) is symmetric definite positive, it can be written as $g(x)=s^2(x)$ for some smooth symmetric definite positive s. Using the change of variable $u=sz$, and denoting $|\cdot|$ the euclidean norm on $\mathbb{R}^n$, we obtain :

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{-i\xi\cdot z}\frac{2\sqrt{\det g(x)}}{g_{ij}(x)z^iz^j}dz &= \int_{\mathbb{R}^n} e^{-i(s^{-1}\xi)\cdot u}\frac{2}{|u|^{n-1}}du \\
&= c_n|s^{-1}\xi|^{-1} \\
&= c_n|\xi|_{g^{-1}}^{-1},
\end{aligned}
$$

where c_n is a constant depending on the dimension, coming from the Fourier transform $\mathcal{F} : \mathcal{H}^\sharp_{-(n-1)}(\mathbb{R}^n) \to \mathcal{H}^\sharp_{-1}(\mathbb{R}^n)$, and $|\cdot|_{g^{-1}}$ denotes the norm given by g^{-1}, which is the canonical norm induced by g on the cotangent bundle. Thus, the principal symbol is

$$\sigma(I_0^* I_0) = c_n |\xi|_{g^{-1}}^{-1}$$

□

4.3. Definition on a manifold without boundary

4.3.1. The point of view of distributions

Let (M, g) be a compact manifold without boundary. Given $\gamma \in \mathcal{G}$, a closed unit speed geodesic on M, we can define the distribution $\delta_\gamma \in \mathcal{D}'(SM)$, which corresponds to the integration along $(\gamma, \dot{\gamma})$ in SM. In other words, we define for $f \in \mathcal{C}^\infty(SM)$,

$$\langle \delta_\gamma, f \rangle = If(\gamma)$$

This corresponds exactly with the definition of the X-ray transform on closed geodesics $\gamma \in \mathcal{G}$.

$\mathcal{D}'(SM)$ (endowed with the weak-* topology) is the topological dual of $\mathcal{C}^\infty(SM)$, so X acts on $\mathcal{D}'(SM)$ by duality (since it acts smoothly on $\mathcal{C}^\infty(SM)$) that is, given $T \in \mathcal{D}'(SM), u \in \mathcal{C}^\infty(SM)$,

$$\langle X \cdot T, u \rangle = -\langle T, X \cdot u \rangle$$

From this, we can define the set of invariant distributions by the flow of X, that is :

$$\mathcal{D}'_{inv}(SM) = \{T \in \mathcal{D}'(SM), X \cdot T = 0\}$$

We can already note that this set of invariant distributions does not contain any L^2 functions (not even L^1 actually) but the constants. This comes from the fact that the geodesic flow is ergodic : as a consequence, any function which is invariant by the flow is constant.

Proposition 4.9. — *If (M, g) is Anosov, then the set $\{\delta_\gamma, \gamma \in \mathcal{G}\}$ is dense in $\mathcal{D}'_{inv}(SM)$.*

Démonstration. — Assume $f \in \mathcal{C}^\infty(SM)$ satisfies $\langle \delta_\gamma, f \rangle = 0$ for all $\gamma \in \mathcal{G}$. By Livcic's theorem C.10, we know that $f = X \cdot u$, for some $u \in \mathcal{C}^\infty(SM)$. Thus, for any $T \in \mathcal{D}'_{inv}(SM)$, we obtain that $\langle T, f \rangle = \langle T, X \cdot u \rangle = -\langle X \cdot T, u \rangle = 0$. □

From now on, we assume that the manifold (M,g) is Anosov. Therefore, we can extend the definition of the X-ray transform to :

$$I : \left| \begin{array}{l} \mathcal{C}^\infty(SM) \to L(\mathcal{D}'_{inv}(SM), \mathbb{R}) \\ f \mapsto (If : \nu \mapsto If(\nu) = \langle \nu, f \rangle) \end{array} \right. ,$$

where $L(\mathcal{D}'_{inv}(SM), \mathbb{R})$ denotes the space of continuous linear forms on $\mathcal{D}'_{inv}(SM)$, endowed with the weak-* topology. The application I is a continuous application between the Fréchet space $\mathcal{C}^\infty(SM)$ (endowed with the canonical norms) and the space $L(\mathcal{D}'_{inv}(SM), \mathbb{R})$. Since $\mathcal{D}'_{inv}(SM)$ is reflexive (as a closed subspace of a reflexive space), the dual of $L(\mathcal{D}'_{inv}(SM), \mathbb{R})$ is $\mathcal{D}'_{inv}(SM)$. The adjoint of I is the map

$$I^* : \left| \begin{array}{l} \mathcal{D}'_{inv}(SM) \to \mathcal{D}'(SM) \\ \nu \mapsto (I^*\nu : \varphi \mapsto \langle \nu, I\varphi \rangle) \end{array} \right.$$

In the sequel, we will mostly be interested in proving the s-injectivity of $I_m : \mathcal{R}_m \subset \mathcal{C}^\infty(SM) \to L(\mathcal{D}'_{inv}(SM), \mathbb{R})$, that is showing that if $I_m(f) = 0$, then there exists $u \in \mathcal{R}_{m-1}$ such that $f = X \cdot u$. It will often be obtained thanks to the surjectivity of its adjoint $I_m^* : \mathcal{D}'_{inv}(SM) \to \mathcal{D}'(SM)$. The orthogonal projector $\pi_k : L^2(SM) \to H_k$ acts by duality on $\mathcal{D}'(SM)$, that is any distribution $\nu \in \mathcal{D}'(SM)$ can be decomposed into a Fourier series where its coefficient $\nu_k = \pi_k(\nu)$ is given by $\langle \nu_k, \varphi \rangle = \langle \nu, \pi_k(\varphi) \rangle$. Thus, formally, the adjoint $I_m^* : \mathcal{D}'_{inv}(SM) \to \mathcal{R}_m^*$ is given by $I_m^*(\nu) = \pi_m(\nu) + \pi_{m-2}(\nu) + ... + \pi_{-m}(\nu)$ (where $\mathcal{R}_m^*$ denotes the dual of $\mathcal{R}_m$). The surjectivity of this operator will be studied on $\mathcal{R}_m$, that is for $f \in \mathcal{R}_m$ (seen as a subspace of $\mathcal{R}_m^*$ by the natural identification of a function with the linear form associated to it via the L^2 scalar product), we will try to find $h \in \mathcal{D}'_{inv}(SM)$ such that $I_m^*(h) = f$.

4.4. A few words about the hypothesis

4.4.1. Strict convexity for manifolds with boundary

For manifolds with boundary, we mentioned in the introduction to this section that we will usually assume that they are strictly convex. Actually, this is not always the optimal hypothesis, depending on the problem we consider. First, let us state the

LEMMA 4.10. — *Assume (M,g) is a compact surface with strictly convex boundary. Then geodesics in $\mathring{M}$ intersect ∂M transversally, i.e. a geodesic coming from $\mathring{M}$ and touching a point $x \in \partial M$ cannot be tangent to ∂M.*

Démonstration. — We can always assume that M is embedded in a larger manifold (N,h) such that $g = i^*h$, where h denotes the inclusion. Consider $x_0 \in \partial M$. It is always possible to choose centered isothermal coordinates (x_1, x_2) in a neighborhood of x_0 such that x_0 is sent on $x = 0$, a neighborhood of x_0 in M corresponds to a neighborhood of 0 in the upper half-plane $\{x_2 \geqslant 0\}$ and $\nu(x_0)$ is sent on the vector $-\partial/\partial x_2(0)$. Indeed, this can be done using the same argument as the one involved in the proof of the existence of isothermal coordinates with the Dirichlet condition that $g = 0$ on $x_2 = 0$ (see Appendix B). Therefore, thanks to the Koszul formula for the Christoffel symbols, one can check that in 0 :

$$\nabla_v \nu = \begin{pmatrix} -\partial_2\lambda & \partial_1\lambda \\ -\partial_1\lambda & -\partial_2\lambda \end{pmatrix} v$$

The bilinear form $(v,w) \mapsto g(\nabla_v \nu, w)$ is positive definite since M is strictly convex. This immediately implies that $\partial_2\lambda(0) < 0$.

Now recall that according to the expression (2.25), we can write in these coordinates :

$$X(x_1,x_2,\theta) = e^{-\lambda}\left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \left(-\frac{\partial\lambda}{\partial x_1}\sin\theta + \frac{\partial\lambda}{\partial x_2}\cos\theta\right)\frac{\partial}{\partial\theta}\right)$$

Consider a smooth geodesic $x(t)$ such that $x(t_0) = 0$, $x_2(t) > 0$ for $t < t_0$ and $\dot{x}_2(t_0) = 0$ (it is a geodesic contained in $\overset{\circ}{M}$ that intersects tangentially ∂M at $t = t_0$). The equation $(\dot{x}, \dot{\theta}) = X(x,\theta)$ tells us (when projected on $\partial/\partial x_2(0)$) that $\theta(t_0) = 0$ (or π, but this case being similar, we forget it) and $\dot{\theta}(t_0) = e^{-\lambda(0)}\cos\theta(t_0)\partial_{x_2}\lambda(0) = e^{-\lambda(0)}\partial_{x_2}\lambda(0) < 0$. So θ decreases as t is close to t_0. Moreover, the equation $\dot{x}_2(t) = e^{-\lambda(x(t))}\sin\theta(t)$ provides $\ddot{x}_2(t_0) = e^{\lambda(0)}\dot{\theta}(t_0) < 0$. Therefore, a Taylor expansion in a vicinity of t_0 yields to :

$$x_2(t) = \frac{1}{2}(t-t_0)^2\ddot{x}_2(t_0) + O((t-t_0)^3)$$

This is negative around t_0 so we obtain a contradiction. □

Remark 4.11. — Note that the arguments applied in the previous proof immediately shows that in the strictly convex case, $\tau = 0$ on $\partial_-(SM)$. If $(x,v) \in \partial(SM)$ satisfies $\langle v, \nu(x)\rangle < 0$, then $\tau(x,v) > 0$. The prototype for such a surface is the unit disk, embedded in $\mathbb{R}^2$, endowed with the euclidean metric. Locally, we have the following picture :

Proposition 4.12. — *Assume (M,g) has a strictly convex boundary. Then τ is zero on $\partial_-(SM)$, smooth on $SM \setminus \partial_0(SM)$, continuous on SM.*

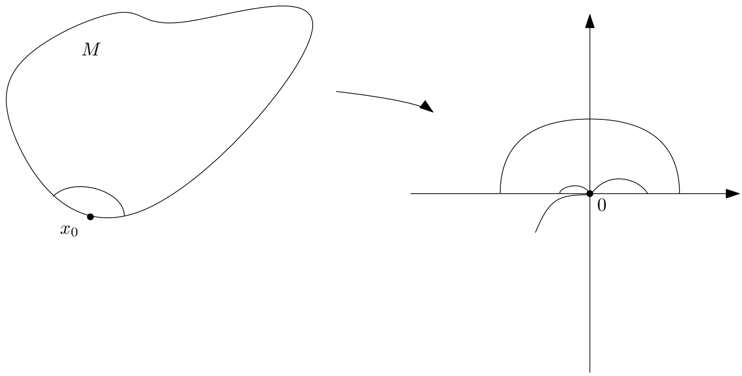

FIGURE *4.2.* The local isothermal coordinates and the form of the geodesics in a neighborhood of the boundary

We see in particular that in the case where the manifold is non-trapping, there exists a maximum exit time for geodesics starting in $\partial_+(SM)$. This will imply the boundedness of the X-ray transform I as an L^2 operator.

Démonstration. — As mentioned before, it is clear that τ is zero on $\partial_-(SM)$. The smoothness of τ on $SM \setminus \partial_0(SM)$ relies on the implicit function theorem. Indeed, consider a point $(x_0, v_0) \in SM \setminus \partial_-(SM)$ and denote by $x_* = \pi(\varphi_{\tau(x_0,v_0)}(x_0, v_0)) \in \partial M$. In particular, $t_0 = \tau(x_0, v_0) > 0$ as we have seen. We can always assume that (M, g) is embedded in (N, h). Consider a smooth function p defined on a vicinity of x_* in N such that $\partial M = \{p = 0\}$ in this neighborhood, and $dp \neq 0$. It can be lifted in the fibers to obtain an application, still denoted by p such that in a vicinity of $\pi^{-1}(x_*)$, we have $\partial(SM) = \{p = 0\}$. We look at the application

$$\Phi : (x, v, t) \mapsto p(\varphi_t(x, v)),$$

defined in a neighborhood of (x_0, v_0, t_0). By definition, τ satisfies the implicit equation

$$\Phi(x, v, \tau(x, v)) = p(\varphi_{\tau(x,v)}(x, v)) \ = 0$$

But, we have

$$\frac{\partial \Phi}{\partial t}(x_0, v_0, t_0) = dp_{\varphi_{t_0}(x_0,v_0)} \left(X(\varphi_{t_0}(x_0, v_0))\right) \neq 0,$$

since the geodesic intersects transversally the boundary at x_* according to Lemma 4.10. Thus, the implicit function theorem allows us to conclude that τ is smooth on $SM \setminus \partial_-(SM)$. It is also smooth on $\partial_-(SM) \setminus \partial_0(SM)$

since τ is zero on it. The continuity can be obtained using local coordinates for instance. $\square$

Remark 4.13. — It can be easily seen that the convexity assumption is necessary to ensure the continuity of τ. Indeed in the following figure, a slight modification of the initial speed can deeply affect τ.

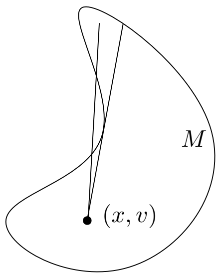

FIGURE *4.3.* The non-continuity of τ without the convexity assumption

Remark 4.14. — The problem of smoothness can be seen on a simple example. If one considers the disk of radius one, with center at $(1,0) \in \mathbb{R}^2$, that is $D = \left\{(x-1)^2 + y^2 \leqslant 1\right\}$, for the euclidean metric, then the exit time of a geodesic starting at $(0,0)$ with direction $(\cos\theta, \sin\theta)$ is given by the function

$$\tau(\theta) = \begin{cases} 2|\cos\theta|, & \theta \in [-\pi/2, \pi/2] \\ 0, & \text{elsewhere} \end{cases}$$

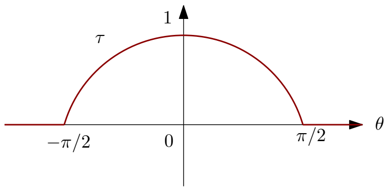

FIGURE *4.4.* The non-smoothness of τ on a disk

THEOREM 4.15. — *Assume (M,g) is non-trapping and has strictly convex boundary. Then, the geodesics in $\mathring{M}$ intersect ∂M transversally and (M,g) satisfies the smooth Livcic property.*

If we consider a smooth function f such that $If = 0$, then it is clear that the function u defined on SM such that $f = X \cdot u$ and $u|_{\partial_-(SM)} = 0$ (and thus $u|_{\partial_+(SM)} = 0$ by $If = 0$) will be smooth on $SM \setminus \partial_0(SM)$. The problem comes from the regularity around $\partial_0(SM)$ because of τ.

Lemma 4.16. — *Consider a smooth symmetric m-tensor $f \in \mathcal{C}^\infty(M, \otimes_S^m T^*M)$ such that $I_m f = 0$. Then there exists a decomposition $f = dh + p$, were $h \in \mathcal{C}^\infty(M, \otimes_S^{m-1} T^*M), h|_{\partial M=0}, p \in \mathcal{C}^\infty(M, \otimes_S^m T^*M)$, with*

$$\forall k \geqslant 0, \frac{\partial^k (f - dh)}{\partial \nu^k} = 0$$

where for $x \in \partial M$, $\nu(x)$ denotes the unit outward vector.

Démonstration. — We consider the coordinates (r, θ) on $(0, \varepsilon) \times \partial M \simeq (0, \varepsilon) \times \mathbb{S}^1$, like in Appendix D. The tensor f can be decomposed in $f = dh + p$, where $h \in \mathcal{C}^\infty(M, \otimes_S^{m-1} T^*M), p \in \mathcal{C}^\infty(M, \otimes_S^m T^*M)$ and $h|_{\partial M} = 0, i_{\frac{\partial}{\partial r}} p = 0$.

First, assume $p(0, \theta_0) = a(0, \theta_0) d\theta^m \neq 0$ for some θ_0. Then, by smoothness, it is also true in a vicinity of $(0, \theta_0)$ and we may assume (the other case being symmetric) that $a(r, \theta) > 0$ in a neighborhood U of $(0, \theta_0)$. Consider a short geodesic starting from $(0, \theta_0)$ (with initial direction v such that $v_r \neq 0, v_\theta > 0$) and staying within U, with exit time τ. We denote it by $(\gamma_t, \dot{\gamma}_t)$. Then

$$I_m f(\gamma) = 0 = \int_0^\tau a(\gamma_t)(\dot{\gamma}_t)_\theta^m dt,$$

which is absurd since $a(\gamma_t)(\dot{\gamma}_t)_\theta^m > 0$.

Now, we reason by induction. Assume that we have proven that $\dfrac{\partial^k p}{\partial \nu^k} = 0$ for any $k < l$ and that there exists a θ_0 such that $\dfrac{\partial^l p}{\partial \nu^l}(0, \theta_0) \neq 0$. As before, we can assume, without loss of generality, that $\dfrac{\partial^l a}{\partial \nu^l}(0, \theta_0) > 0$, which is still true in a neighborhood U of this point. Using a Taylor expansion, we get :

$$p(r, \theta) = \frac{1}{l!} \frac{\partial^l a}{\partial \nu^l}(0, \theta) r^l d\theta^m + O(|r|^{m+1})$$

In particular, there is a vicinity of $(0, \theta_0)$ such that $\dfrac{1}{l!} \dfrac{\partial^l a}{\partial \nu^l}(0, \theta) r^l > 0$ as long as $r > 0$. Thus, considering a small geodesic as we did before, we obtain a contradiction. □

Remark 4.17. — Actually, the proof only works for this derivative because the geodesics that we consider are confined in a conical neighborhood (the value of $(\dot{\gamma})_\theta$ does not change much) and p is of the form $a d\theta^m$. Had we considered a tensor with a term $dr \otimes d\theta^{m-1}$ for instance, this would not have worked, because the short geodesics would have involved a term $(\dot\gamma)_r$ whose sign is not constant along the geodesic.

We can now prove Livcic's theorem in the case M is non-trapping and has strictly convex boundary. What we are going to use is the following

Lemma 4.18. — *Assume $f \in \mathcal{C}^\infty(M, \otimes_S^m T^*M)$ satisfies $If = 0$ and has compact support in $\mathring{M}$. Then $f = X \cdot u$, for some smooth $u \in \mathcal{C}^\infty(SM)$.*

Démonstration. — We set on SM :

$$u(x,v) = -\int_0^{\tau(x,v)} f(\varphi_t(x,v))dt$$

We clearly have $X \cdot u = f$ and u is smooth since τ is smooth on $SM \setminus \partial_0(SM)$ (and the function u is zero in a neighborhood of $\partial_0(SM)$. □

Démonstration. — Consider the annulus $[0,\varepsilon) \times \partial M = U$ like in the previous proof and a cutoff function χ that is 1 in a vicinity of ∂M and with compact support in U. Assume $f \in \mathcal{C}^\infty(M, \otimes_S^m T^*M)$ satisfies $I_m f = 0$. We want to prove that there exists a smooth function u on SM such that $f = X \cdot u$ (where f is seen as a smooth function on SM, that is $\Phi^m(f)$). We will use the two decompositions detailed in Appendix D, namely $f = dh_1 + f^s = dh_2 + p$, where f^s is divergence-free and $i_{\frac{\partial}{\partial r}} p = 0$. Note that the second decomposition is only valid in U.

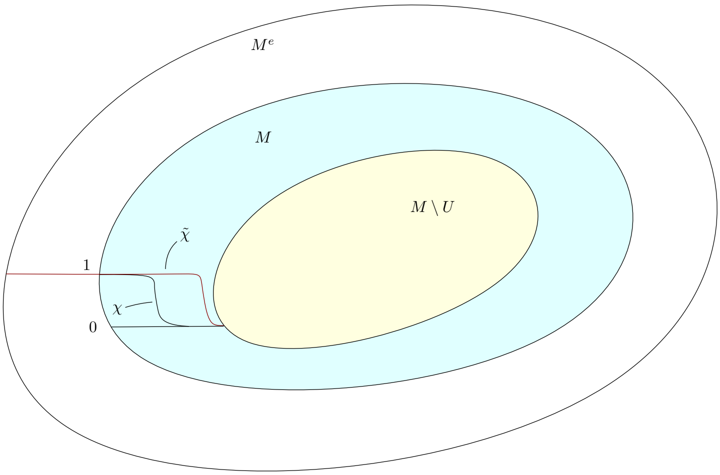

Figure 4.5. The cut-off functions

We can embed (M, g) into a closed manifold (N, h) such that h is a smooth extension of g. We can consider M^e, a tubular neighborhood of M is N which is still strictly convex and non-trapping. Let us denote by $V = (M^e \setminus M) \cup U$. Since p and all its radial derivative vanish, we can extend it smoothly on M^e by the zero section. Consider a cutoff function $\tilde{\chi}$ that is 1 on $M^e \setminus M$ and on the support of χ, with compact support in V. Let us denote by J the X-ray transform on M^e and by τ^e the exit time of the geodesics. We now define :

$$u_1(x, v) = \int_0^{\tau^e(x,-v)} \tilde{\chi}p(\varphi_s(x, -v))ds$$

$$u_2(x, v) = -\int_0^{\tau(x,v)} f^s(\varphi_s(x, v))ds$$

Let us make a few comments. Both functions satisfy by construction $X \cdot u_1 = \tilde{\chi}p, X \cdot u_2 = f^s$. u_1 is defined on SM^e and u_2 is only defined on SM. u_1 is smooth by the previous lemma, whereas u_2 may not be smooth on $\partial_0(SM)$. We now consider :

$$\lambda = \chi(h_1 + u_1) + (1 - \chi)(h_2 + u_2)$$

This is a smooth function defined on SM. Moreover :

$$\begin{aligned} X \cdot \lambda &= \chi X \cdot (h_1 + u_1) + (1 - \chi)X \cdot (h_2 + u_2) + X \cdot \chi\left((h_1 + u_1) - (h_2 + u_2)\right) \\ &= f + X \cdot \chi\left((h_1 + u_1) - (h_2 + u_2)\right) \end{aligned}$$

Let us call $\mu = X \cdot \chi\left((h_1 + u_1) - (h_2 + u_2)\right)$. This is a function supported in an annulus strictly contained in $\overset{\circ}{M}$ and it satisfies $I\mu = 0$ by the previous equality. Thus, by the previous lemma, it can be written $\mu = X \cdot u_3$ for some smooth u_3. Therefore $X \cdot (\lambda - \mu) = f$ and $\lambda - \mu$ is smooth on SM. $\square$

4.4.2. The hypothesis of simplicity and the injectivity of I_0, I_1

We recall that a manifold is *simple* if it is free of conjugate points, simply connected and has strictly convex boundary. It is possible to substitute the hypothesis "simply connected" by "non-trapping" in the definition of a simple manifold because they become equivalent with the other assumptions, as mentioned before. As we have seen in Section 3, the hypothesis that the surface is free of conjugate points provides the crucial identity (3.5)

$$||X \cdot \psi||^2 - (K\psi, \psi) \geqslant 0,$$

for any $\psi \in \mathcal{C}^\infty(SM)$. Note that in the case of an Anosov surface (and therefore, in the case of a surface with negative curvature), this hypothesis is unnecessary since we have seen that an Anosov manifold cannot carry conjugate points (this comes from the existence of the two distinct solutions r^+ and r^- to the Riccati equation). Coupled with the Pestov identity (7.1), the previous identity immediately provides the injectivity of the X-ray transform for 0- and 1-tensors. Thus, we can already state our first theorem of injectivity :

Theorem 4.19. — *Let (M, g) be a simple surface satisfying the smooth Livcic property. Then I_0 and I_1 are s-injective.*

In particular, since a surface with strictly convex boundary is smoothly Livcic, we deduce from this theorem that I_0 and I_1 are injective on such a surface as long as it does not carry conjugate points.

Démonstration. — Let us write once again the Pestov identity for $u \in \mathcal{C}^\infty(SM)$:

$$||XVu||^2 - (KVu, Vu) + ||Xu||^2 - ||VXu||^2 = 0$$

We already know that the sum of the first two terms is positive. And thus, for any $u \in \mathcal{C}^\infty(SM)$:

$$||Xu||^2 - ||VXu||^2 = \sum_k (1-k^2)||(Xu)_k||^2 = 0 \tag{4.3}$$

Let f be a smooth function in $\mathcal{C}^\infty(M)$, that is a 0-tensor, such that $I_0 f = 0$. Since M satisfies the smooth Livcic property, we know that $f = X \cdot u$, for a smooth function u defined on M. Since f is a 0-tensor, it is clear that $(Xu)_k = 0$, as long as $k \neq 0$. Thus the previous equality (4.3) immediately implies that $f = 0$.

Let A be a 1-tensor such that $I_1 f = 0$. We still denote by A the smooth function $\Phi^1(A)$ defined on SM. By the Livcic property, we can write $A = X \cdot u$, for some smooth function u. Since A is a 1-form, a simple computation shows that $||Xu||^2 - ||VXu||^2 = ||A||^2 - ||VA||^2 = 0$. Thus, using the Pestov identity, we obtain the equality in (3.5), i.e. $||XVu||^2 - (KVu, Vu) = 0$. This immediately implies that $Vu = 0$ by Remark 3.17, so $u \in \mathcal{C}^\infty(M)$ is actually constant in the fibers. In other words, $u \in \mathcal{R}_0$. $\square$

5. X-ray transform on negatively curved surfaces and spectral rigidity

5.1. Introduction

5.1.1. The spectrum of the Laplacian

Since Weyl's formula in 1911, — which provides an asymptotic development of the cumulative distribution function of the spectrum of the Laplacian with respect to the volume of the manifold — the link between the spectrum of the Laplacian and the geometric properties of a manifold has been investigated. Recall that on a compact Riemannian manifold (M, g), there exists in any non trivial free homotopy class at least one smooth and closed geodesic whose length is minimal in the class (see [8], Theorem 2.98 for instance). We will call *length spectrum* the collection of lengths of the periodic geodesics counted without multiplicities and *marked length spectrum* the collection of lengths counted with multiplicites. Among the main results concerned with the link between length spectrum and spectrum of the Laplacian, we have the (see [5] and [12]) :

Theorem 5.1. — *The spectrum of the Laplacian determines the length spectrum, that is the lengths of the periodic geodesics without multiplicities.*

Theorem 5.2. — *Let q be a smooth real-valued function. Then, the spectrum of $\Delta + q$ determines the integral of q over every periodic geodesics.*

These two theorems will be used in this form in order to obtain the *spectral rigidity* of manifolds with negative curvature. Let us briefly explain where these two results originate. In the case M is a compact manifold, the Laplace operator $-\Delta$ has compact resolvent so one can find a sequence of positive eigenvalues $\lambda_i \to \infty$ (with associated eigenfunctions f_i). Note that they form an orthonormal basis of $L^2(M)$ and any $f \in L^2(M)$ can be decomposed in its "Fourier" mode $f = \sum_i \alpha_i f_i$. We define the unitary operator $U(t) = e^{it\sqrt{-\Delta}}$. Then, its trace is a distribution on the real line given by :

$$\chi(t) = \sum_i e^{it\sqrt{\lambda_i}}$$

We then have the following result (see [13] for instance) :

Theorem 5.3. — *$T \neq 0$ is in the singular support of χ if and only if there exists a periodic geodesic on M of period T. Thus, the spectrum of the Laplace operator determines the length of the periodic geodesics.*

5.1.2. What about a converse ?

In particular, a long-standing question — now solved — has been to see if two isospectral manifolds (manifolds with same spectrum of the Laplacian counted with multiplicities) were necessarily isometric. This is actually wrong and it is now known that there exists Riemann surfaces with same genus $g \geqslant 4$ and constant negative curvature -1 that are isospectral but not isometric (see [1] for instance). However, J-P. Otal proved in 1990 (see [21] for a reference) a sort of converse : in negative curvature, two manifolds sharing the same marked length spectrum are isometric. Note that it is still ignored whether this result still holds in greater dimensions.

5.1.3. Spectrally rigid manifolds

This section is devoted to the proof of a theorem of V. Guillemin and D. Kazhdan (see [14] for the original paper). Let us first introduce some preliminary notions before stating their result. We fix a Riemannian two-dimensional manifold (M, g). We say that a family of Riemannian metrics $(g_t)_{t\in(0,\varepsilon)}$ is a *deformation* if g_t depends smoothly on t and $g_0 = g$. The deformation is said to be *trivial* if there exists an isotopy $(\phi_t)_{t\in(0,\varepsilon)}$, that is a family of diffeomorphisms depending smoothly on t, such that $g = (\phi_t)^* g_t$.

For each metric g_t, we can consider the Laplace-Beltrami operator Δ_t and its spectrum. We say that the deformation is *isospectral* if the spectrum of the operators Δ_t coincide, counting the multiplicities of the eigenvalues. A manifold is said to be *spectrally rigid* if any isospectral deformation is trivial. We can now state the theorem :

Theorem 5.4 (Guillemin-Kazhdan, 78). — *Let (M, g) be a two-dimensional Riemannian manifold with negative curvature. Then (M, g) is spectrally rigid.*

This result can actually be seen as a linearization of a stronger result one could expect, which is in the spirit of what we obtain in the context of the *boundary rigidity problem*. Namely, it could be reasonable to try to show that two isospectral Riemannian manifolds are isometric (in the sense that there exists a diffeomorphism ψ between the two such that $\psi^* g = g'$), but we have already explained that this is not true.

5.2. Injectivity of the ray transform

In this paragraph, we are going to prove the

THEOREM 5.5. — *Let (M, g) be a surface with negative curvature. Then for any $m \geqslant 0$, the X-ray transform I_m is s-injective.*

Since we assume the curvature to be negative, the geodesic flow is Anosov and the surface satisfies the Livcic property (see Appendix C).

LEMMA 5.6. — *Assume $f : SM \to \mathbb{R}$ is a smooth function such that $f \in \oplus_{k=-n}^{k} \Omega_k$ and f is zero over every periodic integral curve of X. Then, there exists a smooth function $u \in \oplus_{k=-(n-1)}^{n-1} \Omega_k$ such that $f = X \cdot u$.*

Proof of the lemma. — Since M is negatively curved and f is zero over every periodic integral curve of X, we know by Livcic's theorem (see Appendix C) that there exists a smooth function $u : SM \to \mathbb{R}$ such that $X \cdot u = -f$. We decompose the functions as $u = \sum_k u_k$, $f = \sum_k f_k$. Writing $X = \eta^+ + \eta^-$ and projecting on each H_k, we obtain :

$$\eta^+ u_{k-1} + \eta^- u_{k+1} = -f_k$$

Moreover, since we are in negative curvature, the Pestov identity (3.15) gives :

$$||\eta^+ u_k|| \geqslant ||\eta^- u_k|| \tag{5.1}$$

We know that $f_k = 0$ for $k \geqslant n+1$ so :

$$||\eta^+ u_{k-1}|| = ||\eta^- u_{k+1}|| \tag{5.2}$$

Combining (5.1) and (5.2), we obtain :

$$||\eta^+ u_n|| = ||\eta^- u_{n+2}|| \leqslant ||\eta^+ u_{n+2}|| = ||\eta^- u_{n+4}|| \leqslant ||\eta^+ u_{n+4}|| = ...$$

In other words, $(||\eta^+ u_{n+2k}||)_{k \geqslant 0}$ is a non-decreasing sequence and it converges to 0 since $u \in L^2(SM)$. It is therefore constantly zero. We obtain the same result for odd numbers. The same arguments apply for $k \leqslant -n$ so $||\eta^+ u_k|| = 0$ for $k \geqslant n$ and $k \leqslant -n$. Applying once again Pestov identity (3.15), we obtain for $k \geqslant n$ and $k \leqslant -n$ that u_k is zero. □

Proof of Theorem 5.5. — Consider T a symmetric m-tensor such that $I_m T = 0$, that it T is zero over every integral closed curve of the geodesic vector field X. Then, by Lemma 5.6, we know that there exists a smooth function $u \in \mathcal{R}_{m-1}$ such that $T = X \cdot u$. As mentioned in Section 3.2.2, u gives rise to a symmetric $(m-1)$ tensor such that $T = du$ which concludes the proof. □

Remark 5.7. — Actually, the same arguments apply in any dimension, up to a generalization of the technical tools introduced. In [3], it is proved that I_m is s-injective for all m on non-positively curved manifolds of any dimension.

Remark 5.8. — In particular, we see where the proof breaks down when no assumption is made on the curvature. In Pestov's identity, it is no longer possible to obtain the fundamental inequality (5.1) which is at the core of the proof of Lemma 5.6. Therefore, one needs to make more assumptions on the manifold M and/or introduce new tools to tackle this obstacle.

5.3. Proof of Theorem 5.4

Let us first give a heuristical approach. Consider an isospectral deformation $(g_t)_{t\in(-\varepsilon,+\varepsilon)}$. As mentioned in Section 3.2.2, since a metric is a symmetric 2-tensor, we can associate to each metric a smooth real-valued function $f_t : TM \to \mathbb{R}$. The restriction of f_t to the unit tangent bundle SM_t (which depends on t since it is given by the metric g_t) is the constant function 1. Since $\dfrac{\partial g_t}{\partial t}$ is still a symmetric 2-tensor, the restriction to the unit tangent bundle SM_t of the function $\dot{f}_t = \dfrac{\partial f_t}{\partial t} \in \mathcal{C}^\infty(TM)$ canonically associated to it lies in $H_{t,-2} \oplus H_{t,0} \oplus H_{t,2}$, and we can write

$$\dot{f}_t = \dot{f}_{t,-2} + \dot{f}_{t,0} + \dot{f}_{t,2},$$

with $\dot{f}_{t,-2} = \overline{\dot{f}_{t,2}}$. In the following, X_t will denote the geodesic vector field on SM_t with respect to the metric g_t and ∇_t the Levi-Civita connexion. The idea is to prove that for each t, the $\dot{f}_t$ is zero over every integral curve of the geodesic field X_t. Since the surface is negatively curved, we know in particular, according to Anosov's theorem C.8, that the geodesic flow is Anosov. By Livcic's theorem C.10, this will give a smooth real-valued function $u_t : SM \to \mathbb{R}$ such that $\dot{f}_t = -X_t \cdot u_t$. We will actually prove that this function is in $H_{t,-1} \oplus H_{t,1}$ and is therefore associated to a 1-form θ_t. By the musical isomorphism, this provides a vector field Y_t on M, whose family of isotopy is exactly the one we seek.

Consider a periodic geodesic γ for the metric g parametrized by arc-length. Then, we can find a family of closed geodesics $(\gamma_t)_{t\in(0,\varepsilon)}$ depending smoothly on t such that $\gamma_0 = \gamma$. Since the deformation is isospectral and the spectrum of the Laplace-Beltrami operators determines the length of the closed geodesics, according to Theorem 5.1, then we know that the geodesics γ_t have all the same length.

Lemma 5.9. —

$$\int_{\gamma_t} \dot{f}_t = 0,$$

Démonstration. — It is sufficient to prove the lemma in the case $t = 0$ since the result can be deduced from it by reparametrizing the family as (g^{t+s}), with $s \in (0, \varepsilon - t)$. The common length of the geodesic is denoted by L. We have :

$$1 = g_t(\frac{\partial \gamma_t}{\partial s}(s), \frac{\partial \gamma_t}{\partial s}(s)) = f_t(\gamma_t(s), \frac{\partial \gamma_t}{\partial s}(s)),$$

for any $0 \leqslant s \leqslant x, 0 \leqslant t \leqslant \varepsilon$. We see $\Gamma_t : s \mapsto (\gamma_t(s), \dfrac{\partial \gamma_t}{\partial s}(s))$ as a path in TM (and more precisely in SM_t). By definition, we have $\dfrac{\partial \Gamma_t}{\partial s} = X_t(\Gamma_t)$ and the geodesic field X_t is unitary on the unit tangent bundle SM_t. Thus :

$$\forall t \in (0, \varepsilon), \quad \int_\Gamma f_t = L = \int_0^L f_t(\Gamma_t(s)) \, ds$$

Differentiating with respect to t at 0, we obtain :

$$0 = \int_\Gamma \dot{f}_0 + \int_\Gamma d(f_0)$$

Now, the second term is clearly zero since $d(f_0)$ is exact and Γ is a loop. □

Lemma 5.10. — *For each $t \in (0, \varepsilon)$, there exists a smooth real-valued function $u_t : SM_t \to \mathbb{R}$ such that $-X_t \cdot u_t = \dot{f}_t$ and $u_t \in H_{t,-1} \oplus H_{t,1}$ with $u_{t,-1} = \bar{u}_{t,1}$.*

Démonstration. — According to Lemma 5.6, we know that the integral of $\dot{f}_t$ over every periodic integral curve of the geodesic field X_t is zero. Therefore, by the previous lemma, we know that there exists a smooth real-valued function $u_t : SM_t \to \mathbb{R}$ such that $-X_t \cdot u_t = f_t$, $u_t \in H_{t,-1} \oplus H_{t,1}$ and $u_{t,-1} = \bar{u}_{t,1}$. Now, these functions depend smoothly on t. Indeed, we have for instance $\dot{f}_{t,2} = \eta_t^+ u_{t,1}$ and $\dot{f}$ depends smoothly on t and the operator η_t^+ is an injective elliptic operator depending smoothly on t. □

We can now complete the proof of the theorem.

Proof of Theorem 5.4. — Since u_t lies in $H_{t,-1} \oplus H_{t,1}$, it can be seen as a symmetric real-valued 1-tensor, which is nothing but a 1-form $\theta_t \in \Omega^1(M)$. The musical isomorphism $\flat_t$ corresponding to the metric g_t provides a vector field Y_t on M such that $(2Y_t)^{\flat_t} = \theta_t$ (the 2 comes from the operator of symmetrization σ). We now denote by $(\phi_t)_{t \in (0,\varepsilon)}$ the isotopy induced by the family of vector fields $(Y_t)_{t \in (0,\varepsilon)}$. We claim that $(\phi_t)^* g_t = g$. In order to prove this, all we have to do is to check that for any $x \in M, \xi, \eta \in T_x M$, we have :

$$g_t(d(\phi_t)_x(\xi), d(\phi_t)_x(\eta)) = g(\xi, \eta)$$

Since this equality is obviously true for $t = 0$, it is sufficient to prove that the derivative of the left-hand side with respect to t is zero.

By definition of ϕ_t, we have $\dfrac{\partial \phi_t}{\partial t}(x) = Y_t(\phi_t(x))$. Moreover, according to (3.2), the equality $\dot{f}_t = -X_t \cdot u_t$ can be rewritten in terms of tensors :

$$\begin{aligned}\frac{\partial g_t}{\partial t}(\cdot,\cdot) = -\sigma\left(\nabla_t \theta_t\right)(\cdot,\cdot) &= -\frac{1}{2}\left(g_t((\nabla_t)_. (2Y_t), \cdot) + g_t(\cdot, (\nabla_t)_. (2Y_t))\right) \\ &= -\left(g_t((\nabla_t)_. Y_t, \cdot) + g_t(\cdot, (\nabla_t)_. Y_t)\right)\end{aligned}$$

By the Leibnitz rule, we have :

$$\begin{aligned}\left.\frac{\partial}{\partial t'}\right|_{t'=t} g_t\left(d\phi_{t'}(\xi), d\phi_{t'}(\eta)\right) = &\left.\frac{\partial}{\partial t'}\right|_{t'=t} g_t\left(d\phi_{t'}(\xi), d\phi_t(\eta)\right) \\ &+ \left.\frac{\partial}{\partial t'}\right|_{t'=t} g_t\left(d\phi_t(\xi), d\phi_{t'}(\eta)\right)\end{aligned}$$

In order to compute these terms, let us consider a path $\gamma : (a, b) \to M$, parametrized by s such that $\gamma(0) = \phi_t(x)$, $\left.\dfrac{\partial \gamma}{\partial s}\right|_{s=0} = d\phi_t(\xi)$. We thus have a map $c : (t-\varepsilon, t+\varepsilon) \times (a, b) \to M$ such that $c(t', s) = \phi_{t'}(\gamma(s))$. Since ∇ is torsion-free, $\tilde{\nabla} = c^*\nabla$ is torsion-free too and :

$$T^{\tilde{\nabla}}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = 0 = \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} - \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} - \underbrace{\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]}_{=0}$$

In other words, $\tilde{\nabla}_{\frac{\partial}{\partial s}} \dfrac{\partial}{\partial t} = \tilde{\nabla}_{\frac{\partial}{\partial t}} \dfrac{\partial}{\partial s}$ (this can also be seen as a Schwarz lemma when written in coordinates, using a symmetry of the Christoffel symbols). Using the fact that ∇ is the Levi-Civita connection (it is g-adapted), we get :

$$\begin{aligned}\left.\frac{\partial}{\partial t'}\right|_{t'=t} g_t\left(d\phi_{t'}(\xi), d\phi_t(\eta)\right) &= \left.\frac{\partial}{\partial t'}\right|_{t'=t} g_t\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \phi_{t'} \circ \gamma(s), d\phi_t(\eta)\right) \\ &= g_t\left(\tilde{\nabla}_{\frac{\partial}{\partial t'}} \frac{\partial}{\partial s} \cdot \left(\phi_{t'} \circ \gamma(s)\right), d\phi_t(\eta)\right)\Bigg|_{t'=t, s=0} \\ &= g_t\left(\tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t'} \cdot \left(\phi_{t'} \circ \gamma(s)\right), d\phi_t(\eta)\right)\Bigg|_{t'=t, s=0} \\ &= g_t\left(\nabla_{d(\phi_t)_x(\xi)} Y_t(\phi_t(x)), d\phi_t(\eta)\right)\end{aligned}$$

And thus, combining all the previous equalities :

$$\frac{\partial}{\partial t'}\left(g_{t'}(d(\phi_{t'})_x(\xi), d(\phi_{t'})_x(\eta))\right) = 0$$

This conclues the proof of the theorem. $\square$

5.4. A second result

A Riemannian manifold is said to have simple length spectrum if its periodic geodesics are isolated and non-degenerated, and all the periodic geodesics have different periods. By non-degenerate, we mean that Hessian of the energy functional along a periodic geodesic is itself non-degenerate. It is known that the property of having a simple length spectrum is a generic property of Riemannian manifold.

THEOREM 5.11. — *Let M be a compact negatively Riemannian surface with simple length spectrum. Let q_1 and q_2 be two smooth real-valued functions on M and assume $\Delta + q_1$ and $\Delta + q_2$ have the same spectrum. Then $q_1 = q_2$.*

Démonstration. — Let us denote by f the pull-back on SM of the function $q_1 - q_2$. Since the operators $\Delta + q_1,\ \Delta + q_2$ have same spectrum, we know that the integrals of q_1, q_2 over periodic geodesics are equal according to Theorem 5.2. As a consequence, the integral of f over every periodic geodesics if zero. But clearly $f \in H_0$ since f is constant in the fibers. By injectivity of the ray transform I_0, we obtain $f = 0$. $\square$

6. Spectral rigidity on Anosov surfaces

6.1. Introduction

In this part, we are going to prove the injectivity of the X-ray transform on symmetric 2-tensors.

Theorem 6.1 (Paternain, Salo, Uhlmann, 2014). — *If (M,g) is Anosov, then I_2 is s-injective.*

The proof is rather pedestrian and consists in different steps. Recall that we have already established the injectivity of I_0 and I_1 in Section 4.4.2, using the Pestov identity and the inequality provided by the 0-control of a simple surface. In the case of an Anosov surface, we know that we have the inequality (3.6) which betters the α-control for some $0 < \alpha \leqslant 1$. This will be used in order to prove the surjectivity of I_0^*. Then, using the surjectivity of I_0^* we will obtain the surjectivity of I_1^*. Eventually, the surjectivity of this operator will provide the injectivity of I_2. The trick involved in the second proof of surjectivity relies on a successful definition of the product of two invariant (by the geodesic flow) distributions, as long as they are in suitable "mixed" Sobolev spaces (which will be introduced later).

The proof exposed in this part mainly follows [25]. Throughout the first paragraph, it will be assumed that the surface is non-hyperelliptic (see Theorem 3.26), thus providing an explicit description of the kernel of η_-. In the last paragraph, we show how the general case can be recovereed. As a corollary, we will obtain the spectral rigidity of Anosov manifolds, thanks to the same proof as the one provided in Section 5.3.

Corollary 6.2. — *An Anosov surface is spectrally rigid.*

6.2. Surjectivity of I_0^*

Let us first prove the surjectivity of I_0^* on $\mathcal{R}_0 = \mathcal{C}^\infty(M)$. We recall that for $\nu \in \mathcal{D}'_{inv}(SM), I_0^*(\nu) = 2\pi\nu_0$, so we are going to prove that for any $h \in \mathcal{C}^\infty(M)$, there exists a invariant distribution ν such that $2\pi\nu_0 = h$. More precisely, we are going to show the

Theorem 6.3. — *If (M,g) is Anosov, then $I_0^* : H^{-1}_{inv}(SM) \to \mathcal{C}^\infty(M)$ is surjective. Moreover, given $f \in \mathcal{C}^\infty(M)$ and $w \in H^{-1}_{inv}(SM)$ such that $I_0^* w = h$, we have $\pi_k(w) \in \mathcal{C}^\infty(SM)$ for all even k.*

It mainly relies on the following lemma, which looks like an *energy identity*, rather common in PDEs. Throughout the rest of this section, the symbol $\sharp$ in index of a functional set $E_\sharp$ will mean that we consider in E the orthogonal to constant functions for the natural L^2-product on SM.

LEMMA 6.4. — *There exists a constant C such that $||u||_{H^1} \leqslant C||Pu||_{L^2}$, for all $u \in \mathcal{C}^\infty_\sharp(SM)$.*

Démonstration. — We recall that for an Anosov surface, there exists a $0 < \alpha \leqslant 1$ such that it is α-controlled and even better, that is, for all $\psi \in \mathcal{C}^\infty(SM)$, we have :

$$||X\psi||^2 - (K\psi, \psi) \geqslant \alpha \left(||X\psi||^2 + ||\psi||^2\right)$$

Applying this inequality with $\psi = Vu$, and using Pestov identity 7.1, we obtain :

$$||Pu||^2 \geqslant ||Xu||^2 + \alpha \left(||Vu||^2 + ||XVu||^2 \right)$$

By using Cartan's structural equation $[V, X] = H$, we obtain that $Hu = VXu - XVu = Pu - XVu$, thus :

$$||Hu||^2 \leqslant 2(||Pu||^2 + ||XVu||^2)$$

As a consequence, there exists a constant C such that :

$$C||Pu||^2 \geqslant ||Vu||^2 + ||Xu||^2 + ||Hu||^2$$

Eventually, the Poincaré inequality on a closed Riemannian manifold tells us that $||u||^2_{H^1} \approx ||\nabla u||^2_{L^2} = ||Vu||^2 + ||Xu||^2 + ||Hu||^2$, thus leading to :

$$||u||_{H^1} \leqslant C||Pu||_{L^2}$$

□

Now we use a classical argument involving the Hahn-Banach theorem in order to prove the

LEMMA 6.5. — *For any $f \in H^{-1}_\sharp(SM)$, there exists a solution $h \in L^2(SM)$ to the equation $P^*h = f$ in SM. Moreover, we have the inequality $||h||_{L^2} \leqslant C||f||_{H^{-1}}$.*

Démonstration. — Consider $l : P(\mathcal{C}^\infty_\sharp(SM)) \to \mathbb{C}$ defined by $l(Pu) = \langle u, f\rangle$. By the previous lemma, we have :

$$|l(Pu)| \leqslant ||f||_{H^{-1}}||u||_{H^1} \leqslant C||f||_{H^{-1}}||Pu||_{L^2}$$

Thus, $l : P(\mathcal{C}^\infty_\sharp(SM)) \to \mathbb{C}$ is continuous so by the Hahn-Banach theorem, it admits a continuous extension which we still denote by $l : L^2(SM) \to \mathbb{C}$, and satisfies $|l(u)| \leqslant C||f||_{H^{-1}}||u||_{L^2}$, for any $u \in L^2(SM)$. By the Riesz representation theorem, there exists $h \in L^2(SM)$, such that $l(u) = (u, h)_{L^2(SM)}$, for all $u \in L^2(SM)$ and $||h||_{L^2} \leqslant C||f||_{H^{-1}}$. As a consequence, for any $u \in \mathcal{C}^\infty_\sharp(SM)$, we have :

$$\langle u, P^*h\rangle = \langle Pu, h\rangle = l(Pu) = \langle u, f\rangle$$

Now, f is orthogonal to constant functions so $P^*h = f$. □

Proof of the Theorem 6.3. — Let $f \in \mathcal{C}^\infty(M)$. By the previous lemma, we know that there exists $h \in L^2(SM)$ such that $P^*h = XVh = -Xf$. Now set $w = Vh + f \in H^{-1}(SM)$. It is clear that $X \cdot w = 0$, that is $w \in H^{-1}_{inv}(SM)$, and $\pi_0(w) = w_0 = f$. To obtain the smoothness of the w_{2k}, we make use of the ellipticity of the operators $\eta_\pm$, along with a bootstrap argument. Indeed, we know that $X \cdot w = 0$, that is for all $k \in \mathbb{Z}, \eta_+ w_{k-1} + \eta_- w_{k+1} = 0$. Therefore, $\eta_- w_2 = -\eta_+ w_0 = -\eta_+ f$. Since f is smooth, $\eta_+ f$ is smooth and w_2 is smooth by ellipticity of η_-. Inductively, we conclude that all the w_{2k} are smooth. □

6.3. Surjectivity of I_1^*

We recall that a smooth 1-form $a = a_{-1} + a_1 \in \Omega_{-1} \oplus \Omega_1$ is said to be solenoidal if its divergence is zero. As we have seen, this is equivalent to $\eta_+ a_{-1} + \eta_- a_1 = 0$. We still confuse the 1-form and the function in $\mathcal{C}^\infty(SM)$ canonically associated to it.

Theorem 6.6 (Surjectivity of I_1^*). — *Let a be a smooth solenoidal 1-form. Then there exists $w \in H^{-1}_{inv}(SM)$ such that $I_1^*(w) = w_{-1} + w_1 = a$. In particular, such a w can be taken such that $w_0 = 0$.*

Let $T : \mathcal{C}^\infty(SM) \to \bigoplus_{|k| \geqslant 2} \Omega_k$ be the orthogonal projection such that $Tu = u - (\pi_{-1} + \pi_0 + \pi_1)(u)$. We also define $Q := TVX = TP$ and we have $Q^* = XVT$ since $T*$ is self-adjoint and $P^* = XV$. Just like in the previous paragraph, the surjectivity of Q^* is obtained thanks to the following

Lemma 6.7. — *There exists a constant $C > 0$, such that for any $u \in \bigoplus_{|k| \geqslant 1} \Omega_k$, we have :*

$$||u||_{H^1} \leqslant C ||Qu||_{L^2}$$

Démonstration. — First note that

$$||Pu||^2 = ||(Xu)_1||^2 + ||(Xu)_{-1}||^2 + ||Qu||^2 \leqslant ||Xu||^2 + ||Qu||^2$$

We already know by Lemma 6.4 that $||u||_{H^1} \leqslant C||Pu||$. Therefore, an inequality of the form $||Xu|| \leqslant C||Qu||$ would allow us to conclude. We recall the Pestov identity (7.1) :

$$||Pu||^2 = ||VXu||^2 = ||XVu||^2 - (KVu, Vu) + ||Xu||^2$$

Since (M, g) is Anosov, we have for some $0 < \alpha \leqslant 1$:

$$||XVu||^2 - (KVu, Vu) \geqslant \alpha \left(||XVu||^2 + ||Vu||^2\right),$$

which gives :

$$||Pu||^2 \geqslant \alpha||XVu||^2 + ||Xu||^2 \geqslant (1+\alpha)||Xu||^2,$$

since u has no component in Ω_0. Thus :

$$||Qu||^2 = ||Pu||^2 - (||(Xu)_1||^2 + ||(Xu)_{-1}||^2) \geqslant \alpha||Xu||^2$$

$\square$

Remark 6.8. — Actually, this lemma proves the injectivity of I_1. Indeed, assume f is a 1-form such that $I_1(f) = 0$ and define u such that $X \cdot u = f$. We set $v = u - u_0$. $X \cdot v$ has degree 1 and we want to show it is zero. But $Qu = TVXu = 0$ by definition of T. The inequality implies immediately that $u = 0$.

The same trick involving the Hahn-Banach theorem applies here :

Lemma 6.9. — *Consider $f \in H^{-1}(SM)$ such that $f_0 = 0$. Then, there exists $h \in L^2(SM)$ such that $Q^*h = f$. Moreover, we have the inequality $||h||_{L^2} \leqslant C||f||_{H^{-1}}$.*

Démonstration. — In this proof, we will denote by $\Lambda_1 := \bigoplus_{|k|\ \geqslant 1}\ \Omega_k$. Consider the linear form

$$l : \left| \begin{array}{l} Q(\Lambda_1) \to \mathbb{C} \\ Qu \mapsto \langle u, f\rangle \end{array}\right.$$

By the previous lemma, we have :

$$l(Qu) \leqslant ||f||_{H^{-1}}\ ||u||_{H^1} \leqslant C||f||_{H^{-1}}||Qu||_{L^2}$$

Therefore, l is continuous on $Q(\Lambda_1)$, so it admits a continuous extension by the Hahn-Banach theorem (which we still denote l) such that $l : L^2(SM) \to \mathbb{C}$ with $|l(v)| \leqslant C||f||_{H^{-1}}||v||_{L^2}$. By the Riesz representation theorem, we know that there exists a $h \in L^2(SM)$, such that $l = (\cdot, h)_{L^2}$, with $||h||_{L^2} \leqslant C||f||_{H^{-1}}$. Then, for $u \in \mathcal{C}^\infty(SM)$:

$$\langle u, Q^*h\rangle = \langle Qu, h\rangle = \langle Q(u-u_0), h\rangle = l\left(Q(u-u_0)\right) = \langle u - u_0, f\rangle = \langle u, f\rangle,$$

where the last equality holds because $f_0 = 0$. $\square$

Proof of Theorem 6.6. — Consider a smooth solenoidal 1-form a and define $f := -X \cdot (a_{-1} + a_1)$. Since a is solenoidal, $\eta_+ a_{-1} + \eta_- a_1 = f_0 = 0$. By the previous lemma, we know that there exists $h \in L^2(SM)$, such that $Q^*h = XVTh = f$. We define $w = VTh + a_{-1} + a_1 \in H^{-1}(SM)$ which satisfies by construction $X \cdot w = 0$. And $w_{-1} + w_1 = a_{-1} + a_1$, $w_0 = 0$. $\square$

6.4. Injectivity of I_2

As mentioned before, the main idea relies on the fact that it is possible to multiply invariant distributions, as long as they are in some "good" Sobolev spaces, in order to obtain another invariant distribution.

6.4.1. The mixed Sobolev norm

In this paragraph, we introduce the space

$$L^2_x H^s_\theta(SM) = \Big\{u \in \mathcal{D}'(SM),\ \forall k \in \mathbb{Z}, u_k \in L^2(SM), \quad ||u||_{L^2_x H^s_\theta} := \left(\sum_{k=-\infty}^{+\infty} \langle k \rangle^{2s} ||u_k||^2_{L^2} \right)^{1/2} < \infty \Big\}, \tag{6.1}$$

where $\langle k \rangle = (1+k^2)^{1/2}$.

Now that we have introduced the proper Sobolev spaces, let us just show a corollary of Theorem 6.6.

Corollary 6.10 (Corollary of Theorem 6.6). — *Assume $a_1 \in \Omega_1$ and $\eta_- a_1 = 0$. Then there exists $w = \sum_{k \geqslant 1} w_k \in L^2_x H^{-1}_\theta(SM)$ such that $X \cdot w = 0, w_1 = a_1$, each w_k is in $\mathcal{C}^\infty(SM)$ and $||w||_{L^2_x H^{-1}_\theta} \leqslant C||a_1||_{L^2}$.*

It will be important insofar as, on a non-hyperelliptic surface, the product of elements $a_1 \in \Omega_1$ such that $\eta_- a_1 = 0$ generates $\ker \eta_- \cap \Omega_2$.

Démonstration. — Consider $\tilde{w}$ the distribution given by Theorem 6.6 with $a_{-1} = 0$. We denote by w its holomorphic projection. Since $w_{-1} = w_0 = 0$, we have $w = \sum_{k=1}^\infty \tilde{w}_k$ and $X \cdot w = 0$. Now, the smoothness of the w_k comes from a bootstrap argument, using the ellipticity of the operators $\eta_\pm$, just like in the proof of Theorem 6.3. □

We can now prove the

Theorem 6.11. — *Let $u, v \in \mathcal{D}'(SM)$ such that $u = \sum_{k \geqslant 0} u_k, v = \sum_{k \geqslant 0} v_k$, where $u \in L^2_x H^{-s}_\theta, v \in L^2_x H^{-t}_\theta$, for $s, t \geqslant 0$. We define for $k \geqslant 0$ the Cauchy product*

$$w_k = \sum_{j=0}^{k} u_j v_{k-j} \in H_k \tag{6.2}$$

If N is an integer such that $N > s+t+1/2$, then the sum $\sum_k w_k$ converges in $H^{-N-2}(SM)$ to some w with $||w||_{H^{-N-2}} \leqslant C||u||_{L^2_x H^{-s}_\theta}\, ||v||_{L^2_x H^{-t}_\theta}$. Moreover :

$$||w_k||_{L^1(SM)} \leqslant \langle k \rangle^{s+t} ||u||_{L^2_x H^{-s}_\theta}\, ||v||_{L^2_x H^{-t}_\theta} \tag{6.3}$$

If $X \cdot u = X \cdot v = 0$, *then* $X \cdot w = 0$.

Démonstration. — For $k \geqslant 0$, we have by Cauchy-Schwarz

$$\begin{aligned}
||w_k||_{L^1} &\leqslant \sum_{j=0}^{k} ||u_j v_{k-j}||_{L^1} \\
&\leqslant \sum_{j=0}^{k} ||u_j||_{L^2} ||v_{k-j}||_{L^2} \\
&\leqslant \left(\sum_{j=0}^{k} ||u_j||_{L^2}^2 \right)^{1/2} \left(\sum_{j=0}^{k} ||v_j||_{L^2}^2 \right)^{1/2} \\
&\leqslant \left(\sum_{j=0}^{k} \langle j \rangle^{2s} \langle j \rangle^{-2s} ||u_j||_{L^2}^2 \right)^{1/2} \left(\sum_{j=0}^{k} \langle j \rangle^{2t} \langle j \rangle^{-2t} ||v_j||_{L^2}^2 \right)^{1/2} \\
&\leqslant \langle k \rangle^{s+t} ||u||_{L^2_x H^{-s}_\theta} \; ||v||_{L^2_x H^{-t}_\theta}
\end{aligned}$$

Let us define $W^l = \sum_{j=0}^{l} w_j$ and consider $N > s + t + 1/2$. Since we are in dimension three on a compact manifold, by the Kato-Rellich theorem, the continuous embedding $H^2(SM) \hookrightarrow L^\infty(SM)$ holds, that is, there exists a constant $C > 0$, such that for any $\varphi \in H^2(SM), \varphi \in L^\infty(SM)$ and $||\varphi||_{L^\infty(SM)} \leqslant C||\varphi||_{H^2(SM)}$. For $\varphi \in H^{N+2}(SM)$, we have :

$$\begin{aligned}
|\langle W^l, \varphi \rangle| &= \left| \sum_{j=0}^{l} \langle w_j, \varphi_j \rangle \right| \\
&\leqslant \sum_{j=0}^{l} ||w_j||_{L^1(SM)} ||\varphi_j||_{L^\infty(SM)} \\
&\leqslant ||u||_{L^2_x H^{-s}_\theta} \; ||v||_{L^2_x H^{-t}_\theta} \sum_{j=0}^{l} \langle j \rangle^{s+t} ||\varphi_j||_{L^\infty(SM)} \qquad , \\
&\leqslant C||u||_{L^2_x H^{-s}_\theta} \; ||v||_{L^2_x H^{-t}_\theta} \sum_{j=0}^{l} j^{-\delta} j^{s+t+\delta} ||\varphi_j||_{H^2(SM)} \\
&\leqslant C||u||_{L^2_x H^{-s}_\theta} \; ||v||_{L^2_x H^{-t}_\theta} \sum_{j=0}^{l} j^{2(s+t+\delta)} ||\varphi_j||_{H^2(SM)}^2
\end{aligned}$$

using the Cauchy-Schwarz inequality in the last line, for any $\delta > 1/2$. We may take $\delta = N - s - t > 1/2$. Let us define $Y_1 = \eta_+, Y_2 = \eta_-, Y_3 = V$. This generates an equivalent norm on the Sobolev spaces H^k to that generated by the moving frame X, H, V. We have :

$$\begin{aligned} j^{2(s+t+\delta)}||\varphi_j||^2_{H^2} &\leqslant j^N(||\varphi_j||^2_{L^2} + \sum_{q=1}^3 ||Y_q\varphi_j||^2_{L^2} + \sum_{q,r=1}^3 ||Y_qY_r\varphi_j||^2_{L^2}) \\ &\leqslant ||V^N\varphi_j||^2_{L^2} + \sum_{q=1}^3 ||V^NY_q\varphi_j||^2_{L^2} + \sum_{q,r=1}^3 ||V^NY_qY_r\varphi_j||^2_{L^2} \end{aligned}$$

So :

$$\begin{aligned} |\langle W^l, \varphi\rangle| &\leqslant C||u||_{L^2_xH^{-s}_\theta} \ ||v||_{L^2_xH^{-t}_\theta} \sum_{j=-2}^{l+2} \Bigg(||(V^N\varphi)_j||^2_{L^2} + \sum_{q=1}^3 ||(V^NY_q\varphi)_j||^2_{L^2} \\ &\quad + \sum_{q,r=1}^3 ||(V^NY_qY_r\varphi)_j||^2_{L^2} \Bigg) \\ &\leqslant C||u||_{L^2_xH^{-s}_\theta} \ ||v||_{L^2_xH^{-t}_\theta}||\varphi||^2_{H^{2+N}} \end{aligned}$$

Therefore, $W^l \in H^{-N-2}$ and $||W^l||_{H^{-N-2}} \leqslant C||u||_{L^2_xH^{-s}_\theta} \ ||v||_{L^2_xH^{-t}_\theta}$. Using the same inequalities with Cauchy sequences, we can define in a similar fashion, for any $\varphi \in H^{N+2}(SM)$:

$$\langle w, \varphi\rangle = \lim_{l\to\infty} \langle W^l, \varphi\rangle$$

Thus $w \in H^{-N-2}$ and $||w^||_{H^{-N-2}} \leqslant C||u||_{L^2_xH^{-s}_\theta} \ ||v||_{L^2_xH^{-t}_\theta}$. Now, using the definition with the Cauchy product, the conditions $X \cdot u = X \cdot v = 0$ immediately imply $X \cdot w = 0$. □

Now, the problem with this proof is that it is rather hard to generalize to a product of more than three distributions because there is no natural decompositions of functions using Fourier analysis on a space like $L^3(SM)$ (and it is no longer a Hilbert space).

6.4.2. Attempt for a general proof

The general strategy would be to prove the injectivity of the ray transform I_m by induction on m. We are going to try such a proof and show where it breaks down. We assume in this paragraph that the surface is non-hyperelliptic.

Let us fix $m > 0$ and assume that I_l is s-injective for any $l < m$. Consider a smooth real-valued symmetric m-tensor f (which we still confuse with its function $\Phi^m(f) \in \mathcal{R}_m \subset \mathcal{C}^\infty(SM)$) such that $I_m(f) = 0$. By Livcic's theorem, we know that there exists $u \in \mathcal{C}^\infty(SM)$ such that $X \cdot u = f$. We can decompose $f = f_m + f_{m-2} + ... + f_{-(m-2)} + f_{-m} \in \oplus_{j=0}^m H_{m-2j}$. By

Corollary 3.24, we can write $f_m = \eta_+(h_{m-1}) + q_m$, where $h_{m-1} \in \Omega_{m-1}$, $q_m \in H_m \cap \ker\,(\eta_-)$. Now, we use on q_m Max Noether's Theorem 3.26 which allows us to write q_m as a finite sum

$$q_m = \sum_{k \in \mathbb{N}^m} a_{k_1}...a_{k_m}, \tag{6.4}$$

where each a_{k_i} is in $\Omega_1 \cap \ker\,(\eta_-)$. Now, by Theorem 6.6, we know that there exists $w_{k_i} \in H^{-1}_{inv}(SM)$ such that $w_{k_i,1} = a_{k_i}$, $w_{k_i,p} = 0$ for $p \leqslant 0$. We have seen in the previous paragraph that it is possible to give a sense to the multiplication of two distributions, as long as they lie in some "good" mixed Sobolev spaces. The problem is that it is rather unclear that this is still well-defined when there are more than two distributions in the product (and here there are m), and the estimates produced in the previous paragraphs fail to conclude the argument. Actually, this will only allow us to proof the injectivity of I_m for the case $m = 2$.

For the reader's convenience, we simply explain how the proof would end if we were able to give a sense to the product (6.4). Assume $w = \sum_{k \in \mathbb{N}^m} w_{k_1}...w_{k_m} = \sum_{k \geqslant m} w_k$ makes sense as a distribution in some $H^{-N}(SM)$. Then, $X \cdot w = 0$ and $w_m = q_m$. Also note that since f is real-valued, $f_k = \bar{f}_{-k}$. And $q_m = X \cdot (u - h_{m-1}) + \eta_-(h_{m-1}) - (f_{m-2} + ... + f_{-m})$. Thus :

$$||q_m||^2_{L^2} = \langle w_m, q_m \rangle = \langle w, X(u - q_m) \rangle = \langle X \cdot w, u - q_m \rangle = 0,$$

where the penultimate equality holds because $w = \sum_{k \geqslant m} q_k$ and the last equality holds because $u - q_m$ is smooth. As a consequence :

$$\begin{aligned} X \cdot \left(u - h_{m-1} - \bar{h}_{m-1}\right) &= -\eta_+(\bar{h}_{m-1}) - \eta_-(h_{m-1}) + f_{m-2} + ... + f_{-(m-2)} \\ &\in \bigoplus_{j=0}^{m-2} H_{m-2-2j} \end{aligned}$$

The hypothesis tells us that $u - h_{m-1} - \bar{h}_{m-1} \in \bigoplus_{j=0}^{m-3} H_{m-3-2j}$, so $u \in \bigoplus_{j=0}^{m-1} H_{m-1-2j}$, which "proves" the induction.

As explained earlier, we will only be able to carry out this argument with $m = 2$.

6.4.3. Injectivity of I_2

Recall that $\eta_- : \Omega_1 \to \Omega_0$ has a kernel of dimension g, where g is the genus of the surface M, according to Proposition 3.22. We define $\Lambda = \mathrm{Span}\,\{ab, a, b \in \Omega_1 \cap \ker\,(\eta_-)\}$.

Theorem 6.12. — *Assume $q \in \Omega_2$ is in the linear span Λ. Then there exists $w = \sum_{k \geqslant 2} w_k \in H^{-5}(SM)$ such that $X \cdot w = 0, w_2 = q, ||w||_{H^{-5}} \leqslant C||q||_{L^2}$ and each w_k is in $\mathcal{C}^\infty(SM)$.*

Démonstration. — Assume $q \in \Lambda$, that is $q = \sum_{j=1}^N a^j b^j$ (with $N \leqslant \dim(\Lambda)$), where $a^j, b^j \in \Omega_1$ and $\eta_- a^j = \eta_- b^j = 0$. By Corollary 6.10, we know that there exists distributions $u^j, v^j \in L^2_x H^{-1}_\theta$, such that $X \cdot u^j = X \cdot v^j = 0$ and $u_1^j = a^j, v_1^j = b^j$ and $||u^j||_{L^2_x H^{-1}_\theta} \leqslant C||a^j||_{L^2}, ||v^j||_{L^2_x H^{-1}_\theta} \leqslant C||b^j||_{L^2}$. By Theorem 6.11, we know that the product $w^j = u^j v^j$ makes sense a distribution, namely $w^j \in H^{-5}(SM)$ (actually $w^j \in H^{-9/2-\varepsilon}$, for any $\varepsilon > 0$) satisfies $X \cdot w^j = 0, w^j = \sum_{k \geqslant 2} w_k^j, w_2^j = a^j b^j$ and $||w^j||_{H^{-5}} \leqslant C||a^j||_{L^2}||b^j||_{L^2}$. Note that the Fourier coefficients of w^j are in $\mathcal{C}^\infty(SM)$ by (6.2), since the Fourier coefficients of u^j and v^j are in $\mathcal{C}^\infty(SM)$ (by Corollary 6.10). Then, $w = \sum_{j=1}^N w^j$ satisfies the properties stated in the theorem. □

Thanks to this theorem, the argument detailed in the previous paragraph apply here with $m = 2$ (and using the injectivity of I_0 in the end), thus showing the

Theorem 6.13. — *If (M, g) is an Anosov non-hyperelliptic surface, then I_2 is s-injective.*

Let us now explain how we can recover the general case of an Anosov surface and remove the assumption of non-hyperellipticity :

Proof of Theorem 6.1. — Recall that (M, g) has genus $g \geqslant 2$. We admit that for any integer $n \geqslant 1$, the hyperelliptic surface M admits a Galois cover $p : N \to M$ of degree n, where N has genus $n(g-1)+1$. By taking $n \geqslant 5$, it is possible to ensure that N will not be hyperelliptic.

Now, the metric g on M can be lifted on N so that p becomes a local isometry and the geodesic flow will still be Anosov. The transport equation $X \cdot u = f$ also lifts to $\tilde{X} \cdot \tilde{u} = \tilde{f}$ on N and by applying the previous theorem, we can conclude that $\tilde{u}$ has degree one, and so does u. □

Remark 6.14. — This problem has actually been solved very recently by C. Guillarmou in [11], using a rather different approach. Indeed, his proof relies on semi-classical analysis and on the use of "good" Sobolev spaces, that is anistotropic Sobolev spaces (see [7] for a reference) which take into account the splitting into stable and unstable subbundles. Such a technique allows to define properly the product of m distributions.

7. Injectivity of the X-ray transform on simple surfaces

We present in this section the proof of Paternain-Salo-Uhlmann [23] of the injectivity of the X-ray transform on simple surfaces.

7.1. Introduction

We are going to prove the

THEOREM 7.1 (Paternain, Salo, Uhlmann, 2012). — *Let (M, g) be a simple Riemannian surface. Then I_m is s-injective for any $m \geqslant 0$.*

As mentioned in a previous remark, when no assumption is made on the curvature, the Pestov identity — in the form given in (7.1) — can no longer be used to conclude to the injectivity of the ray transform. The problem comes from the fact that we have no control on the sign of the term $-(KVu, Vu)$ in the identity. Let us explain the fundamental idea of Paternain-Salo-Uhlmann [23] at the root of their proof. In order to control the term $-(KVu, Vu)$ in Pestov identity, they introduce an *attenuation* A (which is a 1-form on SM) and compute a new Pestov identity, taking into account this attenuation. This gives birth to a new term in the identity, whose sign can be easily controlled when taking the "good" attenuation A. We will explain this more formally but let us first begin with the Pestov identity in presence of an attenuation.

7.2. Pestov identity in presence of an attenuation

Let A be complex-valued 1-form on M. In the following, we will still denote by A the smooth function $\Phi^1(A)$ associated to A on SM. Therefore, given $u \in \mathcal{C}^\infty(SM)$, what we write Au is just the multiplication of u by $\Phi^1(A)$. On M, $d + A$ is a connection, where d denotes the flat connection. We denote by F_A its curvature.

PROPOSITION 7.2. — *Let $u \in \mathcal{C}^\infty(SM)$. Then :*

$$||(X+A)Vu||^2 - (KVu, Vu) + ||(X+A)u||^2 - ||V(X+A)u||^2 + (\star F_A Vu, u) = 0 \tag{7.1}$$

Remark 7.3. — The last term can be written :

$$(\star F_A Vu, u) = \sum_{k=-\infty}^{+\infty} ik(\star F_A u_k, u_k)$$

Therefore, if A is chosen well enough such that $i\star F_A > 0$ (and big enough), we may obtain a strong positive contribution in the Pestov identity, thus controlling the term $-(KVu, Vu)$.

Démonstration. — Just like for the Pestov identity, the proof relies on the commutant formula

$$||Pu||^2 = (Pu, Pu) = (u, P^*Pu) = (u, [P^*, P]u) + ||P^*u||^2,$$

where $P = V(X + A)$ (and $P^* = (X + A)V$ since $\bar{A} = -A$), $u \in \mathcal{C}^\infty(SM)$. All we have to show is that

$$([P^*, P]u, u) = (KVu, Vu) - ||(X + A)u||^2 - (\star F_A Vu, u)$$

The proof is rather long and tedious so we will only give the main steps. Note that we still confuse the 1-forms and their associated function via the application Φ^1. It relies on the following lemmas, which are easy to obtain by some elementary computations :

Lemma 7.4. —

$$\star F_A = H \cdot A - X \cdot (\star A) + [\star A, A]$$

Lemma 7.5. —

$$[V, A] = - \star A$$
$$[V, \star A] = A$$
$$[X + A, H + \star A] = -KV - \star F_A$$

Now, we have :

$$[P^*, P] = [XV, VX] + [AV, VX] + [XV, VA] + [AV, VA]$$

We already know by a previous computation that $[XV, VX] = -X^2 + VKV$. Using a factorisation trick and the previous formulas, one can prove that :

$$[P^*, P] = -X^2 + VKV + [VA, H] + [\star A, VX] + [\star A, AV]$$

The three last brackets can be computed :

$$[VA, H] = -XA + V[A, H]$$
$$[\star A, VX] = -AX + V[\star A, X]$$
$$[\star A, AV] = [\star A, A]V - A^2$$

Note in particular that $[A, H], [\star A, X]$ are the respective multiplication by $H \cdot A, X \cdot (\star A)$. This yields to :

$$\begin{aligned} [P^*, P] &= VKV - (X^2 + AX + XA + A^2) + [\star A, A]V + V[A, H] + V[\star A, X] \\ &= VKV - (X + A)^2 + (\star F_A - [A, H] + [\star A, X])V + V[A, H] - V[\star A, X] \\ &= VKV - (X + A)^2 + \star F_A V + V[A, H] - [A, H]V + [\star A, X]V - V[\star A, X] \end{aligned}$$

The last four terms can actually be written $V \cdot (X \cdot \star A - H \cdot A)$ but according to the previous lemma $H \cdot A - X \cdot (\star A)$ is in Ω_0 so its V-derivative is zero. Taking the inner product, we obtain the sought identity. $\square$

Let us also mention a lemma which will be useful in the sequel. It is a generalization to the case with attenuation of the inequality for $u \in \mathcal{C}^\infty(SM), u|_{\partial(SM)} = 0$:

$$||XVu||^2 - (KVu, Vu) \geqslant 0$$

LEMMA 7.6. — *If (M, g) is simple, $u \in \mathcal{C}^\infty(SM), u|_{\partial(SM)} = 0$, then :*

$$||(X + A)(Vu)||^2 - (KVu, Vu) \geqslant 0$$

The proof of this Lemma is actually completely similar to that of the inequality without attenuation (using the solutions of the Riccati equation).

7.3. End of the proof

In order to prove the injectivity of the ray transform, we already explained that one has to prove that given $f \in \mathcal{R}_m$, the transport equation $X \cdot u = -f$ in SM and $u|_{\partial(SM)} = 0$ admits a solution $u \in \mathcal{R}_{m-1}$. This problem is solved by the two following propositions :

PROPOSITION 7.7. — *Let (M, g) be a simple surface, and assume that $u \in \mathcal{C}^\infty(SM)$ satisfies $X \cdot u = -f$ in SM with $u|_{\partial(SM)} = 0$. If $m \geqslant 0$ and if $f \in \mathcal{C}^\infty(SM)$ is such that $f_k = 0$ for $k \leqslant -m-1$, then $u_k = 0$ for $k \leqslant -m$.*

PROPOSITION 7.8. — *Let (M, g) be a simple surface, and assume that $u \in \mathcal{C}^\infty(SM)$ satisfies $X \cdot u = -f$ in SM with $u|_{\partial(SM)} = 0$. If $m \geqslant 0$ and if $f \in \mathcal{C}^\infty(SM)$ is such that $f_k = 0$ for $k \geqslant m+1$, then $u_k = 0$ for $k \geqslant m$.*

Now, let us explain the heuristic behind these results. f and u are smooth functions $SM \to \mathbb{C}$ and thus can be seen as sections of the trivial bundle $E = SM \times \mathbb{C}$. Therefore, on can write the transport equation as $\nabla^0_X u = -f$, where $\nabla^0 = d$ is the flat connection on the trivial bundle E. Any connection on E is of the form $\nabla^\Gamma = d + \Gamma$, where Γ is a complex-valued 1-form on SM. Consider a complex-valued 1-form A on M. It can be pulled back via the projection $\pi : SM \to M$ to $\tilde{A} = \pi^* A$. Thus $\nabla^{\tilde{A}}$ is a connection on E. For $c \in \mathcal{C}^\infty(SM)$, one has :

$$\nabla^{\tilde{A}}_X(cu) = c(X \cdot u) + u\nabla^{\tilde{A}}_X(c) = -cf + u(X \cdot c + c\tilde{A}(X))$$

If c is a non-vanishing function on SM and if $\tilde{A} = -c^{-1}dc$, then the previous equality reduces to :

$$\nabla^{\tilde{A}}_X(cu) = -cf$$

For $x \in M$, $d\pi_{(x,v)}(X) = v$ so $\tilde{A}(X) = A(v)$. Therefore, by identifying A with the function $\Phi^1(A)$ defined on SM, the previous equality can be written :

$$(X + A)(cu) = -cf \tag{7.2}$$

In our reasoning, the 1-form A will be prescribed since we want to obtain $i \star F_A > 0$. We therefore need to find a function c such that $A = -c^{-1}dc$. Note that we require c to be holomorphic (or either antiholomorphic). Indeed, when using Pestov identity with attenuation, we will obtain somehow conditions on the Fourier coefficients of cu. In order to recover conditions on the Fourier coefficients of u, we need at least a certain control on c, and the least we can ask is that c be holomorphic.

If we set $c = e^w$ for some $w \in \mathcal{C}^\infty(SM)$, then equality (7.2) becomes

$$X \cdot w = -A \tag{7.3}$$

Now, the following theorem is the key to the proof of the injectivity of the ray transform. We postpone its proof to the next section :

Theorem 7.9. — *Let (M, g) be a simple surface. If A is a smooth 1-form on M, then there exists a holomorphic $w \in \mathcal{C}^\infty(SM)$ and an antiholomorphic $\tilde{w} \in \mathcal{C}^\infty(SM)$, such that $X \cdot w = X \cdot \tilde{w} = A$.*

Remark 7.10. — Actually, one can lower the hypothesis on this theorem to obtain the following statement — which we will not need in our case (it can be found in [23]) :

Theorem 7.11. — *Let (M, g) be a compact nontrapping surface with strictly convex boundary and assume that I_0^* is surjective. If A is a smooth 1-form on M, then there exists a holomorphic $w \in \mathcal{C}^\infty(SM)$ and an antiholomorphic $\tilde{w} \in \mathcal{C}^\infty(SM)$, such that $X \cdot w = X \cdot \tilde{w} = A$.*

Our statement of the Theorem 7.9 comes from the fact that a simple manifold is nontrapping and that I_0^* is surjective on such a simple surface (see the next paragraph).

We will only prove Proposition 7.8 since the proof of Proposition 7.7 is totally equivalent.

Proof of Proposition 7.8. — Note that we can always reduce the proof to the case when f is even or odd. Indeed, we can decompose the transport equation with respect to the odd and the even parts, which gives that $X \cdot u_\mp = -f_\pm$ in SM, $u_\mp|_{\partial(SM)} = 0$. In the sequel, we assume that f is even (so u is odd), and the proof of the other case is similar.

We fix $s > 0$. Let us consider φ, the real-valued 1-form defined on M such that $d\varphi = d\mathrm{vol}^{(3)}$. We define $A = -is\varphi$. Therefore $i \star F_A = s > 0$ since $dA = -is \cdot d\mathrm{vol}$. By Theorem 7.9, we know that we can find an antiholomorphic $w \in \mathcal{C}^\infty(SM)$ such that $X \cdot w = i\varphi$ and we can assume w is even. By the previous remarks, we obtain that the functions $\tilde{u} := e^{sw}u$ and $\tilde{f} := e^{sw}f$ satisfy $(X+A)\tilde{u} = -\tilde{f}$ in SM, with $\tilde{u}|_{\partial(SM)} = 0$. Since w is antiholomorphic and even, e^{sw} is antiholomorphic and even too, $\tilde{f}_k = 0$ for $k \geqslant m+1$ and $\tilde{f}$ is even. Since u is odd and m is even, we know that $u_m = 0$. We define

$$v := \sum_{k=m+1}^{\infty} \tilde{u}_k$$

and our aim is to show that v is zero on SM, which will imply the sought result. We already know that $v \in \mathcal{C}^\infty(SM)$, $v|_{\partial(SM)} = 0$ and v is odd. Moreover, for $k \geqslant m+2$, we have $((X+A)v)_k = ((X+A)\tilde{u})_k = \tilde{f}_k = 0$. For $k \leqslant m-1$, we have $((X+A)v)_k = 0$ since $v_k = 0$ for $k \leqslant m$. By assumption, m is even and v is odd so $(X+A)v$ is even and $((X+A)v)_{m+1} = 0$. Therefore, the only non-vanishing Fourier coefficient is $((X+A)v)_m$ and :

$$(X+A)v = \mu_- v_{m+1}, \quad v|_{\partial(SM)} = 0$$

We now apply the Pestov identity with attenuation (7.1) A to v :

$$||(X+A)Vu||^2 - (KVu, Vu) + ||(X+A)u||^2 - ||V(X+A)u||^2 + (\star F_A Vu, u) = 0$$

By Lemma 7.6, since (M,g) is simple and $v|_{\partial(SM)}$, we know that :

$$||(X+A)Vu||^2 - (KVu, Vu) \geqslant 0$$

Moreover :

$$(\star F_A Vv, v) = \sum_{k=m+1}^{\infty} ik(\star F_A \tilde{u}_k, \tilde{u}_k) = s \sum_{k=m+1}^{\infty} k||\tilde{u}_k||^2 \geqslant 0$$

Now, the remaining two terms are easy to compute

$$||(X+A)u||^2 - ||V(X+A)u||^2 = (1-m^2)||\mu_- v_{m+1}||^2,$$

and using Pestov identity, it is clear that it is non-positive. Moreover, if $m = 0, 1$ the conclusion is immediate, so we may assume from now on that $m \geqslant 2$. In order to conclude, we use the same iterating trick in the spirit of

3. This is always possible since M is simple and therefore diffeomorphic to $\mathbb{R}^2$ (and thus $H^1(M, \mathbb{R}) = 0$)

Guillemin and Kazhdan's proof. First, note that for all $k \in \mathbb{Z}$ (this is the equivalent of equation (3.15) :

$$||\mu_+ v_k||^2 = ||\mu_- v_k||^2 + \frac{i}{2}(KVv_k + (\star F_A)v_k, v_k) = ||\mu_- v_k||^2 + \frac{s}{2}||v_k||^2 - \frac{k}{2}(Kv_k, v_k)$$

For $k \geqslant m+1$, we have

$$\mu_+ v_{k-1} + \mu_- v_{k+1} = ((X+A)v)_k = \tilde{f}_k = 0,$$

thus giving $||\mu_+ v_{k-1}||^2 = ||\mu_- v_{k+1}||^2$ (this is the equivalent of equation (5.2)). As a consequence :

$$\begin{aligned}
&||(X+A)u||^2 - ||V(X+A)u||^2 = (1-m^2)||\mu_- v_{m+1}||^2 \\
&= (1-m^2)\left(||\mu_+ v_{m+1}\ ||^2 - \frac{s}{2}||v_{m+1}||^2 + \frac{m+1}{2}(Kv_{m+1}, v_{m+1})\right) \\
&= .(1-m^2)\left(||\mu_- v_{m+3}\ ||^2 - \frac{s}{2}||v_{m+1}||^2 + \frac{m+1}{2}(Kv_{m+1}, v_{m+1})\right) \\
&= (1-m^2)\left(||\mu_+ v_{m+3}\ ||^2 - \frac{s}{2}\left(||v_{m+1}||^2 + ||v_{m+3}||^2\right)\right. \\
&+ \left.\left(\frac{m+1}{2}(Kv_{m+1}, v_{m+1}) + \frac{m+3}{2}(Kv_{m+3}, v_{m+3})\right)\right)
\end{aligned}$$

We iterate this process. Since $\mu_- v \in L^2(SM)$, we have $||\mu_- v_k\ ||^2 \to 0$. Moreover, we know that this term is non-positive and $m^2 - 1 \geqslant 0$. Thus :

$$-\frac{s}{2}||v||^2 + \sum \frac{k}{2}(Kv_k, v_k) \geqslant 0$$

Since K is bounded, by letting $s \to \infty$, we obtain that $v = 0$. Thus, $\tilde{u}_k = 0$ for $k \geqslant m$ and $u_k = 0$ too, since e^{sw} is antiholomorphic.

□

7.4. Surjectivity of I_0^*

We recall that I_0^* is defined by

$$I_0^* : \left| \begin{array}{l} L^2(\partial_+(SM), \mu) \to L^2(M) \\ h \mapsto \left(x \mapsto I_0^* h(x) = \int_{S_x} h_\psi(x,v) dS_x\right), \end{array} \right. ,$$

and we are here concerned with its restriction $I_0^* : \mathcal{C}_\alpha^\infty(\partial_+(SM)) \to \mathcal{C}^\infty(M)$, where

$$\mathcal{C}_\alpha^\infty(\partial_+(SM)) = \{h \in \mathcal{C}^\infty(\partial_+(SM)), h_\psi \in \mathcal{C}^\infty(SM)\}$$

More precisely, we want to prove that

Theorem 7.12. — *$I_0^* : \mathcal{C}_\alpha^\infty(\partial_+(SM)) \to \mathcal{C}^\infty(M)$ is surjective.*

Recall that $I_0^* I_0$ is an elliptic pseudodifferential operator of order -1 that is formally self-adjoint. We also know, by injectivity of I_0 (see Section 4.4.2) that it is injective. But since M is not a closed manifold, we cannot conclude immediately, thanks to the Fredholm operator theory, to the surjectivity of $I_0^* I_0 : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. Moreover, had we known that $I_0^* I_0$ was surjective, this could not even allow us to draw any conclusion because it is rather unclear that, given $u \in \mathcal{C}^\infty(M)$, $I_0 u \in \mathcal{C}_\alpha^\infty(\partial_+(SM))$. Indeed, according to Lemma 4.18, this is true if u has compact support in $\mathring{M}$, but not in the general case. We thus need to use an embedding trick.

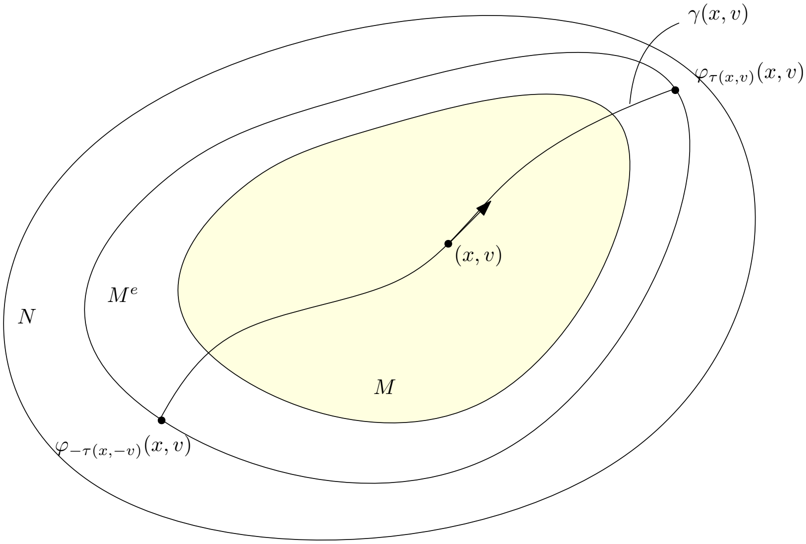

FIGURE *7.1.* The embedding of (M, g) into (N, h).

Démonstration. — Consider (N, h) a smooth Riemannian surface without boundary such that (M, g) embeds into (N, h). Consider a smooth tubular neighborhood of ∂M in N (see below), which we denote by M^e. We denote by J_0 the ray transform on M^e. We take χ, a cut-off function that is constant equal to 1 on M (and even on a compact K which is strictly larger than M), and with support in $\mathring{M}^e$. The operator $\chi J_0^* J_0 \chi$ is elliptic on $\mathrm{supp}(\chi)$. We will use the following

LEMMA 7.13. — *On a closed manifold N, for any $m \in \mathbb{R}$, there exists a self-adjoint and invertible pseudo-differential operator P such that $\sigma(P)(x, \xi) = |\xi|_{g^{-1}}^m$.*

Proof of the lemma. — Consider $B = \mathrm{Op}(|\xi|^{m/2})$. Then B^*B is self-adjoint with principal symbol $|\xi|^m$. It is elliptic, thus Fredholm on N, with index 0. Assume that it has a non-trivial kernel (finitely dimensional since it is Fredholm) and let us call $\{\varphi_1, ..., \varphi_N\}$ an orthonormal basis of the kernel. We consider the operator $P = B^*B + \Pi_{\ker\ B^*B}$, where

$$\Pi_{\ker\ B^*B} = \sum_{i=1}^{N} \langle \varphi_i, u\rangle \varphi_i,$$

is the orthogonal projection on the kernel (it is self-adjoint on $L^2(N)$). By construction, P is invertible. Moreover, the perturbation of $\Pi_{\ker\ B^*B}$ is smooth since the φ_i are smooth and it is finite dimensional (thus compact). □

We take $P = B^2$, defined on N, with B invertible, self-adjoint and $\sigma(B) = |\xi|^{-1/2}$, so that $\sigma(P) = |\xi|^{-1}$. We now define :

$$Q = \chi J_0^* J_0 \chi + (1-\chi) B^2 (1-\chi)$$

Q is clearly self-adjoint with principal symbol

$$\sigma(Q)(x,\xi) = \left(\chi^2(x) + (1-\chi(x))^2\right)\ |\xi|^{-1}$$

and is therefore elliptic of order -1. Let us prove that it is injective. This is rather formal since $Qu = 0$ implies, taking the L^2-scalar product with u :

$$||J_0 \chi u|| = 0, \quad ||B(1-\chi)u|| = 0$$

By injectivity of J_0 and B, we conclude that $u = 0$ on both $\mathrm{supp}(\chi)$ and $\mathrm{supp}(1-\chi)$, that is on N. Since Q — defined on the closed manifold N— is Fredholm of index 0, we deduce that Q is surjective.

Consider $h \in \mathcal{C}^\infty(M)$. We can extend it in order to obtain a smooth function $\tilde{h} \in \mathcal{C}^\infty(N)$ with support in $\chi^{-1}(\{1\}) \subset \mathring{M}^e$. Now, we know that there exists a unique smooth function $f \in \mathcal{C}^\infty(N)$ such that

$$Qf = \chi J_0^* J_0 \chi f + (1-\chi) B^* B (1-\chi) f = \tilde{h} \tag{7.4}$$

Let us call $\tilde{w} = J_0 \chi f$. Since χf is smooth and supported in $\mathring{M}^e$, we know that $\tilde{w}$ is smooth on $\partial_+(SM^e)$ and $\tilde{w}_\psi \in \mathcal{C}^\infty(SM^e)$ (which means, in other words, that $\tilde{w} \in \mathcal{C}_\alpha^\infty(\partial_+(SM^e))$. The function $\tilde{w}_\psi$ is constant along the geodesics (see the figure). We now define on $\partial_+(SM), w = \tilde{w}_\psi|_{\partial_+(SM)}$. Then, by construction, $w \in \mathcal{C}^\infty(\partial_+(SM))$ and $w_\psi = \tilde{w}_\psi|_{SM} \in \mathcal{C}^\infty(SM)$, that is $w \in \mathcal{C}_\alpha^\infty(\partial_+(SM))$.

Now, we go back to (7.4). On M, by construction, the second term vanishes since $\chi \equiv 1$ on $\mathrm{supp}(\tilde{h})$ and we are left with

$$J_0^* J_0 \chi f(x) = J_0^* \tilde{w} = I_0^* w = h(x),$$

which concludes the proof. □

As a corollary, we obtain Theorem 7.9, which we state once again here for the reader's convenience.

Corollary 7.14. — *Let (M, g) be a simple surface. If A is a smooth 1-form on M, then there exists a holomorphic $w \in \mathcal{C}^\infty(SM)$ and an anti-holomorphic $\tilde{w} \in \mathcal{C}^\infty(SM)$, such that $X \cdot w = X \cdot \tilde{w} = A$.*

Démonstration. — Both cases are analogous, so we only deal with the holomorphic case. M being simply connected, the 1-form A can be decomposed into $A = da + \star db$, where $a, b \in \mathcal{C}^\infty(M)$. By Lemma 3.12, we obtain, going back to functions on SM : $A = X \cdot a + H \cdot b$. We can always replace w by $w - a$ so we can now assume that $A = H \cdot b$.

The trick is to try a holomorphic solution of the form $w = (Id + i\mathcal{H})\tilde{w}$, for some even $\tilde{w}$. Then, by Lemma 3.7 :

$$X \cdot w = (Id + i\ \mathcal{H})X \cdot \tilde{w} - [\mathcal{H}, X]\tilde{w} = (Id + i\mathcal{H})X \cdot \tilde{w} - iH \cdot \tilde{w}_0$$

It would be sufficient to find a $\tilde{w}$ invariant by the geodesic flow, i.e. such that $X \cdot \tilde{w} = 0$ and such that $\tilde{w}_0 = -ib$. Now, remember that I_0^* is surjective by the previous theorem and if $h \in \mathcal{C}^\infty(\partial_+(SM)), I_0^* h(x) = \dfrac{1}{2\pi}(h_\psi)_0(x)$. Therefore, we take $h \in \mathcal{C}^\infty(\partial_+(SM))$ such that $I_0^* = -2\pi i b$ and consider the function h_ψ on SM. It satisfies $X \cdot h_\psi = 0, (h_\psi)_0 = -ib$. In order to conclude, we take $\tilde{w} = (h_\psi)_+$. □

7.5. Application to the deformation boundary rigidity problem

In this section, we deduce the deformation boundary rigidity of a simple manifold from Theorem 7.1. This can be seen as a linearized version of the boundary rigidity property (see [26] or [20] for further details). We recall the

Definition 7.15. — *We say that a Riemannian manifold with boundary (M, g) is deformation boundary rigid if any smooth (or at least $\mathcal{C}^1$) family of metrics $(g_s)_{s \in (-\varepsilon, +\varepsilon)}$, such that $d_{g_s} = d_g$ on $\partial M \times \partial M$ and $g_0 = g$, is trivial, that it there exists a smooth isotopy $(\psi_s)_{s \in (-\varepsilon, \varepsilon)}$ such that $\psi_0 = Id$, $\psi_s|_{\partial M} = Id$ and $\psi_s^* g_0 = g_s$.*

Theorem 7.16. — *A simple Riemannian surface is deformation boundary rigid.*

This can be seen as a linearization of the boundary rigidity problem, that is, proving that if d_g and d'_g agree on ∂M, then there exists a diffeomorphism ψ such that $\psi|_{\partial M} = Id$ and $\psi^* g' = g$. The following result is proved in the next paragraph.

Theorem 7.17 (Pestov, Uhlmann, 2003). — *A simple Riemannian surface is boundary rigid.*

Note that the property of being simple for a Riemannian manifold (M, g) is open, that is a small $\mathcal{C}^2$ deformation of the metric will still be simple. We call *simple* a deformation (g_s) such that the (M, g_s) are simple. The proof of Theorem 7.16 only relies on the following fact :

Proposition 7.18. — *Let (M, g) be a simple Riemannian manifold. If I_2 is s-injective for any simple deformation, then (M, g) is deformation boundary rigid. If (M, g) is deformation boundary rigid, then I_2 is s-injective.*

Démonstration. — We assume that I_2 is s-injective. Consider a smooth deformation $(g_s)_{s\in(-\varepsilon,+\varepsilon)}$ such that $d_s = d_0$ on $\partial M \times \partial M$. We define $\beta_s = \dfrac{\partial g_s}{\partial s}$. Now, consider $x, y \in \partial M$ and γ_s the unique (since (M, g_s) is simple) geodesic in M joining x to y under g_s, parametrized by arc-length. We define $T := d_0(x, y)$. Since $d_s = d_0$ on $\partial M \times \partial M$ by assumption, it is immediate that all the γ_s are defined on $[0, T]$. For $\gamma : [a, b] \to M$, any piecewise $\mathcal{C}^1$ curve, we define the usual energy functional

$$E_s(\gamma) := \int_a^b |\dot{\gamma}(t)|_s^2 dt$$

It is clear that $E_s(\gamma_s) = T$ so by differentiating with respect to s, we obtain :

$$0 = \left.\frac{\partial T}{\partial s}\right|_{s=0} = \left.\frac{dE_s(\gamma_s)}{ds}\right|_{s=0} = \int_0^T \left.\frac{dg_s}{ds}\right|_{s=0} (\dot{\gamma}_0(t), \dot{\gamma}_0(t))dt + \left.\frac{dE_0(\gamma_s)}{ds}\right|_{s=0}$$

Since $(\gamma_s)_{s\in(-\varepsilon,+\varepsilon)}$ is a variation of γ_0 which is a geodesic for g_0, the last term vanishes. Therefore, we obtain that $I_2(\beta)(\gamma_0) = 0$. Since x and y were arbitrary, we can conclude that $I_2(\beta_0) = 0$. Of course, this also holds for any s, namely $I_2(\beta_s) = 0$. By assumption, I_2 is s-injective, so there exists a smooth family 1-form $(\alpha_s)_{s\in(-\varepsilon,+\varepsilon)}$ such that $\beta_s = d\alpha_s$. Setting $Y_s = \alpha_s^\flat$ and integrating the family of vector fields (Y_s), we obtain a smooth family of diffeomorphisms $(\psi)_s$ such that $\psi_s^* g_0 = g_s$, just like in the proof of Theorem 5.4.

Assume (M, g) is deformation boundary rigid. Consider a smooth symmetric 2-tensor β such that $I_2(\beta) = 0$. We are going to prove that β is a potential tensor. Let us consider a smooth deformation $(g_s)_{s\in(-\varepsilon,+\varepsilon)}$ such that $g_0 = g$ and $\beta = \left.\dfrac{\partial g_s}{\partial s}\right|_{s=0}$. By repeating the previous argument backwards, it is clear that $d_s = d_0$ on $\partial M \times \partial M$. Thus, by assumption, there exists a smooth family of diffeomorphisms $(\psi_s)_{s\in(-\varepsilon,\varepsilon)}$ such that $\psi_s^* g_0 = g_s$. We define $Y_s := \dfrac{\partial \psi_s}{\partial s}$. Then, using the same computation as in the proof of Theorem 5.4 :

$$\beta_x(v, w) = \left.\frac{\partial}{\partial s}\right|_{s=0} g_0(d\psi_s(v), d\psi_s(w)) = g_0(\nabla_v Y, w) + g_0(\nabla_w Y, v)$$

Setting $\alpha = Y^\flat$, we obtain that $\beta = d\alpha$. This also proves Theorem 7.16. $\square$

8. The boundary rigidity property

This paragraph relates a theorem of L. Pestov and G. Uhlmann proved in [26]. For the reader's conveniance, let us state once again this result :

THEOREM 8.1 (Pestov, Uhlmann, 2003). — *A simple Riemannian surface is boundary rigid.*

We thus consider two simple Riemannian surfaces (M, g_1) and (M, g_2) such that d_{g_1} and d_{g_2} agree on $\partial M \times \partial M$. Let us first introduce some notations that have not been used so far.

8.1. Notations

Let us define on the boundary $\partial(SM)$ the odd part of τ, given by

$$\tau_-(x,\xi) = \frac{1}{2}\left(\tau(x,\xi) - \tau(x,-\xi)\right),$$

for $(x,\xi) \in \partial(SM)$. It is zero on $\partial_-(SM)$ and actually smooth on $\partial(SM)$ (the potential singularity in $\partial_0(SM)$ is killed by the antisymmetrization).

DEFINITION 8.2. — *The scattering relation $\alpha : \partial(SM) \to \partial(SM)$ is given by :*

$$\alpha(x,\xi) = \left(\varphi_{2\tau_-(x,\xi)}(x,\xi), \dot{\varphi}_{2\tau_-(x,\xi)}\right)$$

It actually defines a diffeomorphism $\alpha : \partial(SM) \to \partial(SM)$ such that $\alpha : \partial_-(SM) \to \partial_+(SM), \alpha : \partial_+(SM) \to \partial_-(SM), \alpha|_{\partial_0(SM)} = \mathrm{Id}_{\partial_0(SM)}$ and α is an involution. In other words, it maps an inward pointing vector ξ on ∂M to the outward pointing vector η on ∂M obtained by looking at the exit point of the geodesic passing through (x,ξ).

Given $f \in \mathcal{C}^\infty(M)$, its gradient ∇f (which is given by $df^\sharp$, where $\sharp$ is the musical isomorphism) can also be obtained by XX

ADD DEFINITION OPERATORS A

8.2. A first reduction

A reference for this paragraph is [10], Chapter 2.

PROPOSITION 8.3. — *Assume (M, g_1) and (M, g_2) are two simple Riemannian surfaces such that d_{g_1} and d_{g_2} agree on $\partial M \times \partial M$. Then, there exists a diffeomorphism $\psi : M \to M$, such that $\psi|_{\partial M} = Id$ and $\psi^* g_2 = g_2'$ and $\alpha_{g_1} = \alpha_{g_2'}$.*

Note that this requires to be accurate because even though $d_{g_1} = d_{g_2}$, it may not be obvious that the two unit tangent bundles agree (on ∂M), and there is no reson for it to be the case. We thus need the

Lemma 8.4. — *Assume (M, g_1) and (M, g_2) are two simple Riemannian surfaces such that d_{g_1} and d_{g_2} agree on $\partial M \times \partial M$. Then, there exists a diffeomorphism $\psi : M \to M$, such that $\psi|_{\partial M} = Id$ and $\psi^* g_2 = g_2'$ and $g_1 = g_2'$ on $T_{\partial M} M \times T_{\partial M} M$ (that is on $T_x M \times T_x M$, for any $x \in \partial M$).*

Démonstration. — First, consider $x \in \partial M$ and $\xi \in T_x(\partial M)$, and γ a curve in ∂M adapted to (x, ξ). There exists a unique g_1-geodesic (resp. g_2-geodesic) running from x to $\gamma(s)$ and :

$$|\xi|_{g_1} = \lim_{s\to 0} \frac{d_{g_1}(x, \gamma(s))}{s} = \lim_{s\to 0} \frac{d_{g_2}(x, \gamma(s))}{s} = |\xi|_{g_2}$$

By polarization, this implies that g_1 and g_2 agree on $T_x\partial M \times T_x\partial M$. But there is still one direction missing. Let us denote by $\nu(x)$ the inward normal unit vector field defined for $x \in \partial M$. As seen before, the application $\Psi_i : \partial M \to [0, \varepsilon_i) \to M, \Psi_i(x, t) = \exp_x^i(t\nu_i(x))$ for $i \in \{1, 2\}$ is a diffeomorphism on some collar neighborhood of ∂M and thus $\psi = \Psi_2 \circ \Psi_1^{-1}$ is a diffeomorphism from an annulus to another annulus (in a vicinity of ∂M) such that $\psi|_{\partial M} = \mathrm{Id}$. Now, it is possible to extend ψ to a whole diffeomorphism of the manifold M (this is a rather classical statement is the theory of dynamical systems, see [10], Chapter 3 for instance). We claim that ψ satisfies the required properties. Indeed, consider a geodesic $\gamma_1(t)$ adapted to the point $(x, \nu_1(x))$ and a geodesic γ_2 adapted to $(x, \nu_2(x))$. Then, we have $\gamma_2(t) = \psi(\gamma_1(t))$ and differentiating at $t = 0$, we obtain :

$$d_x\psi(\nu_1(x)) = \nu_2(x)$$

As a consequence, setting $g_2' = \psi^* g_2$, we have for $v \in T_x\partial M$:

$$g_2'(v, \nu_1(x)) = g_2(d_x\psi(v), d_x\psi(\nu_1(x))) = g_2(v, \nu_2(x)) = 0,$$

where the second equality comes from the fact that $d_x\psi = \mathrm{Id}$ on $T_x\partial M$. As a consequence, since $T_x M = T_x\partial M \oplus^{\perp} \mathbb{R}\nu(x)$, we obtain that $g_2' = \psi^* g_2 = g_1$ on $T_{\partial M}\ M \times T_{\partial M} M$. $\square$

We can now go on to proving the previous Proposition.

Démonstration. — From the previous lemma, we can assume that $g_1 = g_2$ on $T_{\partial M}\ M \times T_{\partial M} M$. Consider $(x, v_1) \in \partial_+(S_1 M)$ ($= \partial_+(S_2 M)$ by the previous lemma) and denote by $(y, w_1) = \alpha_1(x, v_1)$. We denote by $\gamma_{x,y}^1$ the unique g_1-geodesic joining x to y and we necessarily have $\dot{\gamma}_{x,y}^1(0) =$

$v_1, \dot{\gamma}^1_{x,y}(\tau_1(x, v_1)) = w_1$. Let us denote by γ^2 the unique g_2-geodesic joining x to y. We want to prove that :

$$v_2 = \dot{\gamma}^2_{x,y}(0) = v_1, \quad w_2 = \dot{\gamma}^2_{x,y}(\tau_2(x, v_2)) = w_1$$

Consider the two smooth functions $r_i : M \to \mathbb{R}$ defined by $r_i(p) = d_i(x, p)$, for $i \in \{1, 2\}$. Now, we have $\nabla r_1(y) = w_1$. Indeed, these two vectors are colinear because they are both orthogonal to the spheres $r_1 = \mathrm{cst}$: this stems from Gauss lemma for w_1 and by definition of the gradient for ∇r_1. But these vectors are both unitary and pointing outward the manifold, so they are equal. This also holds for $\nabla r_2(y) = w_2$.

Consider $h_i = r_i|_{\partial M}$. Then, $\nabla h_i(y)$ is the projection of $\nabla r_i(y)$ (which belongs to the hemisphere $\partial_-(SM)$) onto $T_y(\partial M)$ (see the figure below). Noe in particular that ∇h_i determines ∇r_i. But we have $h_1 = h_2$ by assumption, so $\nabla h_1(y) = \nabla h_2(y)$ and thus $\nabla r_1(y) = w_1 = \nabla r_2(y) = w_2$. Considering the geodesics backwards (with initial vectors $-w_1, -w_2$), we obtain in the same fashion that $v_1 = v_2$. $\square$

In other words, we have proved that if $d_1 = d_2$ on $\partial M \times \partial M$ and $g_1 = g_2$ on $T_{\partial M}M \times T_{\partial M}M$, then $\alpha_1 = \alpha_2$.

8.3. The scattering relation determines the Dirichlet-to-Neumann map

In this paragraph, we are going to prove that the scattering relation determines the Dirichlet-to-Neumann (DN) map and even, more precisely, the traces of conjugate harmonic functions on the boundary. The DN map is defined by the following procedure. Consider $f \in \mathcal{C}^\infty(\partial M)$ and h the solution to the Dirichlet problem :

$$\begin{cases} h|_{\partial M} = f \\ \Delta_g h = 0, \quad \text{in} \quad M \end{cases}$$

Then, the DN map Λ_g is defined by

$$\Lambda_g(f) = \langle \nabla h, \nu \rangle,$$

where ν is the normal unit outward vector field on ∂M. We are going to prove the

Theorem 8.5. — *Assume (M, g_1) and (M, g_2) are simple manifolds. Then $\alpha_1 = \alpha_2$ implies that $\Lambda_1 = \Lambda_2$.*

Now, consider h^* the conjugate harmonic function of h that is the function satisfying $dh^* = \star dh$, where $\star$ is the Hodge operator and denote by f^* its trace on ∂M. It does exist since $H^1(M,\mathbb{R}) = \{0\}$ (because M is diffeomorphic to a closed disk) and $\star dh$ is closed (because $d \star dh = 0$ is equivalent to $\star d \star dh = \Delta_g h = 0$). But we have :

$$\Lambda_g(f) = \langle \nabla h, \nu \rangle = dh(\nu) = - \star dh^*(\nu) = -dh^*(\nu^\perp) = -df^*(\nu^\perp),$$

where we assume that ∂M has been oriented and $\left\{\nu, \nu^\perp\right\}$ form a positively oriented basis of $T_{\partial M}M$. Thus, if we can recover f^* from f and the knowledge of the scattering relation α, then it becomes clear that α determines Λ_g.

Also note, that according to Section 3.2.4, we can see conjugate harmonic functions as smooth functions on SM (constant in the fibers, considering their pullback) and the equality $dh^* = \star dh$ thus becomes $X \cdot h^* = H \cdot h$. We first state the

Lemma 8.6. — *Take $w \in \mathcal{C}^\infty_\alpha(\partial_+(SM))$, then on $\partial_+(SM)$:*

$$2\pi A_-^* \mathcal{H}_+ A_+ w = IHI^* w$$

Démonstration. — For the different notations, we refer to the previous sections. We have $I^*w = 2\pi(w_\psi)_0$ and thus

$$HI^*w = 2\pi H \cdot (w_\psi)_0 = 2\pi(\mathcal{H}_- X \cdot w_\psi - X \cdot \mathcal{H}_+ w_\psi) = -2\pi X \cdot \mathcal{H}_+ w_\psi,$$

using the equality (3.1) and the fact that w_ψ is constant along the geodesics. And eventually, using $w|_{\partial(SM)} = A_+ w$, we obtain :

$$IHI^*w = -2\pi I X \mathcal{H}_+ w_\psi = -2\pi A_-^* \mathcal{H}_+ A_+ w$$

□

Note that the operators A are known (they depend on the scattering relation α), as well as the operator $\mathcal{H}$ because we are on the boundary ∂M and we assume that we know g on ∂M (and the Hilbert transform only depends on the metric). Thus, the lemma proves that the quantity IHI^*w only depends on things we know (it does not depend on g in $\overset{\circ}{M}$ for instance, which is unknown).

Démonstration. — We know that $g_1 = g_2$ on $T_{\partial M}M \times T_{\partial M}M$ and $\alpha_1 = \alpha_2$. Thus, we only have to show that we can determine the traces of conjugate harmonic functions from the knowledge of g on ∂M and α. Both the surjectivity of I^* and the injectivity of I will appear to be crucial in the proof.

Consider a smooth function h_*^0 on ∂M and denote by h_* its harmonic continuation on M and h its conjugate harmonic function (such that $H \cdot h =$

$-X \cdot h_*$). Consider on $\partial_+(SM)$ a function $w \in \mathcal{C}^\infty_\alpha(\partial_+(SM))$ such that $2\pi A_-^* \mathcal{H}_+ A_+ w = -A_-^* h_*^0$. First, we have to prove that such a w exists. Since $I^* : \mathcal{C}^\infty(\partial_+(SM)) \to \mathcal{C}^\infty(M)$ is surjective (see Theorem XX), there exists at least one w such that $I^* w = h$. But then :

$$IHI^*w = -2\pi A_-^* \mathcal{H}_+ A_+ w = IH \cdot h = -IX \cdot h_* = A_-^* h_*^0$$

As a consequence, we can find such a w and it only depends on the data g on ∂M and α (since the equation which defines it only depends on known data).

The converse is also true. Namely, if w satisfies the previous equation, then $h = I^* w$ is a conjugate harmonic function to h^*, the harmonic continuation of h_*^0. Indeed, we only need to prove that $H \cdot h = -X \cdot h_*$. But we have :

$$IHh = IHI^*w = -2\pi A_-^* \mathcal{H}_+ A_+ w = A_-^* h_*^0 = -IXh_*$$

By injectivity of the X-ray transform, we obtain that $H \cdot h + X \cdot h_* = X \cdot p$ for some smooth $p \in \mathcal{C}^\infty(M)$ such that $p|_{\partial M} = 0$ and p is obviously harmonic so $p \equiv 0$. Now, $h^0 = I^* w|_{\partial M} = 2\pi(A_+ w)_0$ is the trace of the conjugate harmonic function h. So h^0 is obtained from h_*^0 and the knowledge of g on ∂M and α/ □

8.4. The DN map determines the conformal class

The reference for this paragraph is [**?**], where the following result is proved.

Theorem 8.7 (Lassas-Uhlmann). — *The DN map Λ_g determines the conformal class of the metric g.*

In other words, this theorem states that given any $f \in \mathcal{C}^\infty(\partial M)$ and u the harmonic solution of the Dirichlet problem $u|_{\partial M} = f$, if one knows the image $\Lambda_g(f) = \langle \nabla u, \nu \rangle$ by the DN map, then it is possible to reconstruct the conformal class of the metric. This is actually the best result one can expect because if g and g' are two conformal and simple metrics on the manifold M, that is $g' = \sigma g$ for some smooth $\sigma > 0$, then $\Delta_{g'} u = \Delta_{\sigma g} u = \sigma^{-1} \Delta_g u$, which means that $\Lambda_g = \Lambda_{\sigma g}$.

8.5. Conclusion of the proof

From the previous sections, we thus know that given (M, g_1) and (M, g_2) two simple Riemannian surfaces such that $d_1 = d_2$ on $\partial M \times \partial M$, we can find a diffeomorphism ψ such that $\psi|_{\partial M} = \mathrm{Id}$ and $g_1 = f\psi^* g_2$, for some smooth $f > 0$ such that $f|_{\partial M} = 1$.

TO BE CONTINUED

9. Conclusion

As a conclusion, let us sum up what is known so far about the injectivity of the X-ray transform. This résumé is partly taken from [24] :

(1) On simple manifolds of dimension $n \geqslant 2$:
   — I_0 is injective and I_1 is s-injective : the proof relies on the extension of the Pestov identity to greater dimension and the use of a "0-control", like in Section 4.4.2,
   — I_m is s-injective for all m if $n = 2$ as we have proved it,
   — I_m is s-injective for all m on manifolds of negative sectional curvature (see [3] for instance), or under certain other curvature assumptions : the proof also relies on the use of Pestov identity like we did in Section 5.

(2) On Anosov manifolds of dimension $n \geqslant 2$:
   — I_0 is injective and I_1 is s-injective,
   — I_m is s-injective for all m if $n = 2$ (see [11]) (we have proved it in the cases $m = 0, 1, 2$) and the proof for $m \geqslant 3$ heavily relies on microlocal analysis,
   — I_m is s-injective for all m on non-positively curved manifolds as mentioned before,
   — But it is not known whether I_m is s-injective on Anosov manifolds of dimension $n \geqslant 3$, and not even I_2.

Annexe A. (Pseudo)differential operators

For this section, we refer to [2] or [29].

A.1. Differential operators

We consider (M,g) a n-dimensional smooth Riemannian manifold. Let E,F be two vector bundles over M. A linear operator $P : \Gamma(M,E) \to \Gamma(M,F)$ is a differential operator of order d if in any local coordinates (x_i), one can write :

$$Pu(x) = \sum_{|\alpha| \leqslant d} a^\alpha(x)\partial_\alpha u(x),$$

where $\alpha = (\alpha_1, ..., \alpha_k) \in \{1, ..., n\}^k$, $|\alpha| = k$, $\partial_\alpha = \partial_{\alpha_1}...\partial_{\alpha_n}$ and $a^\alpha(x)$ is a matrix standing for an linear application $E_x \to F_x$.

Its principal symbol is defined for $x \in M, \xi \in T_x^*M$ by :

$$\sigma_P(x,\xi) = \sum_{|\alpha|=d} a^\alpha(x)\xi_\alpha,$$

with $\xi_\alpha = \xi_{\alpha_1}...\xi_{\alpha_d}$, where $\xi = \xi_i dx^i$. It is a homogeneous polynomial in the variable ξ (of degree d) with values in the linear application $E_x \to F_x$. Actually, one can prove that this principal symbol is well defined, regardless of any coordinate system. For instance, given $f \in \mathcal{C}^\infty(M), t \in \mathbb{R}$ and $u \in \Gamma(M,E)$, one can check that

$$e^{-tf(x)}P(e^{tf(x)}u(x))$$

is a polynomial of degree d in the variable t, whose monomial of degree d is given by :

$$t^d\sigma_P(x, df(x))u(x)$$

We have the two

Lemma A.1. —

$$\sigma_{P\circ Q} = \sigma_P \circ \sigma_Q$$

Lemma A.2. — *Given $P : \Gamma(M,E) \to \Gamma(M,F)$ of order d, it has a formal adjoint P^*, whose principal symbol is given by :*

$$\sigma_{P^*}(x,\xi) = (-1)^d\sigma_P(x,\xi)^*$$

We now introduce the main class of operators, for which the main theorem of this paragraph will be stated.

Definition A.3. — *We say that* $P : \Gamma(M,E) \to \Gamma(M,F)$ *is elliptic if for any* $x \in M, \xi \in T_x^*M$ *with* $\xi \neq 0$*, the principal symbol* $\sigma_P(x,\xi) : E_x \to F_x$ *is injective.*

Theorem A.4. — *Assume* $rg(E) = rg(F)$ *and* $P : \Gamma(M,E) \to \Gamma(M,F)$ *is an elliptic operator. Then ker* (P) *is finite dimensional and there is an* L^2 *orthogonal splitting :*

$$\mathcal{C}^\infty(M,F) = \ker\,(P^*) \oplus P(\mathcal{C}^\infty(M,E))$$

Note that ker (P^*) is also finite dimensional since P^* is elliptic. Thus P is a Fredholm operator of index :

$$\mathrm{ind}(P) = \dim \ker\,(P) - \dim \ker\,(P^*)$$

In particular, a formally self-adjoint elliptic operator has index zero, and the invariance of the Fredholm index under continuous deformation implies that any elliptic operator with the same symbol has also index zero.

One can also prove that a differential operator $P : \Gamma(M,E) \to \Gamma(M,F)$ induces continuous operators on Sobolev spaces as $P : H^{s+d}(M,E) \to H^s(M,F)$, for any $s \in \mathbb{R}$. Given U and V two open sets of M, we will denote $U \subset\subset V$ the fact that there exists a compact K such that $U \subset K \subset V$. We have the following local elliptic estimate :

Lemma A.5. — *Assume* $P : \Gamma(M,E) \to \Gamma(M,F)$ *is elliptic and consider* U, V *two open sets of* M *such that* $U \subset\subset V \subset\subset M$*. If* $u \in \mathcal{D}'(M)$ *is such that* $Pu \in H^s(M)$*, then* $u \in H^{s+m}_{loc}(M)$ *and for each* $\sigma < s+m$*, there exists a constant* $C(U,V,s,\sigma)$ *such that :*

$$||u||_{H^{s+m}(U)} \leqslant C\left(||Pu||_{H^s(V)} + ||u||_{H^\sigma(V)}\right)$$

This is a rather difficult lemma, which can be proved by freezing the coefficients of P, but we will omit its proof. In the case M is closed and compact, this generalizes to the whole manifold :

Lemma A.6. — *Assume* M *is closed compact,* $P : \Gamma(M,E) \to \Gamma(M,F)$ *is elliptic. If* $u \in \mathcal{D}'(M)$ *is such that* $Pu \in H^s(M)$*, then* $u \in H^{s+m}(M)$ *and for each* $\sigma < s+m$*, there exists a constant* $C(s,\sigma)$ *such that :*

$$||u||_{H^{s+m}(M)} \leqslant C\left(||Pu||_{H^s(M)} + ||u||_{H^\sigma(M)}\right)$$

A.2. Pseudodifferential operator

The notion of pseudodifferential operators generalizes that of differential operators. We refer to [29] for further details. On a compact Riemannian n-dimensional manifold (M,g), let us introduce the symbol class $S^m(M)$:

Definition A.7. — *The symbol class $S^m(M)$ of order $m \in \mathbb{R}$ consists of $\mathcal{C}^\infty(T^*M)$ complex-valued functions such that :*

$$\forall \alpha, \beta, \quad |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|},$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$.

There are more general classes of symbols (with three indices $S^m_{\rho,\delta}$ for instance), but they are not used in this memoire, so we will not introduce them. Choosing a chart system and a related partition of unity on M, one can define the pseudodifferential operator $\hat{P} = \mathrm{Op}(p)$ related to the symbol p, acting on $\mathcal{C}^\infty(M)$ and defined locally by :

$$\hat{P} : u \mapsto (\hat{P}u)(x) = \int\int e^{i\xi\cdot(x-y)} p(x,\xi) u(y) dy d\xi$$

The symbol p does not carry a geometrical meaning insofar as a change of coordinate systems will change the symbol. In other words, it is not defined intrinsically. However, this will only change p up to a symbol of subleading order $m-1$, which means that the *principal symbol* of p, denoted by $\sigma(\hat{P}) = p \mod S^{m-1}$ is a well defined function of $\mathcal{C}^\infty(T^*M)$, independent of the charts.

For example, if X is a vector field on M, the operator $\hat{H} = -iX$ admits the symbol $H(x,\xi) = \langle \xi, X \rangle$. It does not depend on the charts, which is something particular to operators of order 1. Let us state a few important operational properties on the symbols :

Proposition A.8. — *(1) If $p \in S^m$, then $\hat{P} : H^s(M) \to H^{s-m}(M)$.*

(2) If $p \in S^{m_1}, q \in S^{m_2}$, then $pq \in S^{m_1+m_2}$ and the principal symbol of $\hat{P} \circ \hat{Q}$ is the product $\sigma(\hat{P})\sigma(\hat{Q})$ (it is an operator of order $m_1 + m_2$ in particular).

(3) If $p \in S^m$, then $\hat{P}^*$ is of order m and has principal symbol $\bar{\sigma}_m(\hat{P}) \in S^m$.

(4) If $p \in S^{m_1}, q \in S^{m_2}$, then $[\hat{P}, \hat{Q}]$ is of order $m_1 + m_2 - 1$ and has principal symbol $\{p,q\} \mod S^{m_1+m_2-2}$.

To an operator $\hat{P}$ of symbol $p \in S^m$, one can associate its kernel K, given in local coordinates by

$$K(x,y) = (2\pi)^{-n} \int p(x,\xi) e^{i(x-y)\cdot\xi} d\xi$$

It is possible to prove rather sharp estimates on the kernel K, but let us simply state the

Proposition A.9. — *K is $\mathcal{C}^\infty$ off the diagonal in $M \times M$.*

The singularities of K along the diagonal actually determine the behaviour of the operator $\hat{P}$.

Whereas the differential operators are local, the pseudodifferential are not, which makes their definition more delicate on manifolds with boundary. This the reason why we use an embedding trick in the study of the operator $I_0^* I_0$.

Annexe B. Existence of isothermal coordinates on a surface

The reference for this part is [29].

We say that a smooth map φ between two Riemannian manifolds (M, g) and (N, h) is *conformal* if there exists $\lambda \in \mathcal{C}^\infty(M)$ such that $\varphi^* h = e^{2\lambda} g$. This is strictly equivalent to the fact that the application φ preserves the angle. We want to show the following theorem

Theorem B.1. — *Let (M, g) be a Riemannian surface and $p \in M$. There exists a local coordinate system around p which is conformal, that is there exists a neighborhood $U \subset M$ of p, a chart $\varphi : U \to \varphi(U)$, and a function $\lambda \in \mathcal{C}^\infty(\varphi(U))$ such that $\varphi^*(e^{2\lambda}(dx^2 + dy^2)) = g$.*

We call this coordinate system an *isothermal* coordinate system. It is rather useful since it simplifies a lot of computations when carried out in local coordinates. The previous definition clearly shows that the composition of two conformal maps is still conformal. In particular, given two local conformal charts $(U, \varphi), (V, \psi)$ on M around p, we see that $\Phi := \psi \circ \varphi^{-1}$ is conformal where it is defined. Since an orientation-preserving conformal map between two oriented open sets of the plane is holomorphic when seen as a function of the complex variable (and note that the converse is also true), we conclude that $\Phi = f + ig$ is holomorphic. Now, this also immediately implies that f and g (which are respectively the real and the imaginary part of Φ) are harmonic functions. As a consequence, the conformal map $\varphi : U \to \varphi(U)$ is harmonic (that is each of its coordinate is harmonic).

Now, let us see that there exists somehow a converse to this fact. Assume that we can find a map $f : U \to \mathbb{R}$ which is harmonic on U, and such that $df_p \neq 0$. We would like to find a function $g : U \to \mathbb{R}$ such that $\varphi = f + ig$ is holomorphic (given an orthonormal basis in U, it satisfies the Cauchy-Riemann equations) and df_p and dg_p are independent (and therefore df and dg will be independent in a vicinity of p). It is easy to see that φ is conformal (i.e. holomorphic) if and only if $\star df = dg$, where $\star$ is the Hodge operator.

This is just a rewriting of the Cauchy-Riemann equations. Since we can always assume U to be simply connected (if not, we can always shrink U so that it becomes true), the existence of g is equivalent by Poincaré's lemma to the fact that $d\star df = 0$ But this is also equivalent to $\star d\star df = -\Delta f = 0$ and since f is assumed to be harmonic, the existence of g is guaranteed. Note that it is clear that df_p and dg_p are independent since $df = \star dg$ and we assumed that $df_p \neq 0$. Thus, for a neighborhood $V \subset U$ small enough around p, df_x and dg_x will still be independent for $x \in V$. In other words, we have obtained that (f, g) is a conformal coordinate system defined on V.

Thus, the proof of Theorem B.1 reduces to proving the following

Proposition B.2. — *Given $p \in M$, there exists a neighborhood $U \subset M$ around p and a real valued function f such that $\Delta f = 0$ and $df_p \neq 0$.*

Démonstration. — The proof mainly relies on a standard Dirichlet problem. Indeed, choose a centered coordinate system $\varphi : x \mapsto (x_1, x_2)$ in a neighborhood U around p. In these coordinates, we have, according to equation (2.3) :

$$\Delta f(x) = g^{ij}(x)\partial_j\partial_k f + b^k(x)\partial_k f,$$

for some smooth functions b^k. Now, let us choose a disk D small enough in $\varphi(U)$ (of radius ε) centered at 0. We could define for instance f to be the solution of the Dirichlet problem $\Delta f = 0$ in D and such that $f = x_1$ on ∂D, but this will not guarantee that $df_0 \neq 0$. In order to ensure this, we consider for $\varepsilon > 0$ the dilated coordinate system $(x_1^\varepsilon, x_2^\varepsilon) = (x_1/\varepsilon, x_2/\varepsilon)$, sending the disk D_ε of the original coordinate system to the unit disk in the dilated coordinate system. Note that in this coordinate system, we have :

$$\Delta_\varepsilon f(x) = g^{ij}(\varepsilon x)\partial_j\partial_k f + \varepsilon b^k(\varepsilon x)\partial_k f,$$

Consider the function f_ε which is harmonic in the unit disk D_1 and equal to x_1^ε on the boundary (in the dilated coordinate system). We define the function v_ε such that $\Delta_\varepsilon v_\varepsilon = b^1(\varepsilon x) = b^1_\varepsilon(x)$ on the unit disk and $v_\varepsilon = 0$ on its boundary. Then we remark that $f_\varepsilon = x_1 - \varepsilon v_\varepsilon$ by unicity of the Dirichlet problem. Now let us see that for ε small enough, $\partial_1 f_\varepsilon(0) = 1 - \varepsilon\partial_1 v_\varepsilon(0)$ is not zero. Since $v_\varepsilon = 0$ on ∂D_1, the Poincaré inequality provides :

$$||v_\varepsilon||^2_{L^2(D_1)} \leqslant C||\nabla v_\varepsilon||^2_{L^2} \leqslant C|\langle \Delta_\varepsilon v_\varepsilon, v_\varepsilon\rangle| = C|\langle b^1_\varepsilon, v_\varepsilon\rangle| \leqslant C||v_\varepsilon||_{L^2}||b^1_\varepsilon||_{L^2}$$

Since b^1_ε is bounded in $L^2(D_1)$ (and in each $H^k(D_1)$) independently of ε, we obtain : $||v_\varepsilon||_{L^2} \leqslant C$. Applying the global elliptic estimate (A.6), we thus obtain :

$$||v||^2_{H^{k+1}} \leqslant C(||\Delta_\varepsilon v_\varepsilon||^2_{H^{k-1}} + ||v_\varepsilon||^2_{L^2}) = C(||b^1_\varepsilon||^2_{H^{k-1}} + ||v_\varepsilon||^2_{L^2}) \leqslant C$$

Since $H^k(D_1) \hookrightarrow C^1(D_1)$ for k large enough, we obtain a uniform bound on the C^1 norm of v_ε, independant of ε, which concludes the proof.

□

Annexe C. On Anosov flows

C.1. Definition

The reference for this section is mainly [20]. Let (M, g) be a Riemannian manifold and $\phi : \mathbb{R} \times M \to M$ a flow.

Definition C.1. — *We say that a closed set $\Lambda \subset M$ is ϕ-invariant (or simply invariant) is $\phi_t(\Lambda) \subset \Lambda$ for all $t \in \mathbb{R}$. A closed invariant set $\Lambda \subset N$ is called hyperbolic if there exists subbundles $E^s, E^u \subset T_\Lambda M = \{(x, v) \in TM,\ x \in \Lambda\}$ such that for all $x \in \Lambda$,*

$$T_x M = \mathbb{R}X(x) \oplus E^s(x) \oplus E^u(x)$$

, and for all $t \in \mathbb{R}$,

$$d\phi_t(E^s(x)) \subset E^s(\phi_t(x)),$$

$$d\phi_t(E^u(x)) \subset E^u(\phi_t(x)),$$

and for all $t \geqslant 0$,

$$||d\phi_t|_{E^s}\ || \leqslant Ce^{-\mu t},$$

$$||d\phi_{-t}|_{E^u}||\ \leqslant Ce^{-\mu t},$$

where $C, \mu > 0$ are constants.

Remark C.2. — Note that it is always possible to find a new metric g' on M such that the constant C is equal to 1.

Definition C.3. — *If M is itself a hyperbolic set, then we say that the flow ϕ is Anosov. The subbundles E^s and E^u are called the stable and unstable bundles respectively.*

Not every manifold can carry an Anosov flow. Actually, this is such a specific property that there is only a limited number of known examples of manifolds which can carry an Anosov flow. For instance, the following result was proved by Ghys (see [9]) in 1984 :

THEOREM C.4 (Ghys, 1984). — *Let M be a closed three-dimensional manifold that is a circle bundle and $\phi : \mathbb{R} \times M \to M$ an Anosov flow. Then there exists a closed surface N of genus $g \geqslant 2$ such that $\pi : M \to SN$ is a finite cover of SN and ϕ is continuously orbit equivalent to the lift to M of the geodesic flow on SN corresponding to a metric g_0 of constant negative curvature -1.*

In our case, we are particularly interested in the situations when the geodesic flow is Anosov on the unit tangent bundle. Ghys' theorem proves in particular that the sphere and the 2-torus cannot carry any metric for which the geodesic flow is Anosov.

DEFINITION C.5. — *We define the* stable *and* unstable *manifold of ϕ at x as the sets :*

$$W^s(x) = \{y \in N,\ d(\phi_t(x), \phi_t(y)) \to_{t\to+\infty} 0\}$$
$$W^u(x) = \{y \in N,\ d(\phi_t(x), \phi_t(y)) \to_{t\to-\infty} 0\}$$

Remark C.6. — Actually, the term *manifold* is too ambitious in so far as these sets are not manifolds but embedded manifolds in M. For instance, on the two torus, given an initial direction with irrational angle, the trace of the geodesic flow starting from $(0,0)$ is a dense set of parallel straight lines, and therefore not a submanifold of the torus.

It is also possible to require the convergence to be exponentially fast in the definition of the previous sets.

We can also define the previous sets locally. For $U \subset M$, we set :

$$W^s(x,U) = \{y \in U,\ (\phi_t(x), \phi_t(y)) \to_{t\to+\infty} 0\}$$
$$W^u(x,U) = \{y \in U,\ d(\phi_t(x), \phi_t(y)) \to_{t\to-\infty} 0\}$$

THEOREM C.7 (Local stable manifold theorem). — *There exists a $\varepsilon > 0$ such that for each $x \in M$, the local stable and unstable manifolds*

$$W^s_{loc}(x) = W^s_{loc}(x, B(x,\varepsilon)), \qquad W^u_{loc}(x) = W^u_{loc}(x, B(x,\varepsilon))$$

are embedded discs such that $T_x W^s_{loc}(x) = E^s(x), T_x W^u_{loc} = E^u(x)$ for all $x \in M$ and

$$\phi_t(W^s_{loc}(x)) \subset W^s_{loc}(\phi_t(x)), \qquad \phi_t(W^u_{loc}(x)) \subset W^u_{loc}(\phi_t(x)),$$

for all $t > 0$.

In particular, one can check that for any neighborhood $U(t)$ of $\phi_t(x)$, the following equalities hold

$$W^s(x) = \cup_{t>0}\phi_{-t}(W^s_{loc}(\phi_t(x), U(t)))$$

$$W^u(x) = \cup_{t>0}\phi_t(W^u_{loc}(\phi_{-t}(x), U(t)))$$

and $T_xW^s(x) = E^s(x), T_xW^u(x) = E^u(x)$.

We will not give the proof of this subtle theorem. It can be found in [15] (Section 17.43).

C.2. The Anosov theorem

We now end this section with a few properties satisfied by Anosov flows. We first state Anosov's celebrated result on the behaviour of the geodesic flow on a negatively curved manifold.

THEOREM C.8 (Anosov, 1967). — *If (M, g) is a closed Riemannian manifold with negative curvature $(K < 0)$, then the geodesic flow on SM is Anosov.*

Even though we will not detail any proof here, we explicit the main ideas leading to this result. There are mainly two ways to prove Anosov's theorem :

— The first one (see [15], Section 17.6) relies on the study of the geodesic variations of a given geodesic γ, that is on the study on the Jacobi fields along this geodesic. One can actually prove, thanks to the hypothesis on the curvature and the use of Jacobi equation, that the vector space of normal (to $\dot{\gamma}$) Jacobi fields along γ which vanish in $\gamma(0)$ (this is a $(n-1)$-dimensional vector space according to the theory of Jacobi vector fields) splits in two vector spaces : one on which the Jacobi fields expand exponentially fast and one on which they contract exponentially fast. Then, from the Jacobi fields, it is easy to recover the derivative of the exponential map on the manifold.

— The second proof is more subtle and relies on a theorem which we state here for cultural purposes. Since $d(\phi_t)_x(X(x)) = X(\phi_t(x))$, the geodesic flow on TM descends to the quotient $\hat{T}M$, defined as vector bundle over M such that $\hat{T}_xM = T_xM/\mathbb{R}X(x)$, and defines a flow on $\hat{T}M$.

THEOREM C.9 (Wojtkowksi, 2000). — *Let M be a closed manifold and $\phi : \mathbb{R} \times M \to M$ a non-singular flow with infinitesimal generator X. We assume that there exists a quadratic form $Q : TM \to \mathbb{R}$ on TM satisfying the following properties :*

— *For each $x \in M$, the form $Q_x : Q|_{T_xM} : T_xM \to \mathbb{R}$ depends continuously on x.*

— *For all* $x \in M, v \in T_xM, a \in \mathbb{R} : Q_x(v + aX(x)) = Q_x(v)$. *This means that* Q *descends to the quotient bundle* $\hat{Q} : \hat{T}M \to \mathbb{R}$.
— $\hat{Q} : \hat{T}M \to \mathbb{R}$ *is non-degenerate.*
— *The Lie derivative* $\mathcal{L}_X Q$ *must be continuous and if* $\hat{L}$ *denotes the projection of the Lie derivative* $\mathcal{L}_X Q$ *to* $\hat{T}M$ *(it is well defined according to the previous point), then* $\hat{L}$ *must be positive definite on* $\hat{T}M$.

Then, ϕ *is Anosov.*

Thus, one only has to find a suitable quadratic form Q in order to prove the Anosov property of the geodesic flow. Somehow, this also relies on a proper use of Jacobi vector fields.

C.3. Livcic's periodic theorem

We now assume that M is a closed compact manifold. We can now state the

Theorem C.10. — *Let* $f : M \to \mathbb{R}$ *be a* $\mathcal{C}^k$ *function such that its integral over every periodic integral curve of* X *is zero. Then, there exists a* $\mathcal{C}^k$ *function* u *such that* $X \cdot u = f$.

Remark C.11. — This result still holds for the α-Hölder regularity. Namely, if $f : M \to \mathbb{R}$ is α-Hölder such that its integral over every periodic integral curve of X is zero, then there exists an α-Hölder function u which is differentiable in the flow direction and such that $X \cdot u = f$.

We will partially prove this result, namely we will only consider the case $k = 1$. For higher regularity, this a very subtle question which was tackled in the 1980's by De la Llave-Marco-Moriyón (see [4]). Using the anisotropic Sobolev spaces introduced in [7], C. Guillarmou (see [11]) was able to extend Livcic's theorem to the Sobolev regularity $H^k(M)$, thus also providing the $\mathcal{C}^k(M)$ regularity.

The main idea in the proof presented below relies on Anosov's closing lemma. We refer to [15] (Corollary 18.1.8) for a proof :

Theorem C.12. — *Let* $\phi : \mathbb{R} \times M \to M$ *be a transitive Anosov flow on a closed manifold* M. *Then there exists* $\varepsilon > 0, K > 0, T_0 > 0$ *such that if for some* $T > T_0$, $d(\phi_T(x), x) < \varepsilon$, *then there exists a unique periodic point* $p \in M$ *with period* $T + \tau$ *such that* $\max\{d(x,p), d(\phi_T(x),p), |\tau|\} \leqslant K\varepsilon$ *and* $W^s_{loc}(p) \cap W^u_{loc}(x) \neq \emptyset$. *In fact, this unique point* p *satisfies in addition :*

$$\max\{d(x,p), d(\phi_T(x),p), |\tau|\} \leqslant Kd(\phi_T(x), x),$$

and there exists a unique point $z \in N$ such that :

$$W^s_{loc}(p) \cap W^u_{loc}(x) = \{z\}$$

From this, we obtain the following proposition :

PROPOSITION C.13. — *Let $\phi : \mathbb{R} \times M \to M$ be a transitive Anosov flow on the closed compact manifold M and $f : M \to \mathbb{R}$ an α-Hölder function. Then there exists $\varepsilon > 0, K_0 > 0, T_0 > 0$ such that if $d(\phi_T(x), x) < \varepsilon$ for some $T > T_0$, then there exists a closed orbit Γ with period $T + \tau$ for some $\tau \geqslant 0$ such that :*

$$\left| \int_0^{T+\tau} f(\phi_t(p))\ dt - \int_0^T f(\phi_t(x))dt \right| \leqslant K_0 d(\phi_T(x), x)^\alpha, \tag{C.1}$$

where p is some point in Γ.

Démonstration. — Note that we can always assume $\varepsilon < 1$ (the theorem is actually used with $\varepsilon \ll 1$). By the previous theorem, there exists a periodic point p of period $T + \tau$ such that $\max\{d(x, p), d(\phi_T(x), p), |\tau|\} \leqslant K \min(\varepsilon, d(\phi_T(x), x))$ and $W^s_{loc}(p) \cap W^u_{loc}(x) = \{z\}$. Moreover, we also have $d(p, z) \leqslant d(x, p), d(\phi_T(x), \phi_T(z)) \leqslant d(\phi_T(x), p)$. Decomposing the integral (C.1), we obtain :

$$\left| \int_0^{T+\tau} f(\phi_t(p))\ dt - \int_0^T f(\phi_t(x))dt \right|$$
$$\leqslant \int_T^{T+\tau} |f(\phi_t(p))|\ dt + \int_0^T |f(\phi_t(p)) - f(\phi_t(z))|dt + \int_0^T |f(\phi_t(z)) - f(\phi_t(x))|dt$$

The first term is bounded by

$$\tau ||f||_\infty \leqslant K||f||_\infty d(x, \phi_T(x)) \leqslant K||f||_\infty d(x, \phi_T(x))^\alpha,$$

while the second is bounded by

$$\int_0^T d(\phi_t(p), \phi_t(z))^\alpha dt \leqslant \int_0^T e^{-\mu\alpha t} d(p, z)^\alpha dt \leqslant \frac{1}{\mu\alpha} d(p, x)^\alpha \leqslant \frac{K}{\mu\alpha} d(x, \phi_T(x))^\alpha$$

and the third by

$$\begin{aligned}
\int_0^T d(\phi_t(x), \phi_t(z))^\alpha dt &\leqslant \int_0^T e^{-\mu\alpha t} d(\phi_T(z), \phi_T(x))^\alpha dt \\
&\leqslant \frac{1}{\mu\alpha} d(p, \phi_T(x))^\alpha \\
&\leqslant \frac{K}{\mu\alpha} d(x, \phi_T(x))^\alpha
\end{aligned}$$

□

Démonstration. — We first prove Theorem C.10 in the α-Hölder case. Since u is differentiable in the flow direction and $X \cdot u = f$, a necessary condition on u is that

$$u(\phi_t(x)) = u(x) + \int_0^t f(\phi_s(x))\, ds,$$

by the fundamental theorem of calculus.

Since ϕ is Anosov, it admits at least a dense orbit Γ_0. We take $x_0 \in \Gamma_0$ and define $u : \Gamma_0 \to \mathbb{R}$ by :

$$u(\phi_t(x_0)) = \int_0^t f(\phi_s(x_0))\, ds \tag{C.2}$$

We are going to prove that u is α-Hölder continuous on Γ_0. Let us take $x, y \in \Gamma_0$ such that $d(x,y) < \varepsilon$, where ε is the one provided by Proposition (C.13). We write $x = \phi_r(x_0), y = \phi_s(x_0)$ and we assume that $s \geqslant r$. We can also assume that $s - r \geqslant T_0$ since Γ_0 is dense. Therefore, we have $d(\phi_{s-r}(x), x) < \varepsilon$. By Proposition (C.13), we know that there exists a closed orbit Γ such that :

$$\left| \int_0^{s-r+\tau} f(\phi_t(z_0))dt - \int_0^{s-r} f(\phi_t(x))dt \right| \leqslant K_0 d(\phi_{s-r}x, x)^\alpha = K_0 d(x,y)^\alpha,$$

for some $\tau \geqslant 0, z_0 \in \Gamma$. By assumption, the first term is zero. Therefore, we obtain :

$$|u(x) - u(y)| = \left| \int_0^{s-r} f(\phi_t(x))dt \right| \leqslant K_0 d(x,y)^\alpha$$

As a consequence, u is α-Hölder on Γ_0 and, in particular, uniformly continuous. Thus, there exists a unique extension $u : \bar{\Gamma}_0 = M \to M$ satisfying (C.2), α-Hölder on M, differentiable along the flow direction and such that $X \cdot u = f$.

We now assume that f is $\mathcal{C}^1$. We may take $\alpha = 1$ in the previous paragraph, so u is Lipschitz on M. If $x \in M, y \in W^s(x)$, we have :

$$u(y) - u(x) = u(\phi_t(x)) - u(\phi_t(y)) + \int_0^t \left(f(\phi_s(x)) - f(\phi_s(y))\right) ds$$

By assumption, since the flow is Anosov and f is $\mathcal{C}^1$, we have

$$|f(\phi_s(x)) - f(\phi_s(y))| \leqslant C d(\phi_s(x), \phi_s(y)) \leqslant C e^{-\lambda s} d(x,y),$$

which allows us to conclude that the integral in the previous line converges as $t \to \infty$. Thus :

$$u(x) - u(y) = \int_0^\infty \left(f(\phi_s(x)) - f(\phi_s(y))\right) ds$$

Now, consider a curve $\gamma : (-\varepsilon, +\varepsilon) \to W^s(x)$ with $\gamma(0) = x, \dot{\gamma}(0) = v \in E^s(x)$ $(= T_x W^s(x))$. Then :

$$\frac{u(\gamma(r)) - u(\gamma(0))}{r} = -\frac{1}{r}\int_0^\infty (f(\phi_s(\gamma(r))) - f(\phi_s(\gamma(0))))\, ds$$

The term in the integral converges to $df(d\phi_s(v))$ as $r \to 0$. Since $d\phi_s(v) \to 0$ converges exponentially fast to 0, we can pass to the limit in the integral and obtain :

$$\left.\frac{d}{dr}\right|_{r=0} u(\gamma(r)) = -\int_0^\infty df \circ d\phi_s(v))ds$$

u is differentiable in the direction of the stable manifold $W^s(x)$ (and a similar argument applies to $W^u(x)$) and the derivative is continuous. Therefore, u is $\mathcal{C}^1$. □

Annexe D. Decomposition of symmetric tensors

D.1. Decomposition in potential and solenoidal parts

The reference for this paragraph is mostly [28]. We consider a smooth compact Riemannian manifold (M, g) with boundary.

Theorem D.1. — *Let $k \geqslant 1$ and $m \geqslant 0$ be integers. For every tensor field $f \in H^k(M, \otimes_S^m T^*M)$, there exists a unique $f^s \in H^k(M, \otimes_S^m T^*M)$ and a unique $v \in H^{k+1}(M, \otimes_S^{m-1} T^*M)$ such that :*

$$f = f^s + dv, \quad \delta f^s = 0, \quad v|_{\partial M} = 0$$

Moreover, there exists a constant C such that :

$$||f^s||_{H^k} \leqslant C||f||_{H^k}, \quad ||v||_{H^{k+1}} \leqslant C||\delta f||_{H^{k-1}}$$

In particular, this theorem immediately implies Theorem 3.8 : if f is smooth, then so are f^s and v.

Démonstration. — Assume $f \in H^k(M, \otimes_S^m T^*M)$ satisfies the previous decomposition. Then $\delta f = \delta f^s + \delta dv = \delta dv$. Setting

$$u = \delta f \in H^{k-1}(M, \otimes_S^m T^*M),$$

we are reduced to solving the problem :

$$\delta dv = u, \quad v|_{\partial M} = 0 \tag{D.1}$$

If we prove, that there exists a unique solution to this equation satisfying the previous estimates, then setting $f^s = f - dv \in H^k(M, \otimes_S^m T^*M)$, we will obtain the sought result.

Thus, we are going to study the differential operator

$$P = \delta d : H^{k+1}(M, \otimes_S^m T^*M) \to H^{k-1}(M, \otimes_S^m T^*M)$$

In particular, we are going to prove that it is elliptic of order 2 with zero kernel. Using Theorem A.4, this will imply unicity and existence of a solution v to the problem (D.1). The estimate will be provided by the global elliptic estimate of Lemma A.6. Note that this estimate is still valid even though the manifold has boundary because we consider a differential operator with Dirichlet conditions on the boundary. The proof relies on a succession of lemmas.

LEMMA D.2. — *The principal symbol of the inner derivative*

$$d : H^k(M, \otimes_S^m T^*M) \to H^{k-1}(M, \otimes_S^{m+1} T^*M)$$

is $\sigma_d(x,\xi)u \mapsto \sigma(\xi \otimes u)$ *(where the second* σ *denotes the operator of symmetrization).*

Proof of the lemma. — Let us do the computation in local coordinates. It is sufficient to do the computation for the operator ∇ since the result is recovered by linearity of the operator of symmetrization σ. Moreover, ∇, is a differential operator of order 1 and since we want to compute its principal symbol, we can forget the part of the operator which is of order 0. Let us denote by $\tilde{\nabla}$ the homogeneous part of order 1 of ∇. If $T = T_{i_1 \ldots i_m} dx^{i_1} \otimes \ldots \otimes dx^{i_m}$, then :

$$\tilde{\nabla} T = \frac{\partial T_{i_1 \ldots i_m}}{\partial x_k} dx^k \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_m} = \sum_{k=1}^n a^k(x) \partial_k T,$$

where $a^k(x) : \Gamma(M, \otimes_S^m T^*M) \to \Gamma(M, \otimes_S^{m+1} T^*M)$ is the operator defined by :

$$a^k(x)(dx^{i_1} \otimes \ldots \otimes dx^{i_m}) = dx^k \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_m}$$

Therefore, we have :

$$\sigma_{\tilde{\nabla}}(x,\xi) = \sum_{k=1}^n a^k(x)\xi_k = \xi \otimes \cdot,$$

which provides the sought result. □

LEMMA D.3. — *The principal symbol of the divergence (the adjoint of* d*)*

$$\delta : H^k(M, \otimes_S^{m+1} T^*M) \to H^{k-1}(M, \otimes_S^m T^*M)$$

is given by :

$$\sigma_\delta(x,\xi) = -(\sigma_d(x,\xi))^* = -C_m(\cdot)(\xi^\sharp)$$

Here, C_m denotes the contraction according to the last factor, namely in coordinates :

$$C_m(dx^{i_1} \otimes ... \otimes dx^{i_m})(X) = dx^{i_m}(X) \cdot dx^{i_1} \otimes ... \otimes dx^{i_{m-1}}$$

Proof of the lemma. — Recall that in coordinates, one has :

$$(\delta T)_{i_1 \dots i_{m-1}} = -\frac{\partial T_{i_1 \dots i_{m-1} j}}{\partial x_k} g^{jk}$$

Thus :

$$\delta T = \sum_k b^k(x) \partial_k T,$$

where $b^k(x) : \Gamma(M, \otimes_S^{m+1} T^*M) \to \Gamma(M, \otimes_S^m T^*M)$ is defined by :

$$b^k(x)(dx^{i_1} \otimes ... \otimes dx^{i_{m-1}} \otimes dx^{i_m}) = -g^{i_m k}(x) dx^{i_1} \otimes ... \otimes dx^{i_{m-1}}$$

Thus its symbol is given by :

$$\sigma_\delta(x,\xi)(T) = \sum_k \xi_k b^k(x) T = -C_m(T)(\xi^\sharp)$$

□

Lemma D.4. — *The principal symbol of $\sigma_{\delta d}$ is given by :*

$$\sigma_{\delta d}(x,\xi) = \sigma_\delta(x,\xi) \circ \sigma_d(x,\xi) = -\frac{1}{m+1} |\xi|^2 Id + \frac{m}{m+1} \sigma_d(x,\xi) \circ \sigma_\delta(x,\xi)$$

Démonstration. — This lemma is proved using the two previous identities. Consider a symmetric m-tensor $T = T_{i_1 \dots i_m} dx^{i_1} \otimes ... \otimes dx^{i_m}$. Then :

$$\sigma_d(x,\xi) = \sigma(\xi \otimes T) = \frac{1}{m+1} \sum_{k=0}^{m} T_{i_1 \dots i_m} \underbrace{dx^{i_1} \otimes ... \otimes \xi}_{k-\text{fold product}} \otimes ... \otimes dx^{i_m}$$

So :

$$\sigma_\delta(x,\xi)\left(\sigma_d(x,\xi) T\right) = -\frac{1}{m+1} \sum_{k=0}^{m-1} T_{i_1 \dots i_m} \xi_l g^{i_m l} dx^{i_1} \otimes ... \otimes \xi \otimes ... \otimes dx^{i_{m-1}}$$
$$- \frac{1}{m+1} |\xi|^2_{g^{-1}} T$$

On the other hand, one can compute :

$$\sigma_d(x,\xi)\left(\sigma_\delta(x,\xi) T\right) = -\frac{1}{m} \sum_{k=0}^{m-1} T_{i_1 \dots i_m} \xi_l g^{i_m l} dx^{i_1} \otimes ... \otimes \xi \otimes ... \otimes dx^{i_{m-1}}$$

Thus :

$$\sigma_\delta \circ \sigma_d = -\frac{1}{m+1} |\xi|^2_{g^{-1}} Id + \frac{m}{m+1} \sigma_d \circ \sigma_\delta$$

□

Since d and δ are formally dual and of order 1, the symbol $\sigma_{\delta d}$ operator is definite negative as long as $\xi \neq 0$. Thus it is elliptic and Fredholm according to Theorem A.4. But δd is formally self-adjoint, so its index is zero. Let us prove that it admits $\{0\}$ as kernel. Consider $v \in \ker(\delta d)$ (it is smooth by ellipticity of the operator). Then, since $v|_{\partial M} = 0$, one has :

$$\langle \delta dv, v\rangle = -\langle dv, dv\rangle = 0$$

So $dv = 0$. Thus, identifying the tensor v with its associated function in SM, one has $v \equiv 0$ by integrating $dv = X \cdot v$ along geodesics (and using the fact that $v|_{\partial M} = 0$).

Moreover, the estimates are provided by Lemma A.6. We have :

$$||v||^2_{H^{k+1}} \leqslant C(||\delta dv||^2_{H^{k-1}} + ||v||^2_{L^2}) = C(||\delta f||^2_{H^{k-1}} + ||v||^2_{L^2})$$

The Poincaré inequality (written for the function canonically associated to the tensor v on SM and still denoted v) holds since $v|_{\partial M} = 0$:

$$||v||^2_{L^2} \leqslant C||dv||^2 = C|\langle \delta dv, v\rangle| = C|\langle \delta f, v\rangle| \leqslant C||\delta f||_{L^2}||v||_{L^2}$$

Thus : $||v||_{L^2} \leqslant C||\delta f||_{L^2} \leqslant C||\delta f||_{H^{k-1}}$, which gives the second estimate. As to the first estimate, we have : $f^s = f - dv$. Thus :

$$\begin{aligned}||f^s||_{H^k} \leqslant ||f||_{H^k} + ||dv||_{H^k} \leqslant ||f||_{H^k} + C||v||_{H^{k+1}} &\leqslant ||f||_{H^k} + C||\delta f||_{H^{k-1}} \\ &\leqslant C||f||_{H^k}\end{aligned}$$

□

D.2. Decomposition in a vicinity of the boundary

Assume (M, g) is an n-dimensional compact manifold with boundary. Then, there exists an annulus $\psi : [0, \varepsilon) \times \partial M \to M$ such that $\psi^* g = dr^2 + h_r$, where $r \in [0, \varepsilon)$ and h_r is a family of metrics on ∂M. Indeed, one can consider

$$\psi(r, u) = \exp_u(r\nu(u)),$$

for ε strictly less than the radius of injectivity of exp on (M, g) and where ν is the inward unitary normal vector to ∂M. This gives the sought result using Gauss' lemma.

Theorem D.5. — *Given $f \in \mathcal{C}^\infty(M, \otimes^m_S T^*M)$, there exists, in a vicinity of ∂M, $h \in \mathcal{C}^\infty(M, \otimes^{m-1}_S T^*M)$ and $p \in \mathcal{C}^\infty(M, \otimes^m_S T^*M)$ such that $h|_{\partial M} = 0$ and $i_{\frac{\partial}{\partial r}} p = 0$ (where i denotes the interior product) and :*

$$f = dh + p,$$

where $d = \sigma\nabla$ is the symmetric covariant derivative.

For the sake of simplicity, and because it is the only case studied in this memoire, we will only prove this result in the two-dimensional case. The arguments given can be extend to any dimension, but their writing may become more tedious.

Démonstration. — Let us assume that such a decomposition exists. In the sequel, our arguments will show that it is indeed uniquely determined. We identify $\partial M \simeq \mathbb{S}^1$ and use the coordinates $(r, \theta) \in [0, \varepsilon) \times \mathbb{S}^1$. We write in short

$$f = f_0(r,\theta)dr^m + f_1 dr^{m-1} \otimes d\theta + ... + f_m d\theta^m,$$
$$h = h_1 dr^{m-1} + h_2 dr^{m-2} \otimes d\theta + ... + h_m d\theta^{m-1},$$

where by $dr^k \otimes d\theta^{m-k}$ we actually mean its symmetrized $\sigma\left(dr^k \otimes d\theta^{m-k}\right)$. Note that, since the tensors are symmetric, we know that the coefficient in front of each term containing the same number of dr (or $d\theta$, which is equivalent) will be equal. The fact that $i_{\frac{\partial}{\partial r}} p = 0$ simply means that $p = a(r,\theta)d\theta^m$ for some smooth function a. The fact that $\nabla_{\frac{\partial}{\partial r}} \dfrac{\partial}{\partial r} = 0$ implies that

$$\nabla(dr) = \alpha(dr \otimes d\theta + d\theta \otimes dr) + \beta d\theta^2$$
$$\nabla(d\theta) = \lambda(dr \otimes d\theta + d\theta \otimes dr) + \mu d\theta^2$$

One can actually show, using the Koszul formula (2.5) that $\alpha = 0$. In particular, ∇ applied on dr or $d\theta$ does not raise the number of dr in the tensor product.

Now, we prove by induction that we recover h in a unique way. The equation $f = dh + p$ projected on the coordinate dr^m provides :

$$f_0(r,\theta) = \frac{\partial h_1}{\partial \theta}(r,\theta)$$

Moreover, since $h|_{\partial M} = 0$, we have $h_1(0,\theta) = 0$ and thus, h_1 is given by the formula :

$$h_1(r,\theta) = \int_0^r f_0(s,\theta)ds$$

By the previous remark made on $\nabla(dr)$ and $\nabla(d\theta)$, using the Leibniz formula for tensors, we see that the coefficient in front of the terms $dr^{m-1} \otimes d\theta$ in the equality $f = dh + p$ will be obtained by :

$$h_1\tilde{\lambda}_2 + \frac{\partial h_1}{\partial r}\tilde{\beta}_2 + h_2\tilde{\gamma}_2 + \omega_2\frac{\partial h_2}{\partial r} = f_1,$$

where $\tilde{\lambda}_2, \tilde{\beta}_2$ and $\tilde{\gamma}_2$ are smooth coefficients which can be expressed in terms of the $\alpha, \beta, \lambda, \mu$ and some constants of symmetrization (independent of the tensors) and $\omega_2 \neq 0$ is a constant of symmetrization. We already know h_1

and since $h_2(0,\theta) = 0$, by intgegrating this ordinary differential equation, we obtain the formula :

$$h_2(r,\theta) = \frac{1}{\omega_2}\int_0^r \exp\left(-\int_0^s \frac{\tilde{\gamma}(u,\theta)}{\omega_2}du\right)\left(f_1 - h_1\tilde{\lambda} - \frac{\partial h_1}{\partial r}\tilde{\beta}\right)(s,\theta)ds$$

Now, iterating this process, we see that we can always write the coefficient in front of the term $dr^k \otimes d\theta^{m-k}$ in the equality $f = dh + p$:

$$h_k\tilde{\lambda}_{k+1} + \frac{\partial h_k}{\partial r}\tilde{\beta}_{k+1} + h_{k+1}\tilde{\gamma}_{k+1} + \omega_{k+1}\frac{\partial h_{k+1}}{\partial r} = f_k,$$

with $h_{k+1}(0,\theta) = 0$. Thus, integrating this differential equation, we get h_{k+1}. Iterating this process, we see that we can recover h and that it is unique.

Setting $p = f - dh$, we see that the only term left will be of the form $a(r,\theta)d\theta^m$, and p is unique.

□

Annexe E. The Riemann-Roch theorem

E.1. Holomorphic line bundles

Let L be a complex line bundle over an oriented Riemannian surface (M,g). We assume L is endowed with a hermitian metric and a metric connection. A holomorphic structure on L is the choice of a covering of M by a collection of charts $\varphi_j : U_j \to \varphi_j(U_j) \subset \mathbb{C}$ and local sections s_j over $\varphi_j(U_j)$ such that $s_k = \psi_{jk}s_j$, where the ψ_{jk} are holomorphic.

We can look at the complexified tangent and cotangent bundles. We denote by $T^*_{\mathbb{C}}M = \mathbb{C}\otimes TM^*$. By Appendix B, we know that the oriented manifold M admits local isothermal coordinates which, as we have seen, is equivalent to the existence of holomorphic charts. Thus, we can define in holomorphic coordinates $dz = dx + idy$ and $d\bar{z} = dx - idy$ and this is independent of the choice of holomorphic coordinates. We define κ to be the holomorphic line bundle generated by dz and $\bar{\kappa}$ the line bundle generated by $d\bar{z}$. In particular, we have :

$$T^*_{\mathbb{C}}M = \kappa \oplus \bar{\kappa}$$

If L is a holomorphic line bundle over M, then we have a naturally defined operator

$$\bar{\partial} : \mathcal{C}^\infty(M,L) \to \mathcal{C}^\infty(M, L\otimes\bar{\kappa}),$$

which we can define as follows. Given a local (nowhere vanishing) holomorphic section S of L on an open set $U \subset M$, then any arbitrary section u in

U is of the form $u = vS$ for some complex-valued smooth function v. We set :

$$\bar{\partial} u = \frac{\partial v}{\partial \bar{z}} S \otimes d\bar{z},$$

and one can check that this is both independent of the choice of holomorphic coordinates and of the local section S.

If L is a holomorphic line bundle, then we can define its inverse L^{-1} (up to an isomorphism) such that $L \otimes L^{-1} \simeq M \times \mathbb{C}$ is the trivial bundle. Actually, since for any vector bundle E over M $E \otimes E^* \simeq \mathrm{End}(E)$, we can take $L^{-1} = L^*$, the dual vector bundle of L. Indeed, $\mathrm{End}(L)$ is trivial (it is isomorphic to the trivial bundle $M \times \mathbb{C}$) because there exists a non-vanishing section (one can take $\mathrm{id} \in \mathrm{End}(L)$ such that $\mathrm{id}(x) = id_{L_x} \in \mathrm{End}(L_x)$).

Note that in this setting, $T_{\mathbb{C}}M = \mathbb{C} \otimes TM$ splits in two holomorphic line bundles, and we have, according to the paragraph above : κ^{-1} is the holomorphic line bundle generated by $\partial/\partial z = 1/2(\partial/\partial x - i\partial/\partial y)$ and $\bar{\kappa}^{-1}$ is the holomorphic line bundle generated by $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$ and

$$T_{\mathbb{C}}M = \kappa^{-1} \ \oplus \bar{\kappa}^{-1}$$

On κ^{-1} the multiplication by i acts as the endomorphism J on TM (the complex structure) induced by the conformal class of the metric and the orientation.

E.2. The first Chern class

Let M be a Riemannian manifold and L be a holomorphic line bundle over M. Consider two connections ∇ and ∇' on L, then since the space of connections is affine, $\nabla - \nabla' \in \Lambda^1(M, \mathrm{End}(L))$. But since $\mathrm{End}(L)$ is trivial (that is isomorphic to $M \times \mathbb{C}$), we can consider $\nabla - \nabla'$ as an element of $\Lambda^1(M, \mathbb{C})$, that is as a complex-valued 1-form on M. We write $\nabla - \nabla' = \alpha$.

In the same fashion, one can prove that F^∇ is a tensor (it is linear in each argument), which means in other words that $F^\nabla \in \Lambda^2(M, \mathrm{End}(L)) \simeq \Lambda^2(M, \mathbb{C})$. Here, using the crucial fact that L is a complex line bundle, a computation yields to $F^\nabla - F^{\nabla'} = d\alpha$. Now, since M is a surface, we automatically have that F^∇ is closed, that is $dF^\nabla = 0$ (actually, this keeps being true even in dimension greater than 2).

As a consequence, $[F^\nabla]$, the class in de Rham cohomology of F^∇, does not depend on ∇ but only on L. We now endow L with a hermitian metric and consider the Levi-Civita connection ∇. Since ∇ is metric, it satisfies the relation

$$\langle F^\nabla(X,Y)s, t\rangle = -\overline{\langle F^\nabla(X,Y)t, s\rangle},$$

and consider a non-vanishing local section s and $t = s$, one can show that F^∇ is imaginary or iF^∇ is real. Thus, $\frac{i}{2\pi}F^\nabla$ is a real-valued 2-form and we define the first Chern class of L as the class in cohomology of $[\frac{i}{2\pi}F^\nabla]$, which is independent of ∇. We define the first Chern number as

$$c_1(L) = \int_M \frac{i}{2\pi}F^\nabla$$

Now if L_1 and L_2 are two holomorphic vector bundles over M endowed with a hermitian metric, the tensorial product $L_1 \otimes L_2$ is still a holomorphic line bundle. If we chose two unitary connections ∇_1, ∇_2, then they naturally give rise to a connection $\nabla = \nabla_1 \otimes \nabla_2$ defined on $L_1 \otimes L_2$ by :

$$\nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$$

From this, and using the definition of the curvature (2.6), a computation leads to :

$$F^\nabla(X,Y)s_1 \otimes s_2 = \left(F^{\nabla_1}(X,Y) + F^{\nabla_2}(X,Y)\right) s_1 \otimes s_2$$

Thus :

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

In particular, if $1 = M \times \mathbb{C}$ denotes the trivial bundle, then, taking the flat connection d, one has $c_1(1) = 0$. Considering $\kappa \otimes \kappa^{-1} = 1$ and the previous equality, we obtain $c_1(\kappa) + c_1(\kappa^{-1}) = c_1(1) = 0$, that is :

$$c_1(\kappa^{-1}) = -c_1(\kappa)$$

But $c_1(\kappa^{-1})$ can be computed thanks to the Gauss-Bonnet formula. Indeed, κ^{-1} is TM considered as a holomorphic line bundle. Thus, $c_1(\kappa^{-1})$ is nothing but the integral of the curvature, which is the Euler characteristic by the Gauss-Bonnet formula :

$$c_1(\kappa^{-1}) = \int_M F^\nabla = \chi(M) = 2(1-g),$$

where ∇ is the metric connection associated to any metric g on M. And we obtain :

$$c_1(\kappa) = -c_1(\kappa^{-1}) = 2(g-1)$$

E.3. The Riemann-Roch formula

Let (M,g) be a compact connected Riemannian surface. We consider the holomorphic line bundle $L_k = \kappa^{\otimes k}$ over M. We recall that a function $u \in \Omega_k = \mathcal{C}^\infty(SM) \cap H_k$ (where H_k is the eigenspace associated to the eigenvalue k of the operator $-iV$) can be identified with a smooth section of L_k. As mentioned in a previous paragraph, we can define the operator

$$\bar{\partial}_k : \mathcal{C}^\infty(M, L_k) \to \mathcal{C}^\infty(M, L_k \otimes \bar{\kappa})$$

and the kernel $\mathcal{O}(k) = \ker\,(L_k)$ of this operator consists of holomorphic sections of L_k. As explained in Section 3.4.1, the dimension of the kernel of the operator $\eta_- : \Omega_k \to \Omega_{k-1}$ is that of the kernel of $\bar{\partial}_k$.

Theorem E.1 (The Riemann-Roch formula). — *If L is a holomorphic line bundle over the compact Riemannian surface M, then :*

$$dim\ \mathcal{O}(L) - dim\ \mathcal{O}(L^{-1} \otimes \kappa) = c_1(L) - \frac{1}{2}c_1(\kappa)$$

The Riemann-Roch formula can be interpreted as a particular case of the general Atiyah-Singer formula, linking the index of an elliptic pseudodifferential operator to the topology of the manifold. In our case, the operator is $\bar{\partial}_L$ (which is elliptic). Let us explain how we can recover the dimensions

If $L_0 = 1 = M \times \mathbb{C}$ denotes the trivial bundle then we have $\dim \mathcal{O}(0) = 1$, since M is connected and the holomorphic functions on M are the constants. It is a standard fact (but not obvious, stemming from Hodge theory) that $\dim \mathcal{O}(1) = g$, where g denotes the genus of M. It can also be seen from the Riemann-Roch formula, using the fact that $c_1(\kappa) = 2(g-1)$. Indeed, from the Riemann-Roch formula, taking $L = \kappa^{-1}$, we obtain :

$$\dim\ \mathcal{O}(1) = \dim\ \mathcal{O}(0) + \frac{1}{2}c_1(\kappa) = 1 + (g-1) = g$$

It is possible to prove, using arguments which involve meromorphic sections on M for instance, that for $k < 0$, $\dim \mathcal{O}(k) = 0$. Thus, taking $L_2 = \kappa \otimes \kappa$, the previous paragraph gives us $c_1(\kappa \otimes \kappa) = 2c_1(\kappa) = 4(g-1)$ and, using the fact that $L_2^{-1} \otimes \kappa \simeq \kappa^{-1}$, the Riemann-Roch formula writes :

$$\dim\ \mathcal{O}(2) = \dim\ \mathcal{O}(-1) + c_1(\kappa\otimes\kappa) - \frac{1}{2}c_1(\kappa) = 0 + 4(g-1) - (g-1) = 3(g-1)$$

Iterating this computation :

$$\dim\ \mathcal{O}(k) = (2k-1)(g-1)$$

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